Solutions to Sample Exam Three
Brendan Owens, 1552

[1]
(a) Find the Maclaurin series for $\sin x$ and $\cos x$.

Start with $f(x) = \sin x$.

$$\begin{array}{rl} f(x) = \sin x & f(0) = 0 \\ f'(x) = \cos x & f'(0) = 1 \\ f''(x) = -\sin x & f''(0) = 0 \\ f^{(3)}(x) = -\cos x & f^{(3)}(0) = -1 \\ f^{(4)}(x) = \sin x & \dots \end{array}$$

Taking derivatives, you see that they repeat every fourth derivative. Then the Maclaurin series for $\sin x$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

You can do the same for $\cos x$ or use the fact that we know $\sin x$ is equal to its Maclaurin series and $\cos x$ is its derivative. Then differentiating the series for $\sin x$ gives the series for $\cos x$ which is

$$1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \cdots = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots$$

(b) Write down the 3rd degree Maclaurin polynomials for each of $\sin x$ and $\cos x$.

Just take the terms up to degree 3 from the answers to part (a).

The 3rd degree Maclaurin polynomial for $\sin x$ is $x - \frac{x^3}{3!}$.

The 3rd degree Maclaurin polynomial for $\cos x$ is $1 - \frac{x^2}{2}$.

(c) Use 3rd degree Maclaurin polynomials to estimate $\sin 1$ and $\cos 1$.

$$\sin 1 \approx 1 - \frac{1^3}{3!} = \frac{5}{6}.$$

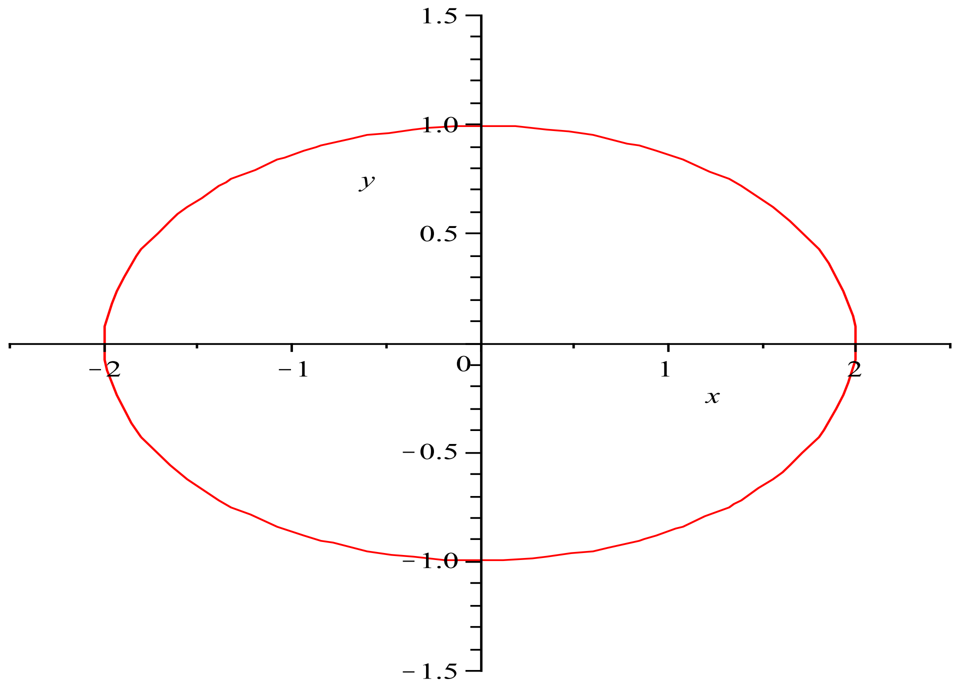
$$\cos 1 \approx 1 - \frac{1^2}{2} = \frac{1}{2}.$$

[2] Consider the parametric curve given by

$$x = 2\cos t,\ y = \sin t.$$

(a) Sketch the curve.

The curve is the ellipse shown.



(b) Find the slope of the tangent line when $t = \pi/6$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \\ &= \frac{\cos t}{-2\sin t}. \end{aligned}$$

At $t = \pi/6$,

$$\frac{dy}{dx} = \frac{\cos(\pi/6)}{-2\sin(\pi/6)} = -\frac{\sqrt{3}}{2}.$$

(c) Write down but do not attempt to evaluate an integral which represents the total arclength of the curve.

The curve is traversed once when t goes from 0 to 2π. The arclength is

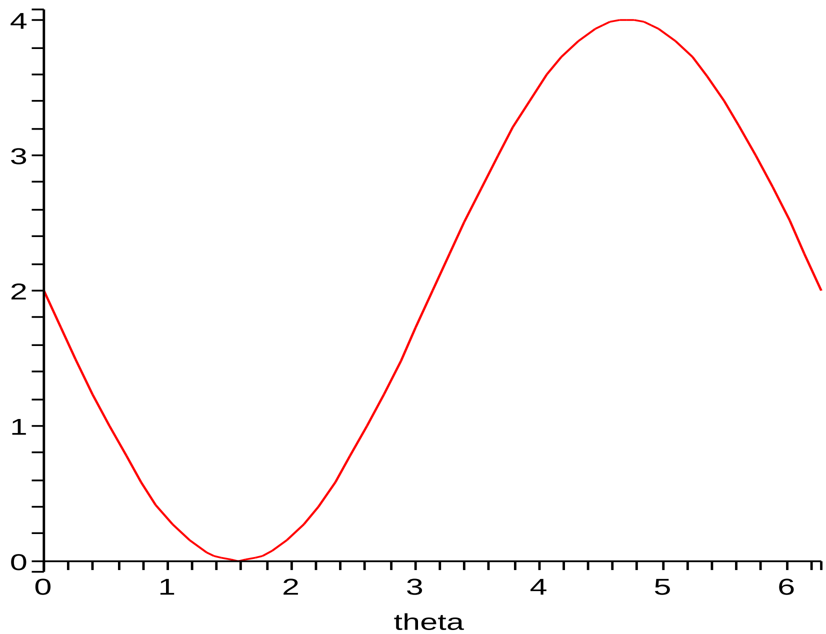
$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2}\, dt \\ &= \int_0^{2\pi} \sqrt{(-2\sin t)^2 + (\cos t)^2}\, dt.q \end{aligned}$$

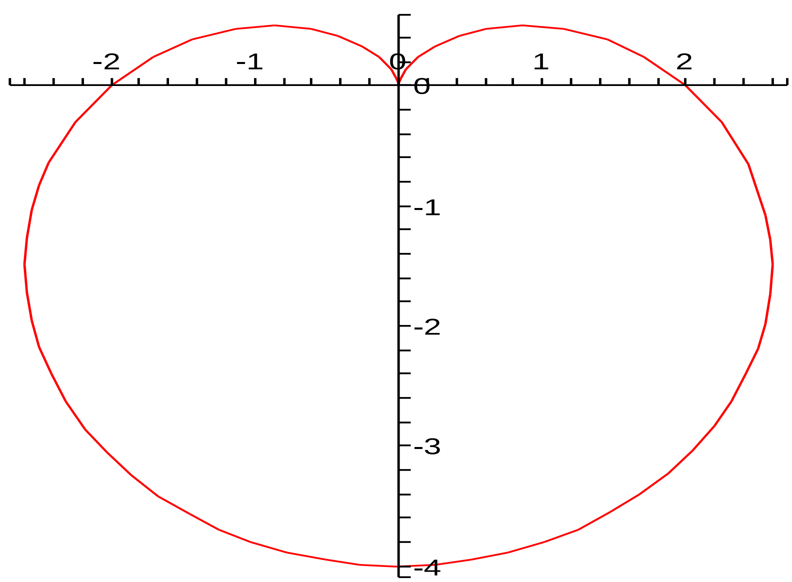
[3]

(a) Sketch the curve with the polar equation

$$r = 2(1 - \sin\theta).$$

(You should first sketch a Cartesian graph in the θ, r plane.)



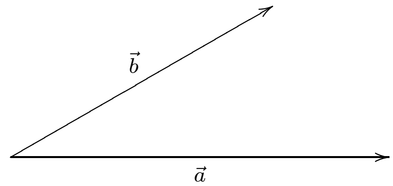


(b) Write down but do not evaluate an integral which represents the area inside this polar curve in the first quadrant.

From plotting the graph we see that the part in the first quadrant corresponds to θ from 0 to $\pi/2$, so the area is

$$\frac{1}{2}\int_0^{\pi/2}(2(1-\sin\theta))^2\,d\theta.$$

[4] Consider the vectors $\vec{a}$ and $\vec{b}$ shown in the following picture. Suppose that $\|\vec{a}\| = 5$, $\|\vec{b}\| = 4$ and the angle between them is thirty degrees.



(a) What is the dot product $\vec{a} \cdot \vec{b}$?

$$\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\| \cos\theta = 20(\sqrt{3}/2) = 10\sqrt{3}$$

(b) What is the length of the cross product $\|\vec{a} \times \vec{b}\|$?

$$\|\vec{a} \times \vec{b}\| = \|\vec{a}\|\|\vec{b}\| \sin\theta = 20(1/2) = 10$$

(c) Does the cross product point into or out of the page? Justify your answer.

It points out of the page, by the right hand rule.