Solution to 7590 exercise: $L(p,q) \cong S^3_{-p/q}(\text{unknot})$

Let p, q be coprime integers, not both zero. The lens space $L(p,q)$ is the quotient of $S^3 = \{(z,w) \in \mathbb{C}^2 \,|\, |z|^2 + |w|^2 = 1\}$ by the action of $\mathbb{Z}/p$ generated by

$$e^{2\pi i/p} \cdot (z,w) = (e^{2\pi i/p} z, e^{2\pi i q/p} w).$$

To demonstrate that $L(p,q) \cong S^3_{-p/q}(\text{unknot})$ (as oriented manifolds) it suffices to see that they may be represented by the same Heegaard diagram. We will illustrate this for the case of $L(3,1)$.

It will be convenient to use cartesian coordinates $w = u + iv$ in the w-plane and both polar and cartesian coordinates $z = re^{i\theta} = x + iy$ in the z-plane. Then S^3 may be parametrized by θ and w. A fundamental domain for the action of $\mathbb{Z}/3$ is given by $\{(\theta, w) \,|\, 0 \leq \theta < \frac{2\pi}{3}\}$; this is illustrated in Figure 1.

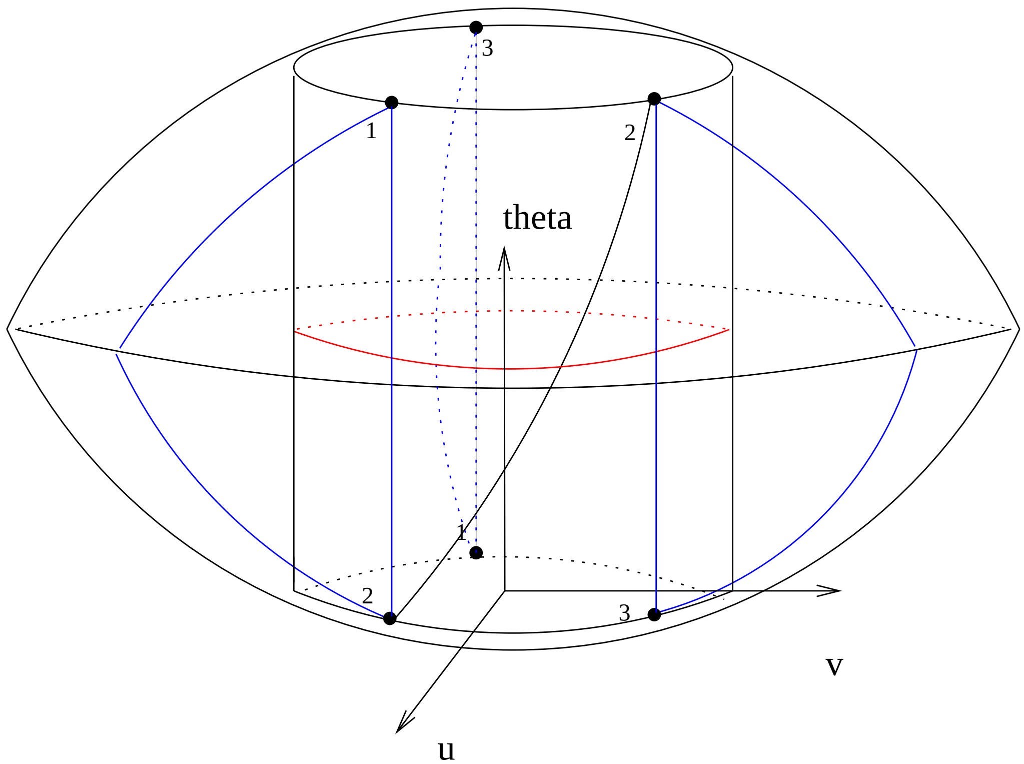

FIGURE 1. The lens space $L(3,1)$.

An oriented basis for $\mathbb{C}^2$ is given by $\{\partial_x, \partial_y, \partial_u, \partial_v\}$ (this does not depend on the choice of complex basis since $GL(2,\mathbb{C}) \hookrightarrow GL^+(4,\mathbb{R})$). At the point $(z,w) = (1,0)$, ∂_x is an outward normal to the unit ball and $\partial_\theta = \partial_y$. Thus an oriented frame for S^3 is given by $\{\partial_\theta, \partial_u, \partial_v\}$ or equivalently $\{\partial_u, \partial_v, \partial_\theta\}$. This basis is indicated in Figure 1.

The vertical cylinder in Figure 1 becomes a torus Σ in $L(3,1)$; each of the points marked 1, 2, 3 on the bottom is identified with that on the top with the same label. Orient Σ as the boundary of the solid torus U_α in which the red curve shown bounds a disk. The black curve shown passing through the point labelled 2 is a longitude. The blue curve, represented by 3 vertical segments, bounds a disk in $S^3 - U_\alpha$; this disk is indicated in the diagram, in which it appears as 3 blue triangles. Thus we arrive at the Heegaard diagram shown in Figure 2.

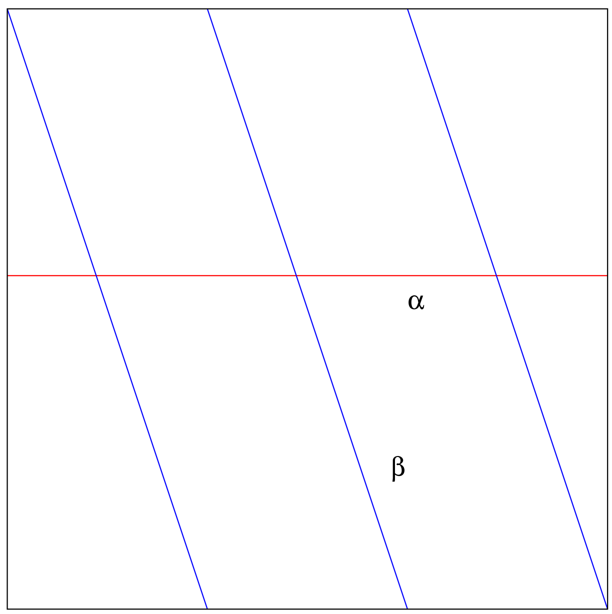

FIGURE 2. Heegaard diagram for $L(3,1)$.

It remains to see that we get the same diagram for the Dehn surgery. The boundary torus of a neighborhood νK of the unknot is shown in Figure 3. The red curve is the 0-framing λ which bounds a disk in $S^3 - \nu K$. The black curve is a right hand meridian μ. The pair (μ, λ) is an oriented basis for $H_1(\partial \nu K)$ and thus (λ, μ) is an oriented basis for the homology of $\Sigma = \partial(S^3 - \nu K)$. The Dehn surgery $S^3_{-p/q}$(unknot) is given by attaching a solid torus to the knot complement so that $-p\mu + q\lambda$ bounds a disk; this gives rise to the blue curve in Figure 2 when $p/q = 3/1$. (Note that the orientation of the Dehn surgery agrees with that of the knot complement.)

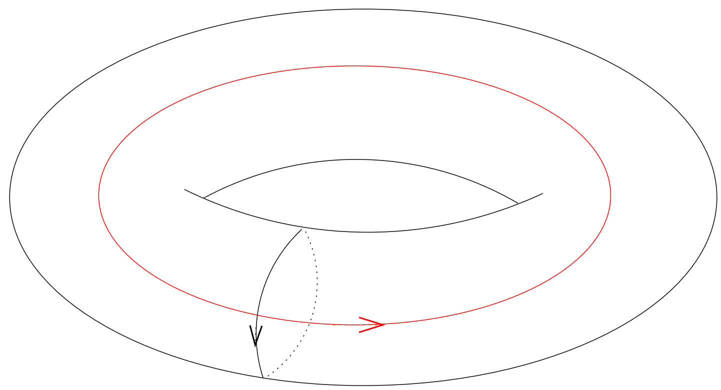

FIGURE 3. Boundary of neighborhood of unknot.