

A CHARACTERISATION OF THE $\mathbb{Z}^n \oplus \mathbb{Z}(\delta)$ LATTICE AND DEFINITE NONUNIMODULAR INTERSECTION FORMS

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ABSTRACT. We prove a generalisation of Elkies' theorem to nonunimodular definite forms (and lattices). Combined with inequalities of Frøyshov and of Ozsváth and Szabó, this gives a simple test of whether a rational homology 3-sphere may bound a definite four-manifold. As an example we show that small positive surgeries on torus knots do not bound negative-definite four-manifolds.

1. INTRODUCTION

The intersection pairing of a smooth compact four-manifold, possibly with boundary, is an integral symmetric bilinear form Q_X on $H_2(X; \mathbb{Z})/Tors$; it is nondegenerate if the boundary of X has first Betti number zero. Characteristic covectors for Q_X are elements of $H^2(X; \mathbb{Z})/Tors$ represented by the first Chern classes of spin^c structures. In the case that Q_X is positive-definite there are inequalities due to Frøyshov (using Seiberg-Witten theory, see [2]) and Ozsváth and Szabó which give lower bounds on the square of a characteristic covector. It is helpful in this context to prove existence of characteristic covectors with small square. The following is the main result of this paper.

Theorem 1. *Let Q be an integral positive-definite symmetric bilinear form of rank n and determinant δ . Then there exists a characteristic covector ξ for Q with*

$$\xi^2 \leq \begin{cases} n - 1 + 1/\delta & \text{if } \delta \text{ is odd,} \\ n - 1 & \text{if } \delta \text{ is even;} \end{cases}$$

moreover the inequality is strict unless $Q = (n - 1)\langle 1 \rangle \oplus \langle \delta \rangle$.

For unimodular forms this theorem was proved by Elkies [1]. To prove Theorem 1 we reinterpret the statement in terms of integral lattices. In Section 2 we introduce the necessary notions and then study short characteristic covectors of some special types of lattices. The proof of the theorem is presented in Section 3. The main idea

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is to use the theory of linking pairings to embed four copies of the lattice into a unimodular lattice and then apply Elkies' theorem along with results of Section 2.

For unimodular forms, a well-known and useful constraint on the square of a characteristic vector is that ξ^2 is congruent modulo 8 to the signature of the form. In Section 4 we give a generalisation of this to nonunimodular forms.

In Section 5 we combine Theorem 1 with an inequality of Ozsváth and Szabó [9, Theorem 9.6] to obtain the following theorem, where $d(Y, \mathfrak{t})$ denotes the correction term invariant defined in [9] for a spin^c structure \mathfrak{t} on a rational homology three-sphere Y :

Theorem 2. *Let Y be a rational homology sphere with $|H_1(Y; \mathbb{Z})| = \delta$. If Y bounds a negative-definite four-manifold X , and if either δ is square-free or there is no torsion in $H_1(X; \mathbb{Z})$, then*

$$\max_{\mathfrak{t} \in \text{Spin}^c(Y)} 4d(Y, \mathfrak{t}) \geq \begin{cases} 1 - 1/\delta & \text{if } \delta \text{ is odd,} \\ 1 & \text{if } \delta \text{ is even.} \end{cases}$$

The inequality is strict unless the intersection form of X is $(n-1)\langle -1 \rangle \oplus \langle -\delta \rangle$.

As an application we consider manifolds obtained as integer surgeries on knots. Knowing which positive surgeries on a knot bound negative-definite manifolds has implications for existence of fillable contact structures and for the unknotting number of the knot (see e.g. [5, 8]). In particular we consider torus knots; it is well known that $(pq-1)$ -surgery on the torus knot $T_{p,q}$ is a lens space. It follows that all large ($\geq pq-1$) surgeries on torus knots bound both positive- and negative-definite four-manifolds (see e.g. [8]). We show in Section 7 that small positive surgeries on torus knots cannot bound negative-definite four-manifolds. In particular, we obtain the following result; for a more precise statement see Corollary 7.2.

Proposition 3. *Let $2 \leq p < q$ and $1 \leq n \leq (p-1)(q-1) + 2$. Then $+n$ -surgery on $T_{p,q}$ cannot bound a negative-definite four-manifold with no torsion in H_1 .*

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2. SHORT CHARACTERISTIC COVECTORS IN SPECIAL CASES

An *integral lattice* L of rank n is the free abelian group \mathbb{Z}^n along with a nondegenerate symmetric bilinear form $Q : \mathbb{Z}^n \times \mathbb{Z}^n \rightarrow \mathbb{Z}$. Let V denote the tensor product

of L with \mathbb{R} ; thus V is the vector space \mathbb{R}^n to which the form Q extends, and L is a sublattice of V . The *signature* $\sigma(L)$ of the lattice L is the signature of Q . We say L is *definite* if $|\sigma(L)| = n$. For convenience denote $Q(x, y)$ by $x \cdot y$ and $Q(y, y)$ by y^2 . A set of generators v_1, \dots, v_n for L forms a basis of V satisfying $v_i \cdot v_j \in \mathbb{Z}$. With respect to such a basis the form Q is represented by the matrix $[v_i \cdot v_j]_{i,j=1}^n$. The *determinant* (or *discriminant*) of L is the determinant of the form (or the corresponding matrix).

The *dual lattice* $L' \cong \text{Hom}(L, \mathbb{Z})$ consists of all vectors $x \in V$ satisfying $x \cdot y \in \mathbb{Z}$ for all $y \in L$. A *characteristic covector* for L is an element $\xi \in L'$ with $\xi \cdot y \equiv y^2 \pmod{2}$ for all $y \in L$.

We say a lattice is *complex* if it admits an automorphism \mathbf{i} with \mathbf{i}^2 given by multiplication by -1 . A lattice is *quaternionic* if it admits an action of the quaternionic group $\{\pm 1, \pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k}\}$, with -1 acting by multiplication. Note that the rank of a complex lattice is even, and the rank of a quaternionic lattice is divisible by 4. For any lattice L , let L^m denote the direct sum of m copies of L . There is a standard way to make $L \oplus L$ into a complex lattice and L^4 into a quaternionic lattice; for example, the quaternionic structure is given by

$$\begin{aligned} \mathbf{i} &: (x, y, z, w) \mapsto (-y, x, -w, z) \\ \mathbf{j} &: (x, y, z, w) \mapsto (-z, w, x, -y). \end{aligned}$$

Let $\mathbb{Z}(\delta)$ denote the rank one lattice with determinant δ ; in particular $\mathbb{Z} = \mathbb{Z}(1)$.

Lemma 2.1. *Let $\delta \in \mathbb{N}$, and let p be a prime. Let L be an index p sublattice of $\mathbb{Z}^{n-1} \oplus \mathbb{Z}(\delta)$. Then L has a characteristic covector ξ with $\xi^2 < n - 1$ unless $L \cong \mathbb{Z}^{n-1} \oplus \mathbb{Z}(p^2\delta)$.*

Proof. We may assume L contains none of the summands of $\mathbb{Z}^{n-1} \oplus \mathbb{Z}(\delta)$; any such summand of L contributes 1 to the right-hand side of the inequality and at most 1 to the left-hand side.

If $n = 1$ then clearly $L \cong \mathbb{Z}(p^2\delta)$. Now suppose $n > 1$. Let $\{e, e_1, \dots, e_{n-2}, f\}$ be a basis for $\mathbb{Z}^{n-1} \oplus \mathbb{Z}(\delta)$, where e, e_1, \dots, e_{n-2} have square 1 and $f^2 = \delta$. Then multiples of e give coset representatives of L in $\mathbb{Z}^{n-1} \oplus \mathbb{Z}(\delta)$; it follows that a basis for L is given by $\{pe, e_1 + s_1e, \dots, e_{n-2} + s_{n-2}e, f + te\}$. Here s_i, t are nonzero residues modulo p in $[1 - p, p - 1]$, whose parities we may choose if p is odd. With respect to this basis, the bilinear form on L has matrix

$$Q = \begin{pmatrix} p^2 & ps_1 & ps_2 & ps_3 & \dots & ps_{n-2} & pt \\ ps_1 & 1 + s_1^2 & s_1s_2 & s_1s_3 & \dots & s_1s_{n-2} & s_1t \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ ps_{n-2} & s_1s_{n-2} & s_2s_{n-2} & s_3s_{n-2} & \dots & 1 + s_{n-2}^2 & s_{n-2}t \\ pt & s_1t & s_2t & s_3t & \dots & s_{n-2}t & \delta + t^2 \end{pmatrix},$$

with inverse

$$Q^{-1} = \begin{pmatrix} \frac{1 + \sum s_i^2 + t^2/\delta}{p^2} & -\frac{s_1}{p} & -\frac{s_2}{p} & -\frac{s_3}{p} & \dots & -\frac{s_{n-2}}{p} & -\frac{t}{p\delta} \\ -\frac{s_1}{p} & 1 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ -\frac{s_{n-2}}{p} & 0 & 0 & 0 & \dots & 1 & 0 \\ -\frac{t}{p\delta} & 0 & 0 & 0 & \dots & 0 & \frac{1}{\delta} \end{pmatrix}.$$

With respect to the dual basis, an element of the dual lattice L' is represented by an n -tuple $\xi \in \mathbb{Z}^n$. An n -tuple corresponds to a characteristic covector if its components have the same parity as the corresponding diagonal elements of Q .

Suppose now that p is odd. Choose s_i to be odd for all i and $t \equiv \delta \pmod{2}$. Then $\xi = (1, 0, \dots, 0)$ is a characteristic covector whose square satisfies

$$\xi^2 = \frac{1 + \sum s_i^2 + t^2/\delta}{p^2} < n - 1,$$

noting that $|s_i|, |t| \leq p - 1$.

Finally if $p = 2$ then $\xi = (0, 0, \dots, 0, (1 + \delta) \bmod 2)$ is a characteristic vector with $\xi^2 < n - 1$. \square

Lemma 2.2. *Let $\delta \in \mathbb{N}$ be odd, and let p be a prime with $p = 2$ or $p \equiv 1 \pmod{4}$. Let M be an index p complex sublattice of $\mathbb{Z}^{2n-2} \oplus \mathbb{Z}(\delta)^2$. Then M has a characteristic covector ξ with $\xi^2 < 2n - 2$ unless $M \cong \mathbb{Z}^{2n-2} \oplus \mathbb{Z}(p\delta)^2$.*

Proof. We may assume M contains none of the summands of $\mathbb{Z}^{2n-2} \oplus \mathbb{Z}(\delta)^2$. Suppose first that $n = 1$. Let $\{e, \mathbf{ie}\}$ be a basis for $\mathbb{Z}(\delta)^2$ with $e^2 = \delta$. Then $\{pe, \mathbf{ie} + se\}$ is a basis for M , for some s . Now

$$\begin{aligned} \mathbf{i}(\mathbf{ie} + se) &= -e + s\mathbf{ie} \in M \\ \implies -e - s^2e &\in M, \end{aligned}$$

from which it follows that p divides $1 + s^2$. The bilinear form on L has matrix

$$Q = \begin{pmatrix} p^2\delta & ps\delta \\ ps\delta & \delta(1 + s^2) \end{pmatrix} = p\delta \begin{pmatrix} p & s \\ s & \frac{1+s^2}{p} \end{pmatrix} \cong p\delta I,$$

since any positive-definite unimodular form of rank 2 is diagonalisable. Thus in this case $M \cong \mathbb{Z}(p\delta)^2$.

Now suppose $n > 1$. Let $\{e, e_1, \dots, e_{2n-3}, f_1, f_2\}$ be a basis for $\mathbb{Z}^{2n-2} \oplus \mathbb{Z}(\delta)^2$, where e, e_i have square 1 and $f_i^2 = \delta$. A basis for M is given by $\{pe, e_i + s_i e, f_i + t_i e\}$, where s_i, t_i are nonzero residues modulo p which we may choose to be odd integers

in $[1 - p, p - 1]$. The matrix in this basis is

$$Q = \begin{pmatrix} p^2 & ps_1 & ps_2 & ps_3 & \dots & ps_{2n-3} & pt_1 & pt_2 \\ ps_1 & 1 + s_1^2 & s_1s_2 & s_1s_3 & \dots & s_1s_{2n-3} & s_1t_1 & s_1t_2 \\ ps_2 & s_1s_2 & 1 + s_2^2 & s_2s_3 & \dots & s_2s_{2n-3} & s_2t_1 & s_2t_2 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ ps_{2n-3} & s_1s_{2n-3} & s_2s_{2n-3} & s_3s_{2n-3} & \dots & 1 + s_{2n-3}^2 & s_{2n-3}t_1 & s_{2n-3}t_2 \\ pt_1 & s_1t_1 & s_2t_1 & s_3t_1 & \dots & s_{2n-3}t & \delta + t_1^2 & t_1t_2 \\ pt_2 & s_1t_2 & s_2t_2 & s_3t_2 & \dots & s_{2n-3}t & t_1t_2 & \delta + t_2^2 \end{pmatrix},$$

with inverse

$$Q^{-1} = \begin{pmatrix} \frac{1 + \sum s_i^2 + \sum t_i^2 / \delta}{p^2} & -\frac{s_1}{p} & -\frac{s_2}{p} & \dots & -\frac{s_{2n-3}}{p} & -\frac{t_1}{p\delta} & -\frac{t_2}{p\delta} \\ -\frac{s_1}{p} & 1 & 0 & \dots & 0 & 0 & 0 \\ -\frac{s_2}{p} & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & & \\ -\frac{s_{2n-3}}{p} & 0 & 0 & \dots & 1 & 0 & 0 \\ -\frac{t_1}{p\delta} & 0 & 0 & \dots & 0 & \frac{1}{\delta} & 0 \\ -\frac{t_2}{p\delta} & 0 & 0 & \dots & 0 & 0 & \frac{1}{\delta} \end{pmatrix}.$$

If $p = 2$ then Q is even, and $\xi = 0$ is characteristic.

Suppose now that p is odd. Then $\xi = (k, l_1, \dots, l_{2n-1})$ is characteristic if k is odd and each l_i is even. By completing squares we have that

$$\xi^2 = \frac{1}{p^2} \left(k^2 + \sum_{i=1}^{2n-3} (ks_i - pl_i)^2 + \frac{1}{\delta} \sum_{i=1}^2 (kt_i - pl_{2n-3+i})^2 \right).$$

It clearly suffices to prove the statement for $\delta = 1$ and $n = 2$; thus the following sublemma completes the proof.

Sublemma 2.3. *Let $F(k, l_1, l_2, l_3) = k^2 + \sum_{i=1}^3 (ks_i - l_i p)^2$, where s_i are odd numbers with $|s_i| \leq p - 2$. Then there exist k odd and l_i even for which $F < 2p^2$.*

Proof. Let $K = \{2 - p, 4 - p, \dots, p - 2\}$, and let

$$K_i = \left\{ k \in K \mid \exists l_i \text{ even with } |ks_i - l_i p| < \frac{p-1}{2} \right\}.$$

Note that $|K_i| = \frac{p-1}{2}$ (since $p \equiv 1 \pmod{4}$).

If there exists $k \in \cap_{i=1}^3 K_i$ then with this choice of k we find

$$F < p^2 + 3 \left(\frac{p-1}{2} \right)^2 < 2p^2.$$

Otherwise let K_{ij} denote $K_i \cap K_j$. Then

$$p-1 \geq |\cup_{i=1}^3 K_i| = \sum_{i=1}^3 |K_i| - \sum_{i<j} |K_{ij}|,$$

from which it follows that $\sum_{i<j} |K_{ij}| \geq \frac{p-1}{2}$. Thus there exists some $k \in K_{ij}$ with $|k| \leq \frac{p+1}{2}$. With this k we find

$$F < p^2 + 2 \left(\frac{p-1}{2} \right)^2 + \left(\frac{p+1}{2} \right)^2 < 2p^2.$$

□

Lemma 2.4. *Let $\delta \in \mathbb{N}$ be odd, and let q be a prime with $q \equiv 3 \pmod{4}$. Let N be an index q^2 quaternionic sublattice of $\mathbb{Z}^{4n-4} \oplus \mathbb{Z}(\delta)^4$, with the quotient group $(\mathbb{Z}^{4n-4} \oplus \mathbb{Z}(\delta)^4)/N$ having exponent q . Then N has a characteristic covector ξ with $\xi^2 < 4n-4$ unless $N \cong \mathbb{Z}^{4n-4} \oplus \mathbb{Z}(q\delta)^4$.*

Proof. We may assume N contains none of the summands of $\mathbb{Z}^{4n-4} \oplus \mathbb{Z}(\delta)^4$. Suppose first that $n = 1$. Let $\{e, \mathbf{i}e, \mathbf{j}e, \mathbf{k}e\}$ be a basis for $\mathbb{Z}(\delta)^4$ with $e^2 = \delta$. Note that e and $\mathbf{i}e$ represent generators of the quotient $\mathbb{Z}/q \oplus \mathbb{Z}/q$; otherwise we have $e + \mathbf{s}\mathbf{i}e \in N$ for some s , and multiplication by \mathbf{i} yields $s^2 + 1 \equiv 0 \pmod{q}$, but -1 is not a quadratic residue modulo q . Now a basis for N is given by $\{qe, q\mathbf{i}e, \mathbf{j}e + s_1e + t_1\mathbf{i}e, \mathbf{k}e + s_2e + t_2\mathbf{i}e\}$. The quaternionic symmetry yields

$$s_1 \equiv t_2, \quad s_2 \equiv -t_1, \quad 1 + s_1^2 + s_2^2 \equiv 0 \pmod{q};$$

it follows that the bilinear form Q on N factors as $q\delta$ times a unimodular form. Thus $N \cong \mathbb{Z}(q\delta)^4$.

We now assume $n = 2$ and $\delta = 1$ (the proof for any other n, δ follows from this case). Let $\{e, \mathbf{i}e, v_1 = \mathbf{j}e, v_2 = \mathbf{k}e, v_3 = f, v_4 = \mathbf{i}f, v_5 = \mathbf{j}f, v_6 = \mathbf{k}f\}$ be a basis of unit vectors for \mathbb{Z}^8 . Then $\{qe, q\mathbf{i}e, v_1 + s_1e + t_1\mathbf{i}e, \dots, v_6 + s_6e + t_6\mathbf{i}e\}$ is a basis for N . In this case the quaternionic symmetry yields

$$s_{i+1} \equiv -t_i, \quad t_{i+1} \equiv s_i, \quad 1 + s_1^2 + s_2^2 \equiv 0 \pmod{q}$$

for $i = 1, 3, 5$, and at most one of s_3, \dots, s_6 is divisible by q . We may choose s_i, t_i in $[1-q, q-1]$ with $s_i + t_i$ odd. Computation of Q, Q^{-1} in the above basis for N now yields that $\xi = (k_1, k_2, l_1, \dots, l_6)$ is characteristic if k_1, k_2 are odd and l_i are even, and

$$\xi^2 = \frac{1}{q^2} \left(k_1^2 + k_2^2 + \sum_{i=1}^6 (k_1 s_i + k_2 t_i - l_i q)^2 \right).$$

There are now two cases to consider, depending on whether or not one of the s_i is zero. Existence of a characteristic ξ with $\xi^2 < 4$ follows in each case from one of the following sublemmas.

Sublemma 2.5. *Let $F = k_1^2 + k_2^2 + \sum_{i=1}^6 (k_1 s_i + k_2 t_i - l_i q)^2$, where s_i are odd and t_i are even integers in $[1 - q, q - 1]$. Then there exist k_1, k_2 odd and l_i even for which $F < 4q^2$.*

Proof. Set $k_2 = 1$. Let $K = \{2 - q, 4 - q, \dots, q - 2\}$, and let

$$K_i = \left\{ k \in K \mid \exists l_i \text{ even with } |k s_i + t_i - l_i q| \leq \frac{q-1}{2} \right\}.$$

Note that $|K_i| \geq \frac{q-1}{2}$ (in fact K_i either contains $\frac{q-1}{2}$ or $\frac{q+1}{2}$ elements, depending on the value of t_i).

If there is a k which is in the intersection of 4 of the K_i 's, then setting $k_1 = k$ we obtain

$$F \leq 1 + 3q^2 + 4 \left(\frac{q-1}{2} \right)^2 < 4q^2.$$

Otherwise let K_{ijk} denote the triple intersection of K_i, K_j, K_k , and let K'_{ij} denote $K_i \cap K_j - \cup_k K_{ijk}$. Then

$$q - 1 \geq |\cup_{i=1}^6 K_i| = \sum_{i=1}^6 |K_i| - \sum_{i < j} |K'_{ij}| - 2 \sum_{i < j < k} |K_{ijk}|,$$

from which it follows that

$$\sum_{i < j} |K'_{ij}| + 2 \sum_{i < j < k} |K_{ijk}| \geq 2(q - 1),$$

and hence

$$\sum_{i < j < k} |K_{ijk}| \geq \frac{q-1}{2}.$$

Thus there exists some $k \in K_{ijk}$ with $|k| \leq \frac{q-1}{2}$. Setting $k_1 = k$ again yields

$$F \leq 1 + 3q^2 + 4 \left(\frac{q-1}{2} \right)^2 < 4q^2.$$

□

Sublemma 2.6. *Let $F = k_1^2 + k_2^2 + \sum_{i=1}^5 (k_1 s_i + k_2 t_i - l_i q)^2 + (k_2 t_6 - l_6 q)^2$, where s_i and t_6 are odd, and t_1, \dots, t_5 are even integers in $[1 - q, q - 1]$. Then there exist k_1, k_2 odd and l_i even for which $F < 4q^2$.*

Proof. First choose k_2 (and l_6) with $|k_2|, |k_2 t_6 - l_6 q| \leq \frac{q-1}{2}$. Again let $K = \{2 - q, 4 - q, \dots, q - 2\}$, and for each $i \leq 5$ let

$$K_i = \left\{ k \in K \mid \exists l_i \text{ even with } |k s_i + k_2 t_i - l_i q| \leq \frac{q-1}{2} \right\};$$

then $|K_i| \geq \frac{q-1}{2}$.

If there is a k which is in the intersection of 4 of the K_i 's, then setting $k_1 = k$ we obtain

$$F \leq 2q^2 + 6 \left(\frac{q-1}{2} \right)^2 < 4q^2.$$

Otherwise (with notation as above) we have

$$q-1 \geq |\cup_{i=1}^5 K_i| = \sum_{i=1}^5 |K_i| - \sum_{i<j} |K'_{ij}| - 2 \sum_{i<j<k} |K_{ijk}|,$$

from which it follows that

$$\sum_{i<j} |K'_{ij}| + 2 \sum_{i<j<k} |K_{ijk}| \geq \frac{3}{2}(q-1),$$

and hence

$$\sum_{i<j<k} |K_{ijk}| \geq \frac{q+1}{4}.$$

Thus there exists some $k \in K_{ijk}$ with $|k| \leq \frac{3(q+1)}{4}$. Setting $k_1 = k$ yields

$$F \leq 2q^2 + 5 \left(\frac{q-1}{2} \right)^2 + \left(\frac{3(q+1)}{4} \right)^2 < 4q^2.$$

□

3. PROOF OF THE MAIN THEOREM

The bilinear form Q on L induces a symmetric bilinear \mathbb{Q}/\mathbb{Z} -valued pairing on the finite group L'/L , called the *linking pairing* associated to L . Such pairings λ on a finite group G were studied by Wall [16] (see also [3]). He observed that λ splits into an orthogonal sum of pairings λ_p on the p -subgroups G_p of G . For any prime p let A_{p^k} (resp. B_{p^k}) denote the pairings on \mathbb{Z}/p^k with p^k times the square of a generator a quadratic residue (resp. nonresidue) modulo p . If p is odd then λ_p decomposes into an orthogonal direct sum of these two types of pairings on cyclic subgroups. The pairing on G_2 may be decomposed into cyclic summands (there are four equivalence classes of pairings on $\mathbb{Z}/2^k$ for $k \geq 3$), plus two types of pairings on $\mathbb{Z}/2^k \oplus \mathbb{Z}/2^k$; these are denoted E_{2^k}, F_{2^k} . For an appropriate choice of generators, in E_{2^k} each cyclic generator has square 0, while in F_{2^k} they each have square 2^{1-k} .

Proposition 3.1. *Let L be a lattice of rank n . Then L^4 may be embedded as a quaternionic sublattice of a unimodular quaternionic lattice U of rank $4n$.*

Proof. Consider an orthogonal decomposition of the linking pairing on L'/L as described above. Let $x \in L'$ represent a generator of a cyclic summand \mathbb{Z}/p^k with $k > 1$. Then $v = p^{k-1}x$ has $v^2 \in \mathbb{Z}$ and $pv \in L$. Adjoining v to L yields a lattice L_1 which contains L as an index p sublattice (so that $\det L = p^2 \det L_1$). Similarly, if $x \in L'$ represents a generator of a cyclic summand of E_{2^k} or F_{2^k} then $v = 2^{k-1}x$ may be adjoined. Finally if x_1, x_2 represent generators of two $\mathbb{Z}/2$ summands then $x_1 + x_2$ may be adjoined. In this way we get a sequence of embeddings

$$L = L_0 \subset L_1 \subset \cdots \subset L_{m_1},$$

where each L_i is an index p sublattice of L_{i+1} for some prime p . Moreover the linking pairing of L_{m_1} decomposes into cyclic summands of prime order, and the determinant of L_{m_1} is either odd or twice an odd number.

Now let $M_0 = L_{m_1} \oplus L_{m_1}$. Note that M_0 is a complex sublattice of $L' \oplus L'$. Let p be a prime which is either 2 or is congruent to 1 modulo 4, so that there exists an integer a with $a^2 \equiv -1 \pmod{p}$. Let $x \in L'$ represent a generator of a \mathbb{Z}/p summand of the linking pairing of L_{m_1} . Then $v = (x, ax) \in L' \oplus L'$ has $v^2 \in \mathbb{Z}$ and $pv \in M_0$. Adjoining v to M_0 yields a lattice M_1 ; since $\mathbf{i}v + av \in M_0$, M_1 is preserved by \mathbf{i} . Continuing in this way we obtain a sequence of embeddings

$$M_0 \subset M_1 \subset \cdots \subset M_{m_2},$$

where each M_i is an index p sublattice of M_{i+1} for some prime p with $p = 2$ or $p \equiv 1 \pmod{4}$. Each M_i is a complex sublattice of $L' \oplus L'$. We may arrange that each M_i with $i > 0$ has odd determinant. The resulting linking pairing of M_{m_2} is the orthogonal sum of pairings on cyclic groups of prime order congruent to 3 modulo 4, and all the summands appear twice.

Finally let $N_0 = M_{m_2} \oplus M_{m_2}$. This is a quaternionic sublattice of $(L')^4$. Let $q \equiv 3 \pmod{4}$ be a prime, and suppose that $x \in L'$ generates a \mathbb{Z}/q summand of the linking pairing of M_{m_2} . There exist integers a, b with $a^2 + b^2 \equiv -1 \pmod{q}$ (take m to be the smallest positive quadratic nonresidue and choose a, b so that $a^2 \equiv m - 1$, $b^2 \equiv -m$). Let $v_1 = (x, 0, ax, bx)$ and let $v_2 = \mathbf{i}v_1 = (0, x, -bx, ax)$. Then $v_i^2, v_1 \cdot v_2 \in \mathbb{Z}$, and $qv_i \in N_0$. Adjoining v_1, v_2 to N_0 yields a quaternionic sublattice N_1 of $(L')^4$; note that $\mathbf{j}v_1 + av_1 - bv_2 \in N_0$. We thus obtain a sequence

$$N_0 \subset N_1 \subset \cdots \subset N_{m_3} = U,$$

where each N_i is an index q^2 sublattice of N_{i+1} for some prime $q \equiv 3 \pmod{4}$, N_{i+1}/N_i has exponent q , each N_i is quaternionic with odd determinant, and U is unimodular. \square

Remark 3.2. *Using the methods of Proposition 3.1 one may show that L^2 embeds in a unimodular lattice if and only if the prime factors of $\det L$ congruent to 3 mod 4*

have even exponents. Applying this to rank 1 lattices one recovers the classical fact that any integer is expressible as the sum of 4 squares, and is the sum of 2 squares if and only if its prime factors congruent to 3 mod 4 have even exponents.

Proof of Theorem 1. Let L be the lattice with form Q . Embed L^4 in a unimodular lattice U of rank $4n$ as in Proposition 3.1. By Elkies' theorem [1] either $U \cong \mathbb{Z}^{4n}$ or U has a characteristic covector ξ with $\xi^2 \leq 4n - 8$. In the latter case, restricting to one of the four copies of L yields a covector $\xi|_L$ with $(\xi|_L)^2 \leq n - 2$, which easily satisfies the statement of the theorem.

This leaves the case that $U \cong \mathbb{Z}^{4n}$. Let $L_0, \dots, L_{m_1}, M_0, \dots, M_{m_2}, N_0, \dots, N_{m_3}$ be as in the proof of Proposition 3.1. By successive application of Lemma 2.4, we see that either $N_0 \cong \mathbb{Z}^{4n-4} \oplus \mathbb{Z}(\delta_1)^4$ for some $\delta_1 \in \mathbb{N}$, or N_0 has a characteristic covector ξ with $\xi^2 < 4n - 4$. In the latter case, restricting ξ to one of the four copies of L yields $(\xi|_L)^2 < n - 1$.

If $N_0 \cong \mathbb{Z}^{4n-4} \oplus \mathbb{Z}(\delta_1)^4$ then $M_{m_2} \cong \mathbb{Z}^{2n-2} \oplus \mathbb{Z}(\delta_1)^2$. By Lemma 2.2 we find that either $M_0 \cong \mathbb{Z}^{2n-2} \oplus \mathbb{Z}(\delta_2)^2$ or M_0 has a characteristic covector ξ with $\xi^2 < 2n - 2$. In the latter case, restricting ξ to one of the two copies of L embedded in M_0 yields $(\xi|_L)^2 < n - 1$.

Finally if $M_0 \cong \mathbb{Z}^{2n-2} \oplus \mathbb{Z}(\delta_2)^2$ then $L_{m_1} \cong \mathbb{Z}^{n-1} \oplus \mathbb{Z}(\delta_2)$. Successive application of Lemma 2.1 now yields that either $L \cong \mathbb{Z}^{n-1} \oplus \mathbb{Z}(\delta)$ or L has a characteristic covector ξ with $\xi^2 < n - 1$. \square

4. A CONGRUENCE CONDITION ON CHARACTERISTIC COVECTORS

Given a positive-definite symmetric bilinear form Q of rank n and determinant δ with $Q \neq (n-1)\langle 1 \rangle \oplus \langle \delta \rangle$, one may ask for an optimal upper bound on the square of a shortest characteristic covector ξ . The square of a characteristic covector of a unimodular form is congruent to the signature modulo 8 (see for example [14]). Thus if $\delta = 1$ we have $\xi^2 \leq n - 8$. In this section we give congruence conditions on the square of characteristic covectors of forms of arbitrary determinant. Lattices in this section are not assumed to be definite. (The results in this section may be known to experts but we have not found them in the literature.)

If ξ_1, ξ_2 are characteristic covectors of a lattice L of determinant δ then their difference is divisible by 2 in L' ; it follows that $\xi_1^2 \equiv \xi_2^2$ modulo $\frac{4}{\delta}$. For a lattice with odd determinant the congruence holds modulo $\frac{8}{\delta}$. We will give a formula for this congruence class in terms of the signature and linking pairing of L . For a lattice with determinant δ even or odd we will determine the congruence class of ξ^2 modulo $\frac{4}{\delta}$ in terms of the signature and determinant.

Lemma 4.1. *Suppose $\delta = \prod_{i=1}^r p_i^{k_i} \cdot \prod_{j=1}^s q_j^{l_j}$ where p_i, q_j are odd primes, not necessarily distinct. Let M be an even lattice of determinant δ with linking pairing isomorphic*

to $\bigoplus_{i=1}^r A_{p_i}^{k_i} \oplus \bigoplus_{j=1}^s B_{q_j}^{l_j}$. Then the signature of M satisfies the congruence

$$\sigma(M) \equiv \sum_{k_i \equiv 1(2)} (1 - p_i) + \sum_{l_j \equiv 1(2)} (5 - q_j) \pmod{8}.$$

(For the definition of A_{p^k} and B_{p^k} see the beginning of Section 3.)

Proof. Let $G(M)$ denote the Gauss sum

$$\frac{1}{\sqrt{\delta}} \sum_{u \in M'/M} e^{i\pi u^2}.$$

(See [15] for more details on Gauss sums and the Milgram Gauss sum formula.) Then $G(M)$ depends only on the linking pairing of M and in fact factors as

$$G(M) = \prod G(A_{p_i}^{k_i}) \cdot \prod G(B_{q_j}^{l_j}).$$

The factors are computed in [15, Theorem 3.9] to be:

$$G(A_{p^k}) = \begin{cases} 1 & \text{if } k \text{ is even,} \\ e^{2\pi i(1-p)/8} & \text{if } k \text{ is odd;} \end{cases}$$

$$G(B_{q^l}) = \begin{cases} 1 & \text{if } l \text{ is even,} \\ e^{2\pi i(5-q)/8} & \text{if } l \text{ is odd.} \end{cases}$$

(Note that in the notation of [15], A_{p^k} corresponds to $(C_p(k); 2)$ and B_{q^l} corresponds to $(C_q(l); 2n_q)$, where n_q is a quadratic nonresidue modulo q .)

The congruence on the signature of M now follows from the Milgram formula:

$$G(M) = e^{2\pi i \sigma(M)/8}.$$

□

Proposition 4.2. *Let L be a lattice of odd determinant δ with linking pairing isomorphic to $\bigoplus_{i=1}^r A_{p_i}^{k_i} \oplus \bigoplus_{j=1}^s B_{q_j}^{l_j}$. Let ξ be a characteristic covector for L . Then*

$$\xi^2 \equiv \sigma(L) - \sum_{k_i \equiv 1(2)} (1 - p_i) - \sum_{l_j \equiv 1(2)} (5 - q_j) \pmod{\frac{8}{\delta}}.$$

Proof. Let M be an even lattice with the same linking pairing as L . (Existence of M is proved by Wall in [16].) Then by [4, Satz 3] L and M are stably equivalent; that is to say, there exist unimodular lattices U_1, U_2 such that $L \oplus U_1 \cong M \oplus U_2$. Let ξ be a characteristic covector for $L \oplus U_1$. Then we have decompositions

$$\xi = \xi_L + \xi_{U_1} = \xi_M + \xi_{U_2}.$$

Taking squares we find

$$\begin{aligned}
\xi_L^2 &\equiv \xi_M^2 + \xi_{U_2}^2 - \xi_{U_1}^2 \\
&\equiv \sigma(U_2) - \sigma(U_1) \\
&\equiv \sigma(L) - \sigma(M) \\
&\equiv \sigma(L) - \sum_{k_i \equiv 1(2)} (1 - p_i) - \sum_{l_j \equiv 1(2)} (5 - q_j) \pmod{\frac{8}{\delta}},
\end{aligned}$$

where the last line follows from Lemma 4.1. \square

Corollary 4.3. *Let L be a lattice of determinant $\delta \in \mathbb{N}$, and let ξ be a characteristic covector for L . Then*

$$\xi^2 \equiv \begin{cases} \sigma(L) - 1 + 1/\delta & \text{if } \delta \text{ is odd} \\ \sigma(L) - 1 & \text{if } \delta \text{ is even} \end{cases} \pmod{\frac{4}{\delta}}.$$

Proof. If δ is odd then Proposition 4.2 shows that the congruence class of ξ^2 modulo $\frac{4}{\delta}$ depends only on the signature and determinant; the formula then follows by taking L to be the lattice with the form $r\langle 1 \rangle \oplus s\langle -1 \rangle \oplus \langle \delta \rangle$ where $r + s = n - 1$ and $r - s = \sigma(L) - 1$.

If δ is even then as in the proof of Proposition 3.1 we find that either L or $L \oplus \mathbb{Z}(2)$ embeds as an index 2^k sublattice of a lattice with odd determinant. It again follows that the congruence class of ξ^2 modulo $\frac{4}{\delta}$ depends only on the signature and determinant. \square

5. PROOF OF THEOREM 2

We begin by noting the following restatement of Theorem 1 for negative-definite forms:

Theorem 5.1. *Let Q be an integral negative-definite symmetric bilinear form of rank n and determinant of absolute value δ . Then there exists a characteristic covector ξ for Q with*

$$\xi^2 \geq \begin{cases} -n + 1 - 1/\delta & \text{if } \delta \text{ is odd,} \\ -n + 1 & \text{if } \delta \text{ is even;} \end{cases}$$

moreover the inequality is strict unless $Q = (n - 1)\langle -1 \rangle \oplus \langle -\delta \rangle$.

Let Y be a rational homology three-sphere and X a smooth negative-definite four-manifold bounded by Y , with $b_2(X) = n$. For any Spin^c structure \mathfrak{t} on Y let $d(Y, \mathfrak{t})$ denote the correction term invariant of Ozsváth and Szabó [9]. It is shown in [9, Theorem 9.6] that for each Spin^c structure $\mathfrak{s} \in \text{Spin}^c(X)$,

$$(1) \quad c_1(\mathfrak{s})^2 + n \leq 4d(Y, \mathfrak{s}|_Y).$$

The image of $c_1(\mathfrak{s})$ in $H^2(X; \mathbb{Z})/Tors$ is a characteristic covector for the intersection pairing Q_X on $H_2(X; \mathbb{Z})/Tors$. Let δ denote the absolute value of the determinant of

Q_X and $\text{Im}(\text{Spin}^c(X))$ the image of the restriction map from $\text{Spin}^c(X)$ to $\text{Spin}^c(Y)$. Then combining (1) with Theorem 5.1 yields

$$\max_{\mathfrak{t} \in \text{Im}(\text{Spin}^c(X))} 4d(Y, \mathfrak{t}) \geq \begin{cases} 1 - 1/\delta & \text{if } \delta \text{ is odd,} \\ 1 & \text{if } \delta \text{ is even,} \end{cases}$$

with strict inequality unless the intersection form of X is $(n-1)\langle -1 \rangle \oplus \langle -\delta \rangle$.

Theorem 2 follows immediately since if either δ is square-free or if there is no torsion in $H_1(X; \mathbb{Z})$, then the restriction map from $\text{Spin}^c(X)$ to $\text{Spin}^c(Y)$ is surjective, and $|H_1(Y; \mathbb{Z})| = \delta$.

Similar reasoning yields the following variant of Theorem 2:

Proposition 5.2. *Let Y be a rational homology sphere with $|H_1(Y; \mathbb{Z})| = rs^2$, with r square-free. If Y bounds a negative-definite four-manifold X , then*

$$\max_{\mathfrak{t} \in \text{Spin}^c(Y)} 4d(Y, \mathfrak{t}) \geq \begin{cases} 1 - 1/r & \text{if } r \text{ is odd,} \\ 1 & \text{if } r \text{ is even.} \end{cases}$$

Remark 5.3. *Suppose Y and X are as in the statement of Theorem 2 with δ even. If in fact*

$$\max_{\mathfrak{t} \in \text{Im}(\text{Spin}^c(X))} 4d(Y, \mathfrak{t}) = 1,$$

then it is not difficult to see that this maximum must be attained at a spin structure.

6. SURGERIES ON L -SPACE KNOTS

Let K be a knot in the three-sphere and

$$\Delta_K(T) = a_0 + \sum_{j>0} a_j(T^j + T^{-j})$$

its Alexander polynomial. Torsion coefficients of K are defined by

$$t_i(K) = \sum_{j>0} ja_{|i|+j}.$$

Note that $t_i(K) = 0$ for $i \geq N_K$, where N_K denotes the degree of the Alexander polynomial of K . For any $n \in \mathbb{Z}$ denote the n -surgery on K by K_n . K is called an L -space knot if for some $n > 0$, K_n is an L -space. Recall from [11] that a rational homology sphere is called an L -space if $\widehat{HF}(Y, \mathfrak{s}) \cong \mathbb{Z}$ for each spin^c structure \mathfrak{s} (so its Heegaard Floer homology groups resemble those of a lens space).

There are a number of conditions coming from Heegaard Floer homology that an L -space knot K has to satisfy. In particular (see [11, Theorem 1.2]), its Alexander polynomial has the form

$$(2) \quad \Delta_K(T) = (-1)^k + \sum_{j=1}^k (-1)^{k-j} (T^{n_j} + T^{-n_j}).$$

From this it follows easily that the torsion coefficients $t_i(K)$ are given by

$$(3) \quad \begin{aligned} t_i(K) &= n_k - n_{k-1} + \cdots + n_{k-2j} - i & n_{k-2j-1} \leq i \leq n_{k-2j} \\ t_i(K) &= n_k - n_{k-1} + \cdots + n_{k-2j} - n_{k-2j-1} & n_{k-2j-2} \leq i \leq n_{k-2j-1}, \end{aligned}$$

where $j = 0, \dots, k-1$ and $n_{-j} = -n_j$, $n_0 = 0$. In particular, $t_i(K)$ is nonincreasing in i for $i \geq 0$.

The following formula for d -invariants of surgeries on such a knot is based on results in [12] and [11] (see also [13, Theorem 1.2]). The proof was outlined to us by Peter Ozsváth.

Theorem 6.1. *Let $K \subset S^3$ be an L -space knot and let $t_i(K)$ denote its torsion coefficients. Then for any $n > 0$ the d -invariants of the $\pm n$ surgery on K are given by*

$$d(K_n, i) = d(U_n, i) - 2t_i(K), \quad d(K_{-n}, i) = -d(U_n, i)$$

for $|i| \leq n/2$, where U_n denotes the n surgery on the unknot U .

Before sketching a proof of the theorem we need to explain the notation. The d -invariants are usually associated to spin^c structures on the manifold. The set of spin^c structures on a three-manifold Y is parametrized by $H^2(Y; \mathbb{Z}) \cong H_1(Y; \mathbb{Z})$. In the case of interest we have $H_1(K_{\pm n}; \mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$, hence spin^c structures can be labelled by the elements of $\mathbb{Z}/n\mathbb{Z}$. Such a labelling is assumed in the above theorem and described explicitly as follows. Attaching a 2-handle with framing n to $S^3 \times [0, 1]$ along $K \subset S^3 \times \{1\}$ gives a cobordism W_n from S^3 to K_n . Note that $H_2(W_n; \mathbb{Z})$ is generated by the homology class of the core of the 2-handle attached to a Seifert surface for K ; denote this generator by f_n . A spin^c structure \mathfrak{t} on K_n is labelled by i if it admits an extension \mathfrak{s} to W_n satisfying

$$\langle c_1(\mathfrak{s}), f_n \rangle - n \equiv 2i \pmod{2n}.$$

If $K = U$ is the unknot, the surgery is the lens space which bounds the disk bundle over S^2 with Euler number n . Using either [9, Proposition 4.8] or [10, Corollary 1.5], the d -invariants of U_n for $n > 0$ and $|i| < n$ are given by

$$d(U_n, i) = \frac{(n - 2|i|)^2}{4n} - \frac{1}{4}.$$

Proof of Theorem 6.1. According to [12, Theorem 4.1] the Heegaard Floer homology groups $HF^+(K_n, i)$ for $n \neq 0$ can be computed from the knot Floer homology of K . The knot complex $C = CFK^\infty(S^3, K)$ is a \mathbb{Z}^2 -filtered chain complex, which is a finitely generated free module over $\mathcal{T} := \mathbb{Z}[U, U^{-1}]$. We write (i, j) for the components of bidegree. Here U denotes a formal variable in degree -2 that decreases the bidegree by $(1, 1)$. The homology of the complex C is $HF^\infty(S^3) = \mathcal{T}$, the homology of the quotient complex $B^+ := C\{i \geq 0\}$ is $HF^+(S^3) = \mathbb{Z}[U^{-1}] =: \mathcal{T}^+$ and the homology of $C\{i = 0\}$ is $\widehat{HF}(S^3) = \mathbb{Z}$. The complexes $C\{j \geq 0\}$ and B^+ are quasi-isomorphic and we fix a chain homotopy equivalence from $C\{j \geq 0\}$ to B^+ .

We now recall the description of $HF^+(K_{\pm n})$ ($n > 0$) in the case of an L -space knot K . For $s \in \mathbb{Z}$, let A_s^+ denote the quotient complex $C\{i \geq 0 \text{ or } j \geq s\}$. Let $v_s^+ : A_s^+ \rightarrow B^+$ denote the projection and $h_s^+ : A_s^+ \rightarrow B^+$ the chain map defined by first projecting to $C\{j \geq s\}$, then applying U^s to identify with $C\{j \geq 0\}$ and finally applying the chain homotopy equivalence to B^+ . For any $\sigma \in \{0, 1, \dots, n-1\}$ let $\mathbb{A}_\sigma^+ = \bigoplus_{s \in \sigma + n\mathbb{Z}} A_s^+$ and $\mathbb{B}_\sigma^+ = \bigoplus_{s \in \sigma + n\mathbb{Z}} B_s^+$, where $B_s^+ = B^+$ for all $s \in \mathbb{Z}$. Let $D_{\pm n}^+ : \mathbb{A}_\sigma^+ \rightarrow \mathbb{B}_\sigma^+$ be a homomorphism that on A_s^+ acts by $D_{\pm n}^+(a_s) = (v_s^+(a_s), h_s^+(a_s)) \in B_s^+ \oplus B_{s \pm n}^+$. Then $HF^+(K_{\pm n}, \sigma)$ is isomorphic to the direct sum of the kernel and the cokernel of the map that $D_{\pm n}^+$ induces between the homologies of \mathbb{A}_σ^+ and \mathbb{B}_σ^+ (which are direct sums of the homologies of A_s^+ and B_s^+). Moreover, this isomorphism is a homogeneous map of degree $\pm d(U_n, \sigma)$, where \mathbb{A}_σ^+ is graded so that $D_{\pm n}^+$ has degree -1 . When computing for K_n , the grading on B_s^+ , where $s = \sigma + nk$, is such that $U^0 \in B_s^+$ has grading $2k\sigma + nk(k-1) - 1$. In case of $-n$ -surgery $U^0 \in B_{\sigma+kn}^+$ has grading $-2(k+1)\sigma - nk(k+1)$. (See [12] for more details.)

Suppose now that K is an L -space knot with Alexander polynomial as in (2). Define $\delta_k := 0$ and

$$\delta_l := \begin{cases} \delta_{l+1} - 2(n_{l+1} - n_l) + 1 & \text{if } k - l \text{ is odd} \\ \delta_{l+1} - 1 & \text{if } k - l \text{ is even} \end{cases}$$

for $l = k-1, k-2, \dots, -k$, where as above $n_{-l} = -n_l$. Then by [11, Theorem 1.2] $C\{i = 0\}$ is (up to quasi-isomorphism) equal to the free abelian group with one generator x_l in bidegree $(0, n_l)$ for $l = -k, \dots, k$ and the grading of x_l is δ_l . To determine the differentials note that the homology of $C\{i = 0\}$ is \mathbb{Z} in grading 0, so generated by (the homology class of) x_k . It follows that the differential on $C\{i = 0\}$ is a collection of isomorphisms $C_{0, n_k-2l+1} \rightarrow C_{0, n_k-2l}$ for $l = 1, \dots, k$. Similarly we see that the differential on $C\{j = 0\}$ is given by a collection of isomorphisms $C_{n_k-2l+1, 0} \rightarrow C_{n_k-2l, 0}$ for $l = 1, \dots, k$. This, together with U -equivariance, determines the complex C . (For an example, see Figure 1.)

Suppose that the homology H of a quotient complex of C is isomorphic to $\mathcal{T}^+ = \mathbb{Z}[U^{-1}]$. We say H starts at (i, j) if the element U^0 has a representative of bidegree (i, j) . With this notation $H_*(B_s)$ starts at $(0, n_k)$ and $H_*(C\{j \geq s\})$ starts at $(s, s + n_k)$. It remains to consider $H_*(A_s^+)$. Note that these groups are also isomorphic to \mathcal{T}^+ . For $s \geq n_k$ the homology of A_s^+ starts at $(0, n_k)$, so $v_s^+ = id$ and $h_s^+ = U^s$. For $n_{k-1} \leq s < n_k$, $H_*(A_s^+)$ starts at $(s - n_k, s)$, so $v_s^+ = U^{n_k-s}$ and $h_s^+ = U^{n_k}$. For $n_{k-2} \leq s < n_{k-1}$ it starts at $(n_{k-1} - n_k, n_{k-1})$, hence $v_s^+ = U^{n_k-n_{k-1}}$ and $h_s^+ = U^{n_k-n_{k-1}+s}$. It is now easy to observe that for $s \geq 0$ the homology $H_*(A_s^+)$ starts at $(t_s, t_s + n_k)$ and thus $v_s^+ = U^{t_s}$ and $h_s^+ = U^{t_s+s}$, where t_s is a torsion coefficient of K (compare with equation (3)). Similarly for $s < 0$ we obtain $v_s^+ = U^{t_s-s}$ and $h_s^+ = U^{t_s}$.

Note that since the spin^c structures corresponding to i and $-i$ are conjugate (so their d invariants are equal) we may restrict to those in the range $\{0, 1, \dots, \lfloor n/2 \rfloor\}$. Consider now K_n ($n > 0$) and choose some $\sigma \in \{0, 1, \dots, \lfloor n/2 \rfloor\}$. Writing $D_n^+ : \mathbb{A}_\sigma^+ \rightarrow$

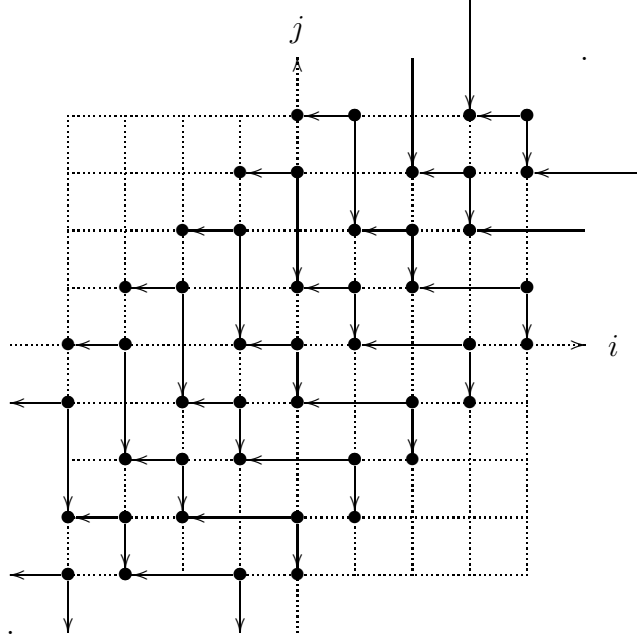


FIGURE 1. **The knot Floer complex** $CFK^\infty(T_{3,5})$. Each bullet represents a \mathbb{Z} , and each arrow is an isomorphism. The groups and differentials on the axes are determined by the Alexander polynomial

$$\Delta_{T_{3,5}}(T) = -1 + (T + T^{-1}) - (T^3 + T^{-3}) + (T^4 + T^{-4}),$$

and these in turn determine the entire complex.

\mathbb{B}_σ^+ in components we obtain for $l > 0$:

$$\begin{aligned} b_{\sigma+ln} &= U^{t_{\sigma+ln}} a_{\sigma+ln} + U^{t_{\sigma+(l-1)n+\sigma+(l-1)n}} a_{\sigma+(l-1)n} \\ &\implies U^{t_{\sigma+ln}} a_{\sigma+ln} = b_{\sigma+ln} - U^{t_{\sigma+(l-1)n+\sigma+(l-1)n}} a_{\sigma+(l-1)n} \\ b_\sigma &= U^{t_\sigma} a_\sigma + U^{t_{n-\sigma}} a_{\sigma-n} \\ &\implies U^{t_{n-\sigma}} a_{\sigma-n} = b_\sigma - U^{t_\sigma} a_\sigma \\ b_{\sigma-ln} &= U^{t_{ln-\sigma+ln-\sigma}} a_{\sigma-ln} + U^{t_{(l+1)n-\sigma}} a_{\sigma-(l+1)n} \\ &\implies U^{t_{(l+1)n-\sigma}} a_{\sigma-(l+1)n} = b_{\sigma-ln} - U^{t_{ln-\sigma+ln-\sigma}} a_{\sigma-ln} \end{aligned}$$

from which it is easily seen that D_n^+ is surjective and its kernel contains one \mathcal{T}^+ summand, isomorphic to $H_*(A_\sigma)$. Since D_n^+ shifts grading by -1 , $U^0 \in H_*(B_\sigma)$ has

grading -1 , and the component of D_n^+ from $H_*(A_\sigma)$ to $H_*(B_\sigma)$ is equal to U^{t_σ} , it follows that $U^0 \in \mathcal{T}^+ \subset \ker D_n^+$ has grading $-2t_\sigma$. The formula for $d(K_n, \sigma)$ now follows using the degree shift between $HF^+(K_n, \sigma)$ and $\ker D_n^+$.

Finally consider K_{-n} ($n > 0$) and choose $\sigma \in \{0, 1, \dots, \lfloor n/2 \rfloor\}$. Writing D_{-n}^+ in components yields:

$$\begin{aligned} b_{\sigma+ln} &= U^{t_{\sigma+ln}} a_{\sigma+ln} + U^{t_{\sigma+(l+1)n} + \sigma + (l+1)n} a_{\sigma+(l+1)n} & (l \geq 0) \\ &\implies U^{t_{\sigma+ln}} a_{\sigma+ln} = b_{\sigma+ln} - U^{t_{\sigma+(l+1)n} + \sigma + (l+1)n} a_{\sigma+(l+1)n} \\ b_{\sigma-n} &= U^{t_{\sigma+\sigma}} a_\sigma + U^{t_{n-\sigma} + n - \sigma} a_{\sigma-n} \\ &\implies U^{t_{\sigma+\sigma}} a_\sigma = b_{\sigma-n} - U^{t_{n-\sigma} + n - \sigma} a_{\sigma-n} \\ b_{\sigma-ln} &= U^{t_{ln-\sigma} + ln - \sigma} a_{\sigma-ln} + U^{t_{(l-1)n-\sigma}} a_{\sigma-(l-1)n} & (l \geq 2) \\ &\implies U^{t_{(l-1)n-\sigma}} a_{\sigma-(l-1)n} = b_{\sigma-ln} - U^{t_{ln-\sigma} + ln - \sigma} a_{\sigma-ln}. \end{aligned}$$

The top and bottom equations determine a_s (or more precisely $U^N a_s$ for $N \gg 0$) for all $s \in \sigma + n\mathbb{Z}$, so the middle equation cannot be fulfilled in general. It follows that D_{-n}^+ has cokernel isomorphic to $H_*(B_{\sigma-n}) = \mathcal{T}^+$; the grading of U^0 in this group is 0. The formula for $d(K_{-n}, \sigma)$ again follows from the degree shift. \square

Combining Theorems 2 and 6.1 yields

Theorem 6.2. *Let $n > 0$ and let K be an L -space knot whose torsion coefficients satisfy*

$$t_i(K) > \begin{cases} \frac{(n-2i)^2+1}{8n} - \frac{1}{4} & \text{if } n \text{ is odd} \\ \frac{(n-2i)^2}{8n} - \frac{1}{4} & \text{if } n \text{ is even} \end{cases}$$

for $0 \leq i \leq n/2$. Then for $0 < m \leq n$, K_m cannot bound a negative-definite four-manifold with no torsion in H_1 .

Proof. The formulas follow from the above-mentioned Theorems. To see that obstruction for n -surgery to bound a negative-definite manifold implies obstruction for all m -surgeries with $0 < m \leq n$ observe that for fixed i the right-hand side of the inequality is an increasing function of n for $n \geq 2i$. \square

Note that the surgery coefficient m in Theorem 6.2 is an integer. In [8] we will consider in more detail the question of which surgeries (including Dehn surgeries) on a knot K can be given as the boundary of a negative-definite four-manifold. In particular we will show that Theorem 6.2 holds as stated with $m \in \mathbb{Q}$.

7. EXAMPLE: SURGERIES ON TORUS KNOTS

In this section we consider torus knots $T_{p,q}$; we assume $2 \leq p < q$. Right-handed torus knots are L -space knots since for example $pq - 1$ surgery on $T_{p,q}$ yields a lens space [6]. Let $N = (p-1)(q-1)/2$ denote the degree of the Alexander polynomial of $T_{p,q}$. The following proposition gives a simple function which approximates the torsion coefficients of a torus knot.

Proposition 7.1. *The torsion coefficients of $T_{p,q}$ are given by*

$$t_i = \#\{(a, b) \in \mathbb{Z}_{\geq 0}^2 \mid ap + bq < N - i\}$$

and they satisfy $t_i \geq g(N - i)$ for $0 \leq i \leq N$, where $x \mapsto g(x)$ is a piecewise linear continuous function, which equals 0 for $x \leq 0$ and whose slope on the interval $[(k - 1)q, kq]$ is k/p .

Proof. The (unsymmetrised) Alexander polynomial of $K = T_{p,q}$ is

$$\tilde{\Delta}_K(T) = \frac{(1 - T^{pq})(1 - T)}{(1 - T^p)(1 - T^q)},$$

which is a polynomial of degree $2N$. Writing $\tilde{\Delta}_K$ as a formal power series in T we obtain

$$\begin{aligned} \tilde{\Delta}_K(T) &= (1 - T^{pq})(1 - T) \sum_{a \geq 0} T^{ap} \sum_{b \geq 0} T^{bq} \\ &= (1 - T) \sum_{a \geq 0} T^{ap} \sum_{b \geq 0} T^{bq} - T^{pq}(1 - T) \sum_{a \geq 0} T^{ap} \sum_{b \geq 0} T^{bq}. \end{aligned}$$

Clearly only the terms in the first product of the last line contribute to the nonzero coefficients of Δ_K . A power T^k appears with coefficient $+1$ in this term whenever $k = ap + bq$ for some $a, b \in \mathbb{Z}_{\geq 0}$, and with coefficient -1 whenever $k - 1 = ap + bq$ for some $a, b \in \mathbb{Z}_{\geq 0}$. Since the coefficient a_j in $\Delta_K(T)$ is the coefficient of T^{N-j} in $\tilde{\Delta}_K(T)$, we obtain $a_j = m_j - m_{j+1}$, where

$$m_j := \#\{(a, b) \in \mathbb{Z}_{\geq 0}^2 \mid ap + bq = N - j\}.$$

Then for $i \geq 0$

$$t_i = \sum_{j > 0} ja_{i+j} = \sum_{j > 0} m_{i+j} = \#\{(a, b) \in \mathbb{Z}_{\geq 0}^2 \mid ap + bq < N - i\}.$$

In what follows it is convenient to replace t_i by

$$s(i) := t_{N-i} = \#\{(a, b) \in \mathbb{Z}_{\geq 0}^2 \mid ap + bq < i\}.$$

Define

$$\bar{g}(i) := \begin{cases} i/p & \text{if } i \geq 0 \\ 0 & \text{if } i \leq 0 \end{cases}$$

and $\bar{s}(i) := \lceil g(i) \rceil$; clearly $\bar{s}(i) \geq \bar{g}(i)$ for every i . Separating the set appearing in the definition of $s(i)$ into subsets with fixed value of b we obtain

$$s(i) = \sum_{b \geq 0} \#\{a \in \mathbb{Z}_{\geq 0} \mid ap < i - bq\} = \sum_{b \geq 0} \bar{s}(i - bq) \geq \sum_{b \geq 0} \bar{g}(i - bq) =: g(i).$$

□

The following corollary describes the range of surgeries on $T_{p,q}$ that cannot bound negative-definite manifolds according to Theorem 6.2 (and using Proposition 7.1). To obtain the result of Proposition 3 note that all the lower bounds in the corollary allow for $m = 2$. Note that Lisca and Stipsicz have shown that Dehn surgery on $T_{2,2n+1}$ with positive framing r bounds a negative-definite manifold (possibly with torsion in H_1) if and only if $r \geq 4n$ [5].

Corollary 7.2. *Let $2 \leq p < q$ and $N = (p-1)(q-1)/2$.*

- *If p is even and m is less than the minimum of*

$$\frac{1 + \sqrt{4N}}{2 + \frac{1}{2}\sqrt{\alpha(\alpha + 4\beta)} - 4 - \frac{1}{2}\alpha} - \frac{1}{q - p + 3}$$

where $\alpha = q(p-2) + 2$ and $\beta = q - p + 1$

- *or if p is odd and m is less than the minimum of*

$$\frac{1 + \sqrt{4N}}{2 + \frac{1}{2}\sqrt{\alpha(\alpha + 4\beta)} - 4 - \frac{1}{2}\alpha - \frac{q(p-3)}{p}} - \frac{1}{q - p + 5 - \frac{q+2}{p}}$$

where $\alpha = q(p-4) + 2 + 3q/p$ and $\beta = 2q - p + 1$

- *and $1 \leq n \leq 2N + m$,*

then $+n$ -surgery on $T_{p,q}$ cannot bound a negative definite four-manifold with no torsion in H_1 .

Proof. Write $n = 2N + m$; we may assume $n < pq - 1$. It suffices to show that

$$(4) \quad h(x) := \frac{(m + 2x)^2 + 1}{8n} - \frac{1}{4} < g(x)$$

for $-m/2 \leq x \leq N$, where g is the function appearing in Proposition 7.1.

Since h is convex and g is piecewise linear, it is enough to check the inequality for $x = kq$ with $k = 0, 1, \dots, \lfloor N/q \rfloor$ and for $x = N$. Consider first $x = kq$. From the definition of g we obtain $g(kq) = qk(k+1)/2p$. Substituting this into (4) we obtain

$$4k^2(q^2 - n\frac{q}{p}) + 4k(qm - n\frac{q}{p}) + m^2 + 1 - 2n < 0.$$

Since $q - n/p > 1/p$, it suffices to consider only $k = 0$ and $k = \lfloor N/q \rfloor$. For $k = 0$ the last inequality yields $m < 1 + \sqrt{4N}$.

Assume first p is even; then $\lfloor N/q \rfloor = p/2 - 1$ and substituting this into (4) gives

$$(m + q(p-2))^2 + 1 < n(2 + q(p-2)).$$

Writing $\mu = m + q(p-2)$, $\alpha = q(p-2) + 2$ and $\beta = q - p + 1$ the last inequality becomes

$$\mu^2 - \alpha\mu + 1 - \alpha\beta < 0,$$

which implies

$$\mu < \frac{\alpha}{2} + \frac{1}{2}\sqrt{\alpha(\alpha + 4\beta) - 4};$$

this is equivalent to the second condition on m in the statement of the corollary.

Since the slope of g on the interval from $(p/2 - 1)q$ to N is $1/2$, we get $g(N) = g((p/2 - 1)q) + (N - (p/2 - 1)q)/2 = (pq - 2p + 2)/8$. Substituting this into (4) gives

$$n^2 - n(pq - 2p + 4) + 1 < 0,$$

which holds if $n < pq - 2p + 4$ or $m < q - p + 3$.

Assume now p is odd; then $\lfloor N/q \rfloor = (p - 3)/2$ and substituting this into (4) gives

$$(m + q(p - 3))^2 + 1 < n(2 + \frac{q}{p}(p - 1)(p - 3)).$$

Writing $\mu = m + q(p - 3)$, $\alpha = q(p - 4) + 2 + 3q/p$ and $\beta = 2q - p + 1$ the last inequality becomes

$$\mu^2 - \alpha\mu + 1 - \alpha\beta < 0,$$

from which the second condition on m in the statement of the corollary follows.

Now the slope of g on the interval from $q(p - 3)/2$ to N is $(p - 1)/2p$ and $g(N) = (pq - 2p + 4 - (q + 2)/p)/8$. Substituting this into (4) gives

$$n^2 - n(pq - 2p + 6 - \frac{q + 2}{p}) + 1 < 0,$$

which holds if $n < pq - 2p + 6 - \frac{q+2}{p}$ or $m < q - p + 5 - \frac{q+2}{p}$. □

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