

TOWARDS A SPLITTER THEOREM FOR INTERNALLY 4-CONNECTED BINARY MATROIDS VI

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ABSTRACT. Let M be a 3-connected binary matroid; M is called internally 4-connected if one side of every 3-separation is a triangle or a triad, and M is $(4, 4, S)$ -connected if one side of every 3-separation is a triangle, a triad, or a 4-element fan. Assume M is internally 4-connected and that neither M nor its dual is a cubic Möbius or planar ladder or a certain coextension thereof. Let N be an internally 4-connected proper minor of M . Our aim is to show that M has a proper internally 4-connected minor with an N -minor that can be obtained from M either by removing at most four elements, or by removing elements in an easily described way from a special substructure of M . When this aim cannot be met, the earlier papers in this series showed that, up to duality, M has a good bowtie, that is, a pair, $\{x_1, x_2, x_3\}$ and $\{x_4, x_5, x_6\}$, of disjoint triangles and a cocircuit, $\{x_2, x_3, x_4, x_5\}$, where $M \setminus x_3$ has an N -minor and is $(4, 4, S)$ -connected. We also showed that, when M has a good bowtie, either $M \setminus x_3, x_6$ has an N -minor; or $M \setminus x_3/x_2$ has an N -minor and is $(4, 4, S)$ -connected. In this paper, we show that, when $M \setminus x_3, x_6$ has an N -minor but is not $(4, 4, S)$ -connected, M has an internally 4-connected proper minor with an N -minor that can be obtained from M by removing at most three elements, or by removing elements in a well-described way from one of several special substructures of M . This is a significant step towards obtaining a splitter theorem for the class of internally 4-connected binary matroids.

1. INTRODUCTION

Seymour's Splitter Theorem [12] established that if N is a proper 3-connected minor of a 3-connected matroid M , then M has a proper 3-connected minor M' with an N -minor such that $|E(M) - E(M')| = 1$ unless $r(M) \geq 3$ and M is a wheel or a whirl. The current paper is the sixth in a series whose aim is to obtain a splitter theorem for the class of internally 4-connected binary matroids. Specifically, we believe we can prove that if M and N are internally 4-connected binary matroids, and M has a proper N -minor, then M has a proper minor M' such that M' is internally 4-connected with an N -minor, and M' can be produced from M by a small number of simple operations.

Any unexplained matroid terminology used here will follow [11]. The only 3-separations allowed in an internally 4-connected matroid have a triangle or a triad on one side. A 3-connected matroid M is $(4, 4, S)$ -connected if, for every 3-separation (X, Y) of M , one of X and Y is a triangle, a triad, or a 4-element fan,

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that is, a 4-element set $\{x_1, x_2, x_3, x_4\}$ that can be ordered so that $\{x_1, x_2, x_3\}$ is a triangle and $\{x_2, x_3, x_4\}$ is a triad.

To provide a context for our main theorem, we briefly describe our progress towards obtaining the desired splitter theorem. Johnson and Thomas [9] showed that, even for graphs, a splitter theorem in the internally 4-connected case must take account of some special examples. For $n \geq 3$, let G_{n+2} be the *biwheel* with $n+2$ vertices, that is, G_{n+2} consists of an n -cycle $v_1, v_2, \dots, v_n, v_1$, the *rim*, and two additional vertices, u and w , both of which are adjacent to every v_i . Thus the dual of G_{n+2} is a cubic planar ladder. Let M be the cycle matroid of G_{2n+2} for some $n \geq 3$ and let N be the cycle matroid of the graph that is obtained by proceeding around the rim of G_{2n+2} and alternately deleting the edges from the rim vertex to u and to w . Both M and N are internally 4-connected but there is no internally 4-connected proper minor of M that has a proper N -minor. We can modify M slightly and still see the same phenomenon. Let G_{n+2}^+ be obtained from G_{n+2} by adding a new edge z joining the hubs u and w . Let Δ_{n+1} be the binary matroid that is obtained from $M(G_{n+2}^+)$ by deleting the element $v_{n-1}v_n$ and adding the third element on the line spanned by wv_n and uv_{n-1} . This new element is also on the line spanned by uv_n and wv_{n-1} . For $r \geq 3$, Mayhew, Royle, and Whittle [10] call Δ_r the *rank- r triangular Möbius matroid* and note that $\Delta_r \setminus z$ is the dual of the cycle matroid of a cubic Möbius ladder. The following is the main result of [4, Theorem 1.2].

Theorem 1.1. *Let M be an internally 4-connected binary matroid with an internally 4-connected proper minor N such that $|E(M)| \geq 15$ and $|E(N)| \geq 6$. Then*

- (i) *M has a proper minor M' such that $|E(M) - E(M')| \leq 3$ and M' is internally 4-connected with an N -minor; or*
- (ii) *for some (M_0, N_0) in $\{(M, N), (M^*, N^*)\}$, the matroid M_0 has a triangle T that contains an element e such that $M_0 \setminus e$ is $(4, 4, S)$ -connected with an N_0 -minor; or*
- (iii) *M or M^* is isomorphic to $M(G_{r+1}^+)$, $M(G_{r+1})$, Δ_r , or $\Delta_r \setminus z$ for some $r \geq 5$.*

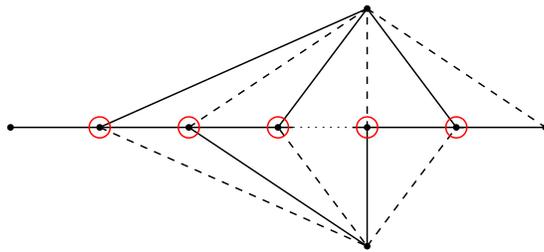


FIGURE 1. All the elements shown are distinct. There are at least three dashed elements; and all dashed elements are deleted.

That theorem prompted us to consider those matroids for which the second outcome in the theorem holds. In order to state the next result, we need to define some special structures. Let M be an internally 4-connected binary matroid and N be an internally 4-connected proper minor of M . Suppose M has disjoint triangles T_1 and T_2 and a 4-cocircuit D^* contained in their union. We call this structure

a *bowtie* and denote it by (T_1, T_2, D^*) . If D^* has an element d such that $M \setminus d$ has an N -minor and $M \setminus d$ is $(4, 4, S)$ -connected, then (T_1, T_2, D^*) is a *good bowtie*. Motivated by (ii) of the last theorem, we seek to determine more about the structure of M when it has a triangle containing an element e such that $M \setminus e$ is $(4, 4, S)$ -connected with an N -minor. One possible outcome here is that M has a good bowtie. Indeed, as the next result shows, if that outcome or its dual does not arise, we get a small number of easily described alternatives. We shall need two more definitions. A *terrahawk* is the graph that is obtained by adjoining a new vertex to a cube and adding edges from the new vertex to each of the four vertices that bound some fixed face of the cube. Figure 1 shows a modified graph diagram, which we use to keep track of some of the circuits and cocircuits in M . Each of the cycles in that diagram corresponds to a circuit of M while a (red) circled vertex indicates a known cocircuit of M . Occasionally, a red curve is used to indicate a cocircuit that consists of the edges that are crossed by the curve. For example, in Figure 3, $\{w_{k-2}, u_{k-1}, v_{k-1}, u_k, v_k\}$ is a cocircuit. At the end of Section 3, we shall say more about what can be inferred from such a diagram. We shall call a structure of the form shown in Figure 1 an *open rotor chain* noting that all of the elements in the figure are distinct and, for some $n \geq 3$, there are n dashed elements. We will refer to deleting the dashed elements from Figure 1 as *trimming an open rotor chain*. The following is a special case of [6, Corollary 1.4].

Theorem 1.2. *Let M and N be internally 4-connected binary matroids such that $|E(M)| \geq 16$ and $|E(N)| \geq 6$. Suppose that M has a triangle T containing an element e for which $M \setminus e$ is $(4, 4, S)$ -connected with an N -minor. Then one of the following holds.*

- (i) M has an internally 4-connected minor M' that has an N -minor such that $1 \leq |E(M) - E(M')| \leq 4$; or
- (ii) M or M^* has a good bowtie; or
- (iii) M is the cycle matroid of a terrahawk; or
- (iv) for some (M_0, N_0) in $\{(M, N), (M^*, N^*)\}$, the matroid M_0 contains an open rotor chain that can be trimmed to obtain an internally 4-connected matroid with an N_0 -minor.

We remark that there is a small error in [6, Theorem 1.1] since it requires at least five elements to be removed when trimming an open rotor chain. But, as the proof there makes clear, trimming exactly four elements is a possibility. Trimming exactly three elements is also possible but that is included under (i) of [6, Theorem 1.1].

This theorem leads us to consider a good bowtie $(\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}, \{x_2, x_3, x_4, x_5\})$ in an internally 4-connected binary matroid M where $M \setminus x_3$ is $(4, 4, S)$ -connected with an N -minor. In $M \setminus x_3$, we see that $\{x_5, x_4, x_2\}$ is a triad and $\{x_6, x_5, x_4\}$ is a triangle, so $\{x_6, x_5, x_4, x_2\}$ is a 4-element fan. It follows, by [5, Lemma 2.5], that either

- (i) $M \setminus x_3, x_6$ has an N -minor; or
- (ii) $M \setminus x_3, x_6$ does not have an N -minor, but $M \setminus x_3/x_2$ is $(4, 4, S)$ -connected with an N -minor.

In this paper, we focus on the first of these two cases and assume, in addition, that $M \setminus x_6$ is not $(4, 4, S)$ -connected. In [7], we treat the second of these two cases. Finally, in [8], we treat the remaining subcase of (i).

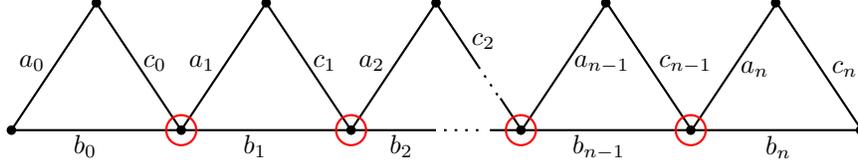


FIGURE 2. A string of bowties. All elements are distinct except that a_0 may be the same as c_n .

In a matroid M , a *string of bowties* is a sequence $\{a_0, b_0, c_0\}, \{b_0, c_0, a_1, b_1\}, \{a_1, b_1, c_1\}, \{b_1, c_1, a_2, b_2\}, \dots, \{a_n, b_n, c_n\}$ with $n \geq 1$ such that

- (i) $\{a_i, b_i, c_i\}$ is a triangle for all i in $\{0, 1, \dots, n\}$;
- (ii) $\{b_j, c_j, a_{j+1}, b_{j+1}\}$ is a cocircuit for all j in $\{0, 1, \dots, n-1\}$; and
- (iii) the elements $a_0, b_0, c_0, a_1, b_1, c_1, \dots, a_n, b_n, c_n$ are distinct except that a_0 and c_n may be equal.

The reader should note that this differs slightly from the definition we gave in [2] in that here we allow a_0 and c_n to be equal instead of requiring all of the elements to be distinct. Figure 2 illustrates a string of bowties, but this diagram may obscure the potential complexity of such a string. Evidently $M \setminus c_0$ has $\{c_1, b_1, a_1, b_0\}$ as a 4-fan. Indeed, $M \setminus c_0, c_1, \dots, c_i$ has a 4-fan for all i in $\{0, 1, \dots, n-1\}$. We shall say that the matroid $M \setminus c_0, c_1, \dots, c_n$ has been obtained from M by *trimming a string of bowties*. This operation plays a prominent role in our main theorem, and is the underlying operation in trimming an open rotor chain. Before stating this result, we introduce the other operations that incorporate this process of trimming a string of bowties. Such a string can attach to the rest of the matroid in a variety of ways. In most of these cases, the operation of trimming the string will produce an internally 4-connected minor of M with an N -minor. But, in three cases, when the bowtie string is embedded in a modified quartic ladder in certain ways, we need to adjust the trimming process.

Consider the three configurations shown in Figure 3 and Figure 4 where the elements in each configuration are distinct except that d_2 may equal w_k . We refer to each of these configurations as an *enhanced quartic ladder*. Indeed, in each configuration, we can see a portion of a quartic ladder, which can be thought of as two interlocking bowtie strings, one pointing up and one pointing down. In each case, we focus on $M \setminus c_2, c_1, c_0, v_0, v_1, \dots, v_k$ saying that this matroid has been obtained from M by an *enhanced-ladder move*. In Figure 5, the configuration in Figure 4 has been redrawn omitting the triangles $\{c_0, b_1, b_2\}$ and $\{v_{k-2}, t_{k-1}, t_k\}$ as well as the cocircuits $\{b_2, c_0, c_2, t_0, u_0\}$ and $\{s_{k-2}, u_{k-2}, v_{k-2}, t_k, v_k\}$. The ladder structure is evident there and the enhanced ladder move corresponds to deleting all of the dashed edges.

For some $n \geq 2$, let $\{a_0, b_0, c_0\}, \{b_0, c_0, a_1, b_1\}, \{a_1, b_1, c_1\}, \dots, \{a_n, b_n, c_n\}$ be a bowtie string in a matroid M . Assume, in addition, that $\{b_n, c_n, a_0, b_0\}$ is a cocircuit. Then the string of bowties has wrapped around on itself as in Figure 6. We call the resulting structure a *ring of bowties* and denote it by $(\{a_0, b_0, c_0\}, \{b_0, c_0, a_1, b_1\}, \{a_1, b_1, c_1\}, \dots, \{a_n, b_n, c_n\}, \{b_n, c_n, a_0, b_0\})$. We also require that the elements in a bowtie ring are distinct, although this is guaranteed if

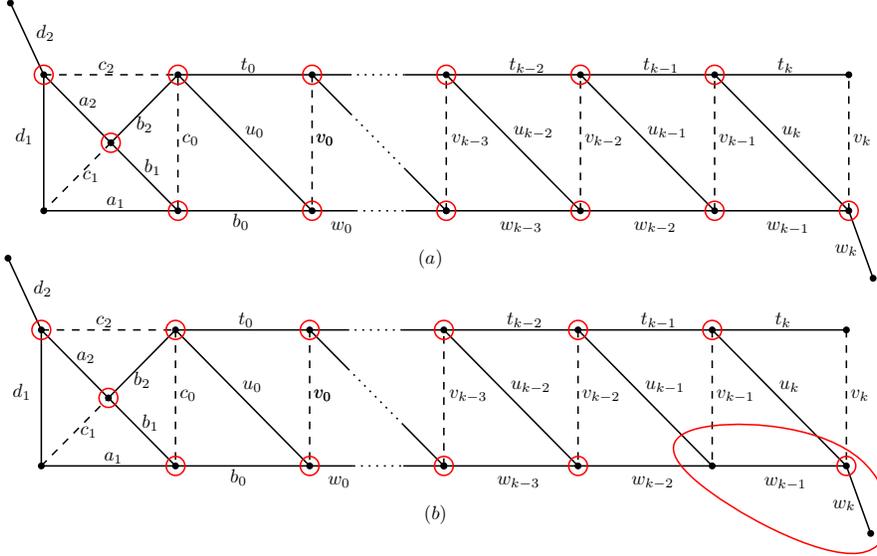


FIGURE 3. In both (a) and (b), all elements shown are distinct, except that d_2 may be w_k . Furthermore, in (a), $k \geq 0$; and, in (b), $k \geq 1$ and $\{w_{k-2}, u_{k-1}, v_{k-1}, u_k, v_k\}$ is a cocircuit.

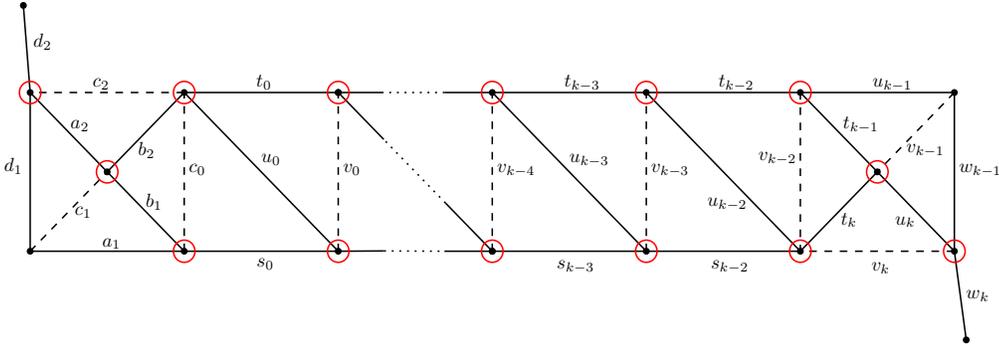


FIGURE 4. In this configuration, $k \geq 2$ and the elements are all distinct except that d_2 may be w_k .

M is internally 4-connected. We refer to each of the structures in Figure 7 as a *ladder structure* and we refer to removing the dashed elements in Figure 6 and Figure 7 as *trimming a ring of bowties* and *trimming a ladder structure*, respectively.

In the case that trimming a string of bowties in M yields an internally 4-connected matroid with an N -minor, we are able to ensure that the string of bowties belongs to one of the more highly structured objects shown in one of Figures 3, 4, 6, or 7. The following theorem is the main result of this paper.

Theorem 1.3. *Let M and N be internally 4-connected binary matroids such that $|E(M)| \geq 13$ and $|E(N)| \geq 7$. Assume that M has a bowtie $(\{x_0, y_0, z_0\}, \{x_1, y_1, z_1\}, \{y_0, z_0, x_1, y_1\})$, where $M \setminus z_0$ is $(4, 4, S)$ -connected,*

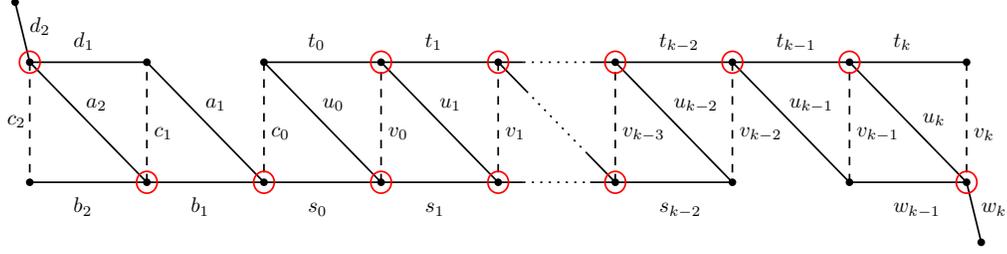


FIGURE 5. The configuration in Figure 4 redrawn omitting two triangles and two 5-cocircuits.

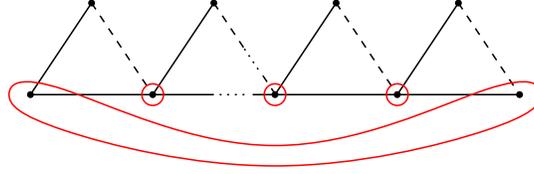


FIGURE 6. A bowtie ring. All elements are distinct. The ring contains at least three triangles.

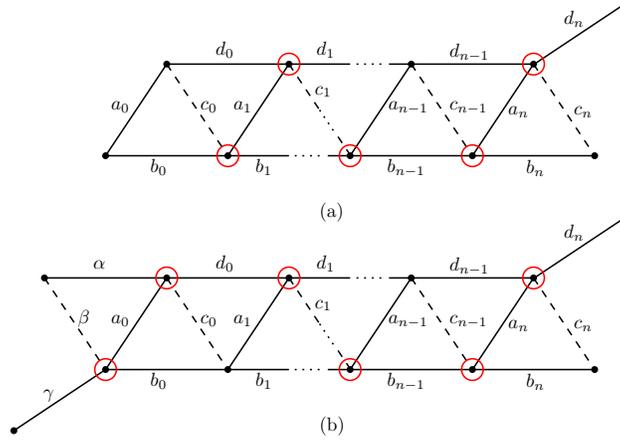


FIGURE 7. In (a) and (b), the elements shown are distinct, with the exception that d_n may be the same as γ in (b). Either $\{d_{n-2}, a_{n-1}, c_{n-1}, d_{n-1}\}$ or $\{d_{n-2}, a_{n-1}, c_{n-1}, a_n, c_n\}$ is a cocircuit in (a) and (b). Either $\{b_0, c_0, a_1, b_1\}$ or $\{\beta, a_0, c_0, a_1, b_1\}$ is also a cocircuit in (b). Furthermore, $n \geq 2$.

$M \setminus z_0, z_1$ has an N -minor, and $M \setminus z_1$ is not $(4, 4, S)$ -connected. Then one of the following holds.

- (i) M has a proper minor M' such that $|E(M)| - |E(M')| \leq 3$ and M' is internally 4-connected with an N -minor; or

- (ii) M contains an open rotor chain, a ladder structure, or a ring of bowties that can be trimmed to obtain an internally 4-connected matroid with an N -minor; or
- (iii) M contains an enhanced quartic ladder from which an internally 4-connected minor of M with an N -minor can be obtained by an enhanced-ladder move.

2. PRELIMINARIES

In this section, we give some basic definitions mainly relating to matroid connectivity. The subsequent section contains some straightforward properties of connectivity along with a lemma concerning bowties that distinguishes various cases whose analysis is fundamental to completing our work on the splitter theorem. The main result of this paper completely resolves what happens in one of these cases. In Section 3, we outline the proof of the main result, Theorem 1.3.

Let M and N be matroids. We shall sometimes write $N \preceq M$ to indicate that M has an N -minor, that is, a minor isomorphic to N . Now let E be the ground set of M and r be its rank function. The *connectivity function* λ_M of M is defined on all subsets X of E by $\lambda_M(X) = r(X) + r(E - X) - r(M)$. Equivalently, $\lambda_M(X) = r(X) + r^*(X) - |X|$. We will sometimes abbreviate λ_M as λ . For a positive integer k , a subset X or a partition $(X, E - X)$ of E is *k-separating* if $\lambda_M(X) \leq k - 1$. A *k-separating partition* $(X, E - X)$ of E is a *k-separation* if $|X|, |E - X| \geq k$. If n is an integer exceeding one, a matroid is *n-connected* if it has no *k-separations* for all $k < n$. This definition [13] has the attractive property that a matroid is *n-connected* if and only if its dual is. Moreover, this matroid definition of *n-connectivity* is relatively compatible with the graph notion of *n-connectivity* when n is 2 or 3. For example, when G is a graph with at least four vertices and with no isolated vertices, $M(G)$ is a 3-connected matroid if and only if G is a 3-connected simple graph. But the link between *n-connectivity* for matroids and graphs breaks down for $n \geq 4$. In particular, a 4-connected matroid with at least six elements cannot have a triangle. Hence, for $r \geq 3$, neither $M(K_{r+1})$ nor $PG(r - 1, 2)$ is 4-connected. This motivates the consideration of other types of 4-connectivity in which certain 3-separations are allowed.

A matroid is *internally 4-connected* if it is 3-connected and, whenever (X, Y) is a 3-separation, either $|X| = 3$ or $|Y| = 3$. Equivalently, a 3-connected matroid M is internally 4-connected if and only if, for every 3-separation (X, Y) of M , either X or Y is a triangle or a triad of M . A graph G without isolated vertices is *internally 4-connected* if $M(G)$ is internally 4-connected.

In a matroid M , a subset S of $E(M)$ is a *fan* if $|S| \geq 3$ and there is an ordering (s_1, s_2, \dots, s_n) of S such that $\{s_1, s_2, s_3\}, \{s_2, s_3, s_4\}, \dots, \{s_{n-2}, s_{n-1}, s_n\}$ alternate between triangles and triads. We call (s_1, s_2, \dots, s_n) a *fan ordering* of S . For convenience, we will often refer to the fan ordering as the fan. We will be mainly concerned with 4-element and 5-element fans. By convention, we shall always view a fan ordering of a 4-element fan as beginning with a triangle and we shall use the term *4-fan* to refer to both the 4-element fan and such a fan ordering of it. Moreover, we shall use the terms *5-fan* and *5-cofan* to refer to the two different types of 5-element fan where the first contains two triangles and the second two triads. Let (s_1, s_2, \dots, s_n) be a fan ordering of a fan S . When M is 3-connected having at least five elements and $n \geq 4$, every fan ordering of S has its first and last

elements in $\{s_1, s_n\}$. We call these elements the *ends* of the fan while the elements of $S - \{s_1, s_n\}$ are called the *internal elements* of the fan. When (s_1, s_2, s_3, s_4) is a 4-fan, our convention is that $\{s_1, s_2, s_3\}$ is a triangle, and we call s_1 the *guts element* of the fan and s_4 the *coguts element* of the fan since $s_1 \in \text{cl}(\{s_2, s_3, s_4\})$ and $s_4 \in \text{cl}^*(\{s_1, s_2, s_3\})$.

In a matroid M , a set U is *fully closed* if it is closed in both M and M^* . The *full closure* $\text{fcl}(Z)$ of a set Z in M is the intersection of all fully closed sets containing Z . Let (X, Y) be a partition of $E(M)$. If (X, Y) is k -separating in M for some positive integer k , and y is an element of Y that is also in $\text{cl}(X)$ or $\text{cl}^*(X)$, then it is well known and easily checked that $(X \cup y, Y - y)$ is k -separating, and we say that we have *moved y into X* . More generally, $(\text{fcl}(X), Y - \text{fcl}(X))$ is k -separating in M . Let n be an integer exceeding one. If M is n -connected, an n -separation (U, V) of M is *sequential* if $\text{fcl}(U)$ or $\text{fcl}(V)$ is $E(M)$. In particular, when $\text{fcl}(U) = E(M)$, there is an ordering (v_1, v_2, \dots, v_m) of the elements of V such that $U \cup \{v_m, v_{m-1}, \dots, v_i\}$ is n -separating for all i in $\{1, 2, \dots, m\}$. When this occurs, the set V is called *sequential*. Moreover, if $n \leq m$, then $\{v_1, v_2, \dots, v_n\}$ is a circuit or a cocircuit of M . A 3-connected matroid is *sequentially 4-connected* if all of its 3-separations are sequential. It is straightforward to check that, when M is binary, a sequential set with 3, 4, or 5 elements is a fan. Let (X, Y) be a 3-separation of a 3-connected binary matroid M . We shall frequently be interested in 3-separations that indicate that M is, for example, not internally 4-connected. We call (X, Y) or X a $(4, 3)$ -*violator* if $|Y| \geq |X| \geq 4$. Similarly, (X, Y) is a $(4, 4, S)$ -*violator* if, for each Z in $\{X, Y\}$, either $|Z| \geq 5$, or Z is non-sequential.

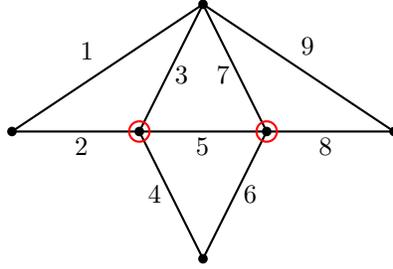


FIGURE 8. A quasi rotor, where $\{2, 3, 4, 5\}$ and $\{5, 6, 7, 8\}$ are cocircuits.

Next we note another special structure from [14], which has already arisen frequently in our work towards the desired splitter theorem. In an internally 4-connected binary matroid M , we shall call $(\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}, \{2, 3, 4, 5\}, \{5, 6, 7, 8\}, \{3, 5, 7\})$ a *quasi rotor* with *central triangle* $\{4, 5, 6\}$ and *central element* 5 if $\{1, 2, 3\}$, $\{4, 5, 6\}$, and $\{7, 8, 9\}$ are disjoint triangles in M such that $\{2, 3, 4, 5\}$ and $\{5, 6, 7, 8\}$ are cocircuits and $\{3, 5, 7\}$ is a triangle. Section 4 is dedicated to results concerning bowties and quasi rotors.

For all non-negative integers i , it will be convenient to adopt the convention throughout the paper of using T_i and D_i to denote, respectively, a triangle $\{a_i, b_i, c_i\}$ and a cocircuit $\{b_i, c_i, a_{i+1}, b_{i+1}\}$. Let M have $(T_0, T_1, T_2, D_0, D_1, \{c_0, b_1, a_2\})$ as a quasi rotor. Now T_2 may also be the central triangle of a quasi rotor. In fact, we may have a structure like one of the two depicted in Figure 9. If $T_0, D_0, T_1, D_1, \dots, T_n$ is a string of bowties in M , for some $n \geq 2$, and M has the additional structure

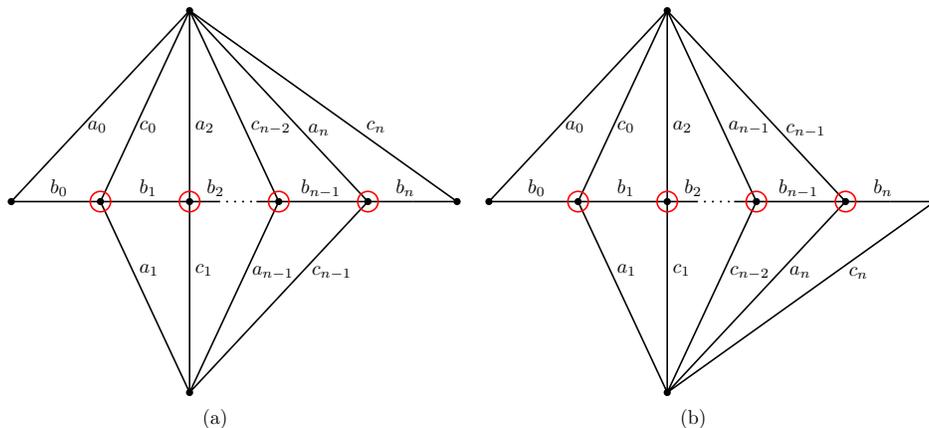


FIGURE 9. Right-maximal rotor chain configurations. In the case that n is even, the rotor chain is depicted on the left. If n is odd, then the rotor chain has the form on the right.

that $\{c_i, b_{i+1}, a_{i+2}\}$ is a triangle for all i in $\{0, 1, \dots, n-2\}$, then we say that $((a_0, b_0, c_0), (a_1, b_1, c_1), \dots, (a_n, b_n, c_n))$ is a *rotor chain*. Clearly, deleting a_0 from a rotor chain gives an open rotor chain. Observe that every three consecutive triangles within a rotor chain have the structure of a quasi rotor; that is, for all i in $\{0, 1, \dots, n-2\}$, the sequence $(T_i, T_{i+1}, T_{i+2}, D_i, D_{i+1}, \{c_i, b_{i+1}, a_{i+2}\})$ is a quasi rotor. Zhou [14] considered a similar structure called a *double fan of length $n-1$* ; it consists of all of the elements in the rotor chain except for a_0, b_0, b_n , and c_n .

If a rotor chain $((a_0, b_0, c_0), (a_1, b_1, c_1), \dots, (a_n, b_n, c_n))$ cannot be extended to a rotor chain of the form $((a_0, b_0, c_0), (a_1, b_1, c_1), \dots, (a_{n+1}, b_{n+1}, c_{n+1}))$, then we call it a *right-maximal rotor chain*.

In the introduction, we defined a string of bowties. We say that such a string $T_0, D_0, T_1, D_1, \dots, T_n$ is a *right-maximal bowtie string* in M if M has no triangle $\{u, v, w\}$ such that $T_0, D_0, T_1, D_1, \dots, T_n, \{x, c_n, u, v\}, \{u, v, w\}$ is a bowtie string for some x in $\{a_n, b_n\}$. Now let $(\{a_0, b_0, c_0\}, \{b_0, c_0, a_1, b_1\}, \{a_1, b_1, c_1\}, \dots, \{a_n, b_n, c_n\}, \{b_n, c_n, a_0, b_0\})$ be a ring of bowties. It is tempting to assume that, in such a ring, the set $\{b_0, b_1, \dots, b_n\}$ is a circuit of M . Indeed, when M is internally 4-connected, if $M = M(G)$ for some graph G , then it is not difficult to check that each of the cocircuits in the bowtie ring corresponds to the set of edges meeting some vertex of G . It follows that $\{b_0, b_1, \dots, b_n\}$ is indeed a circuit of M . However, if M is not graphic, then $\{b_0, b_1, \dots, b_n\}$ need not be a circuit of M . To see this, observe that $(\{a_0, b_0, c_0\}, \{b_0, c_0, a_1, b_1\}, \{a_1, b_1, c_1\}, \dots, \{a_3, b_3, c_3\}, \{b_3, c_3, a_0, b_0\})$ is a bowtie ring in the bond matroid of the graph G shown in Figure 10. However, $\{b_0, b_1, b_2, b_3\}$ is not a bond of G .

3. OUTLINE OF THE PROOF

This section gives an outline of the strategy used to prove the main theorem of the paper. The rest of the paper is concerned with implementing that strategy. Recall that a matroid M is $(4, 4, S)$ -connected if it is 3-connected and every 3-separation in M has, as one of its sides, a triad, a triangle, or a 4-fan. The hypotheses of

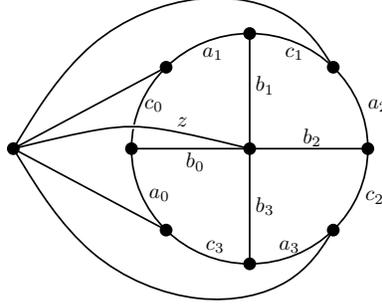


FIGURE 10. The set $\{b_0, b_1, b_2, b_3\}$ is not a bond of this graph.

the theorem present us with a bowtie $(\{a_0, b_0, c_0\}, \{a_1, b_1, c_1\}, \{b_0, c_0, a_1, b_1\})$ where $M \setminus c_0$ is $(4, 4, S)$ -connected, $M \setminus c_0, c_1$ has an N -minor, and $M \setminus c_1$ is not $(4, 4, S)$ -connected.

Because $M \setminus c_1$ is not $(4, 4, S)$ -connected, Lemma 4.3 gives us that $\{a_1, b_1, c_1\}$ is the central triangle of a quasi rotor $(T_0, T_1, T_2, D_0, D_1, \{c_0, b_1, a_2\})$ where we recall that T_i denotes a triangle $\{a_i, b_i, c_i\}$, and D_i denotes a 4-cocircuit $\{b_i, c_i, a_{i+1}, b_{i+1}\}$. Thus, for some $n \geq 2$, we have a right-maximal rotor chain $((a_0, b_0, c_0), (a_1, b_1, c_1), \dots, (a_n, b_n, c_n))$. In Lemma 9.2, we prove that either the theorem holds, or M has such a right-maximal rotor chain in which $M \setminus c_0, c_1, \dots, c_n$ is sequentially 4-connected with an N -minor, M/b_i has no N -minor for all i in $\{1, 2, \dots, n-1\}$, and $M \setminus c_n$ is $(4, 4, S)$ -connected, while M has a triangle T_{n+1} and a 4-cocircuit D_n such that $T_0, D_0, T_1, D_1, \dots, T_n, D_n, T_{n+1}$ is a bowtie string in M , and $M \setminus c_0, c_1, \dots, c_{n+1}$ has an N -minor.

We extend this bowtie string to a right-maximal bowtie string $T_0, D_0, T_1, D_1, \dots, T_n, D_n, \dots, T_k$. Then $k \geq n+1$. In Lemma 5.7, we show that $M \setminus c_0, c_1, \dots, c_k$ has an N -minor. In Lemma 10.1, we deal with the case when this bowtie string does not wrap around on itself to form a bowtie ring, and we show that the theorem holds in this case.

We may now assume that $(T_k, T_0, \{b_k, a_k, a_0, b_0\})$ is a bowtie, so $(T_0, D_0, T_1, D_1, \dots, T_k, D_k)$ is a bowtie ring. In that case, Lemma 5.5 shows that, when the theorem does not hold, $M \setminus c_0, c_1, \dots, c_k$ is sequentially 4-connected M has a triangle $\{e_1, f_1, g_1\}$ that is disjoint from $T_0 \cup T_1 \cup \dots \cup T_k$, and M has a cocircuit $\{e_1, f_1, c_j, z_j\}$ for some j in $\{0, 1, \dots, k\}$ where z_j is b_j or a_j . Situations corresponding to the two possibilities for z_j are illustrated in Figure 11. The proof of the theorem is completed when these two cases are treated in Lemma 10.4.

In the two graph diagrams shown in Figure 11, the diagrams suggest that the set $\{b_0, b_1, \dots, b_n\}$ is a circuit and this is certainly true when M is graphic. But this need not be so for an arbitrary internally 4-connected binary matroid M . This means that the reader needs to exercise some caution in dealing with such diagrams when they wrap around. Each of the triangles shown is certainly a circuit of the matroid M , and one can infer that other cycles in the graphs are circuits in the matroid when, for example, such cycles can be built by taking symmetric differences of sets of overlapping triangles.

In the next section, we note some properties of bowties and quasi rotors. In Section 5, we prove some results for strings of bowties, while Section 6 presents

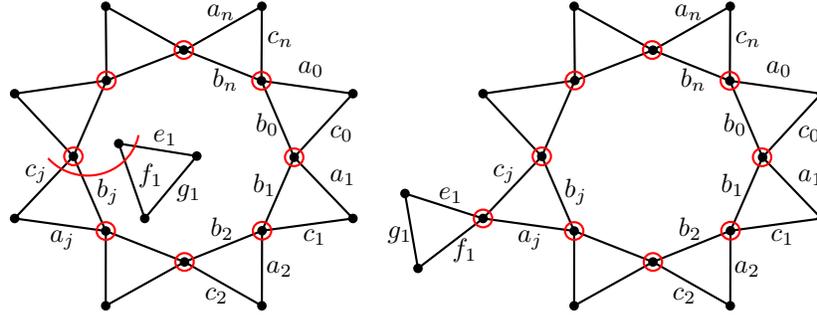


FIGURE 11. The two quintessential examples of obstructions to $M \setminus c_0, c_1, \dots, c_n$ being internally 4-connected when M contains a bowtie ring. The curved line on the left indicates that $\{e_1, f_1, b_j, c_j\}$ is a cocircuit.

some results for quartic ladder segments. The rest of the proof of our splitter theorem both here and in the sequels to this paper will essentially amount to an analysis of the behavior of bowtie strings and ladder segments.

4. SOME RESULTS FOR BOWTIES AND QUASI ROTORS

The following lemma will be used repeatedly throughout the paper. It extends [3, Lemma 2.2] to include seven-element internally 4-connected binary matroids, noting that the only such matroids are F_7 and its dual, and neither of these matroids has a 4-element fan.

Lemma 4.1. *Let N be an internally 4-connected matroid having at least seven elements and M be a binary matroid with an N -minor. If (s_1, s_2, s_3, s_4) is a 4-fan in M , then $M \setminus s_1$ or M/s_4 has an N -minor. If $(s_1, s_2, s_3, s_4, s_5)$ is a 5-fan in M , then either $M \setminus s_1, s_5$ has an N -minor, or both $M \setminus s_1/s_2$ and $M \setminus s_5/s_4$ have N -minors.*

We will use the following elementary result frequently when considering bowtie structures.

Lemma 4.2. *Let M be an internally 4-connected matroid having at least ten elements. If $(\{1, 2, 3\}, \{4, 5, 6\}, \{2, 3, 4, 5\})$ is a bowtie in M , then $\{2, 3, 4, 5\}$ is the unique 4-cocircuit of M that meets both $\{1, 2, 3\}$ and $\{4, 5, 6\}$.*

Proof. Let $X = \{1, 2, 3, 4, 5, 6\}$. As X contains two circuits, $r(X) \leq 4$. Suppose M has a 4-cocircuit C^* other than $\{2, 3, 4, 5\}$ that meets both $\{1, 2, 3\}$ and $\{4, 5, 6\}$. Then, by orthogonality, $C^* \subseteq X$. Hence $r^*(X) \leq 4$. Thus $\lambda(X) = r(X) + r^*(X) - |X| \leq 4 + 4 - 6 = 2$. This is a contradiction as $|E(M)| \geq 10$. \square

Observe that the last result need not hold if $|E(M)| < 10$. Indeed, every bowtie of $M^*(K_{3,3})$ has three distinct 4-cocircuits meeting both of its triangles.

The following result uses Lemma 4.2 to make a straightforward modification of [5, Lemma 2.6].

Lemma 4.3. *Let $(\{1, 2, 3\}, \{4, 5, 6\}, \{2, 3, 4, 5\})$ be a bowtie in an internally 4-connected binary matroid M with $|E(M)| \geq 13$. Then $M \setminus 6$ is*

$(4, 4, S)$ -connected unless $\{4, 5, 6\}$ is the central triangle of a quasi rotor $(\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}, \{2, 3, 4, 5\}, \{y, 6, 7, 8\}, \{x, y, 7\})$ for some x in $\{2, 3\}$ and some y in $\{4, 5\}$. In addition, when $M \setminus 6$ is $(4, 4, S)$ -connected, one of the following holds.

- (i) $M \setminus 6$ is internally 4-connected; or
- (ii) M has a triangle $\{7, 8, 9\}$ disjoint from $\{1, 2, 3, 4, 5, 6\}$ such that $(\{4, 5, 6\}, \{7, 8, 9\}, \{a, 6, 7, 8\})$ is a bowtie for some a in $\{4, 5\}$; or
- (iii) every $(4, 3)$ -violator of $M \setminus 6$ is a 4-fan of the form (u, v, w, x) , where M has a triangle $\{u, v, w\}$ and a cocircuit $\{v, w, x, 6\}$ where u and v are in $\{2, 3\}$ and $\{4, 5\}$, respectively, and $|\{1, 2, 3, 4, 5, 6, w, x\}| = 8$; or
- (iv) $M \setminus 1$ is internally 4-connected and M has a triangle $\{1, 7, 8\}$ and a cocircuit $\{a, 6, 7, 8\}$ where $|\{1, 2, 3, 4, 5, 6, 7, 8\}| = 8$ and $a \in \{4, 5\}$.

In this paper, we are focussing on the case in which $(\{1, 2, 3\}, \{4, 5, 6\}, \{2, 3, 4, 5\})$ is a bowtie and $M \setminus 6$ has an N -minor but is not $(4, 4, S)$ -connected; that is, we are concerned with the case when $\{4, 5, 6\}$ is the central triangle of a quasi rotor. The remaining cases that arise from this lemma will be treated in [7] and [8].

We proved the following result in [2, Theorem 6.1].

Theorem 4.4. *Let M be an internally 4-connected binary matroid having $(\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}, \{2, 3, 4, 5\}, \{5, 6, 7, 8\}, \{3, 5, 7\})$ as a quasi rotor and having at least thirteen elements. Then either*

- (i) $M \setminus 1$, $M \setminus 9$, $M \setminus 1/2$, or $M \setminus 9/8$ is internally 4-connected; or
- (ii) M has triangles $\{6, 8, 10\}$ and $\{2, 4, 11\}$ such that $|\{1, 2, \dots, 11\}| = 11$, and $M \setminus 3, 4/5$ is internally 4-connected.

We show next that if M contains an element e that is in two triangles of a quasi rotor and M/e has an N -minor, then M has an internally 4-connected minor M' that has an N -minor and satisfies $|E(M) - E(M')| \in \{1, 2, 3\}$.

Lemma 4.5. *Let M be an internally 4-connected binary matroid having $(\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}, \{2, 3, 4, 5\}, \{5, 6, 7, 8\}, \{3, 5, 7\})$ as a quasi rotor and having at least thirteen elements. Let N be an internally 4-connected matroid containing at least six elements such that M/e has an N -minor for some e in $\{3, 5, 7\}$. Then one of $M \setminus 1$, $M \setminus 9$, $M \setminus 1/2$, $M \setminus 9/8$, or $M \setminus 3, 4/5$ is internally 4-connected with an N -minor.*

Proof. First suppose that $|E(N)| = 6$. Then $N \cong M(K_4)$. Since each internally 4-connected binary matroid with at least six elements has an $M(K_4)$ -minor, the lemma follows by Theorem 4.4. We may now assume that $|E(M)| \geq 7$. By symmetry, it suffices to prove the result for e in $\{5, 7\}$. We prove it first for $e = 5$ and then use that case to prove it for $e = 7$.

Assume $N \preceq M/5$. As $M/5$ has $\{3, 7\}$ and $\{4, 6\}$ as circuits, we deduce that $N \preceq M/5 \setminus 3, 4$. The theorem holds for $e = 5$ if $M/5 \setminus 3, 4$ is internally 4-connected, so assume it is not. Then, by Theorem 4.4, one of $M \setminus 1$, $M \setminus 9$, $M \setminus 1/2$, or $M \setminus 9/8$ is internally 4-connected. By symmetry, if we can show that $N \preceq M \setminus 1/2$, then the theorem will follow for $e = 5$. Now $N \preceq M/5 \setminus 3, 4$ and, in $M \setminus 3, 4$, the elements 2 and 5 are in series. Hence $M/5 \setminus 3, 4 \cong M/2 \setminus 3, 4$. But, in $M/2$, the elements 1 and 3 are in parallel. Hence $M/2 \setminus 3, 4 \cong M/2 \setminus 1, 4$. We deduce that $N \preceq M \setminus 1/2$, so the theorem holds for $e = 5$.

Now suppose that $N \preceq M/7$ but $N \not\preceq M/5$. Then $N \preceq M/7 \setminus 5, 8$. Now $M/7 \setminus 5, 8 \cong M \setminus 5, 8/6 \cong M/6 \setminus 4, 8$. Since $(1, 2, 3, 5, 7)$ is a fan of $M \setminus 4, 8$, Lemma 4.1 implies that $N \preceq M \setminus 4, 8 \setminus 1, 7$ or $N \preceq M \setminus 4, 8/5 \setminus 7$. The second possibility is excluded because $N \not\preceq M/5$. Thus $N \preceq M \setminus 7, 8$. As $M \setminus 7, 8$ has $\{5, 6\}$ as cocircuit, we deduce that $N \preceq M/5$; a contradiction. \square

5. STRINGS AND RINGS OF BOWTIES

Strings and rings of bowties will feature prominently throughout the rest of the paper. This section develops some properties of such structures. Recall that, for each natural number i , we are using T_i and D_i to denote a triangle $\{a_i, b_i, c_i\}$ and a cocircuit $\{b_i, c_i, a_{i+1}, b_{i+1}\}$, respectively.

When bowtie strings appear in our theorems, they do so embedded in more highly structured configurations, specifically, open rotor chains, ladder structures, enhanced quartic ladders, and bowtie rings. Of the last four structures, bowtie rings allow for the most variation in the surrounding structure. Suppose that we have a bowtie string as shown in Figure 2 and that, in addition, $n \geq 2$ and $\{b_n, c_n, a_0, b_0\}$ is a cocircuit, D_n . Then $(T_0, D_0, T_1, D_1, \dots, T_n, D_n)$ is a ring of bowties. Suppose that this ring occurs in an internally 4-connected graphic matroid $M(G)$. As observed in the introduction, for all i in $\{0, 1, \dots, n\}$, there is a vertex v_i of G such that $\{b_i, c_i, a_{i+1}, b_{i+1}\}$ is the set of edges of G meeting v_i . Hence $\{b_0, b_1, \dots, b_n\}$ is a cycle of G . For each i , let w_i be the vertex that meets a_i and c_i . The definition of a ring of bowties does not require w_0, w_1, \dots, w_n to be distinct. Indeed, subject to some constraints to ensure that G stays internally 4-connected, they need not be. As an example, take a copy of K_{10} with vertex set $\{u_0, u_1, \dots, u_9\}$ and let $n = 99$, so we have a bowtie ring with 100 triangles. Suppose that, for all j in $\{0, 1, \dots, 99\}$, the vertex w_j is identified with u_t where $j \equiv t \pmod{10}$. Let G be the resulting 110-vertex graph. Then $M(G)$ is easily shown to be internally 4-connected. For $N = M(G) \setminus c_0, c_1, \dots, c_{99}$, we see that N is internally 4-connected while there is no internally 4-connected proper minor of M that has an N -minor.

As another example, let H be the octahedron graph. Clearly $M(H)$ contains a bowtie ring $(T_0, D_0, T_1, D_1, T_2, D_2, T_3, D_3)$ with exactly four triangles. In addition, $M(H)$ has a 4-cocircuit D' such that $(T_0, D_0, T_1, D_1, T_2, D')$ is a bowtie ring. A bowtie ring is *minimal* exactly when no proper subset of its set of triangles is the set of triangles of a bowtie ring. Next we show that when we trim a bowtie ring to produce an internally 4-connected matroid, that bowtie ring must be minimal.

Lemma 5.1. *Let $(T_0, D_0, T_1, D_1, \dots, T_n, D_n)$ be a bowtie ring in an internally 4-connected binary matroid M where $|E(M)| \geq 10$. If $M \setminus c_0, c_1, \dots, c_n$ is internally 4-connected, then $(T_0, D_0, T_1, D_1, \dots, T_n, D_n)$ is a minimal bowtie ring.*

Proof. Suppose the lemma does not hold. Then some proper subset of $\{T_0, T_1, \dots, T_n\}$ is the set of triangles of a bowtie ring. Hence M has a 4-cocircuit that meets two triangles in this set and that is not contained in $\{D_0, D_1, \dots, D_n\}$. Choose such a 4-cocircuit C^* to maximize $|C^* \cap \{c_0, c_1, \dots, c_n\}|$.

Take two distinct integers i and j in $\{0, 1, \dots, n\}$ such that C^* meets T_i and T_j . Lemma 4.2 implies that T_i and T_j are not consecutive triangles in the ring. By orthogonality, $C^* \subseteq T_i \cup T_j$. If $C^* \supseteq \{a_i, b_i\}$, then $C^* \triangle D_{i-1}$ is a 4-cocircuit that is not contained in $\{D_0, D_1, \dots, D_n\}$ and that has a larger intersection with $\{c_0, c_1, \dots, c_n\}$ than C^* does; a contradiction. Thus $c_i \in C^*$ and, by symmetry,

$c_j \in C^*$. Then $M \setminus c_0, c_1, \dots, c_n$ has a 2-cocircuit contained in $\{a_i, b_i, a_j, b_j\}$. This is a contradiction as $|E(M \setminus c_0, c_1, \dots, c_n)| \geq 6$ since $n \geq 2$. \square

The following property of bowtie strings will be frequently used.

Lemma 5.2. *Let $T_0, D_0, T_1, D_1, \dots, T_n$ be a string of bowties in a matroid M . Then, for all k in $\{1, 2, \dots, n\}$,*

$$\begin{aligned} M \setminus c_0, c_1, \dots, c_n / b_n &\cong M \setminus c_0, c_1, \dots, c_{k-1} / b_k \setminus a_k, a_{k+1}, \dots, a_n \\ &\cong M \setminus c_0, c_1, \dots, c_{k-1} / b_{k-1} \setminus a_k, a_{k+1}, \dots, a_n \\ &\cong M \setminus a_0, a_1, \dots, a_n / b_0. \end{aligned}$$

Proof. Evidently, $M \setminus c_0, c_1, \dots, c_n / b_n \cong M \setminus c_0, c_1, \dots, c_{n-1} / b_n \setminus a_n \cong M \setminus c_0, c_1, \dots, c_{n-1} / b_{n-1} \setminus a_n$. The lemma follows by repeatedly applying this observation. \square

Lemma 5.3. *Let $T_0, D_0, T_1, D_1, \dots, T_n$ be a string of bowties in an internally 4-connected binary matroid M , where $|E(M)| \geq 8$. Then either $M \setminus c_0, c_1, \dots, c_n$ is 3-connected, or it has a_j or b_j in a cocircuit of size at most two for some j in $\{2, 3, \dots, n\}$.*

Proof. As $|E(M)| \geq 8$ and M is internally 4-connected having c_0 in a triangle, [2, Lemma 2.5] implies that $M \setminus c_0$ is 3-connected. Now if $M \setminus c_0, c_1, \dots, c_n$ is 3-connected, then the lemma holds. Hence we may assume there is an element i in $\{1, 2, \dots, n\}$ such that $M \setminus c_0, c_1, \dots, c_i$ is not 3-connected but $M \setminus c_0, c_1, \dots, c_j$ is 3-connected for all j in $\{0, 1, \dots, i-1\}$. Then Tutte's Triangle Lemma [13] (or see [11, Lemma 8.7.7]) implies that either $M \setminus c_0, c_1, \dots, c_{i-1} \setminus x$ is 3-connected for some x in $\{a_i, b_i\}$, or c_i is in a triad of $M \setminus c_0, c_1, \dots, c_{i-1}$. The former does not occur because $M \setminus c_0, c_1, \dots, c_{i-1}$ has $\{b_{i-1}, a_i, b_i\}$ as a cocircuit. Thus c_i is in a triad of $M \setminus c_0, c_1, \dots, c_{i-1}$. By orthogonality, this triad meets $\{a_i, b_i\}$. Then $M \setminus c_0, c_1, \dots, c_i$ has a cocircuit $\{x, y\}$ where $x \in \{a_i, b_i\}$ but $y \notin \{a_i, b_i\}$. Thus M has a cocircuit C^* such that $\{c_i, x, y\} \subseteq C^* \subseteq \{y, a_i, b_i, c_i, c_{i-1}, \dots, c_0\}$. We know that C^* contains at least four elements, since it meets a triangle in M . Suppose $c_i \neq a_0$. Then, by orthogonality between C^* and each of T_0, T_1, \dots, T_{i-1} , and T_i , we deduce that $C^* = \{x, c_i, y, c_j\}$ for some $j \neq i$, where $y \in \{a_j, b_j\}$. Now Lemma 4.2 implies that $\{i, j\} \neq \{0, 1\}$ so the lemma holds when $c_i \neq a_0$. We may now assume that $c_i = a_0$. Then $i = n$. Moreover, $n \geq 2$, otherwise the 4-element set D_0 is both a circuit and a cocircuit of M ; a contradiction. Orthogonality between C^* and each of T_0, T_1, \dots, T_{i-1} , and T_i implies that $C^* = \{x, y, c_n, c_0\}$. Then $M \setminus c_0, c_1, \dots, c_n$ has a_n or b_n in a cocircuit of size at most two and again the lemma holds. \square

The following modifies the argument used to prove [2, Lemma 11.4].

Lemma 5.4. *Let M be an internally 4-connected matroid having at least ten elements. Suppose that M has $T_0, D_0, T_1, D_1, \dots, T_n$ as a string of bowties where T_0 and T_n are disjoint. Assume that (T_n, T_{n+1}, D) is a bowtie of M for some 4-cocircuit $D \neq D_{n-1}$ where $\{c_n, a_{n+1}, b_{n+1}\} \subseteq D$. If T_{n+1} meets T_i for some $i \leq n$, then*

- (i) $T_{n+1} = T_j$ for some j with $0 \leq j \leq n-2$; or
- (ii) $T_{n+1} \cap (T_0 \cup T_1 \cup \dots \cup T_n) = T_{n+1} \cap T_0 = \{a_0\} = \{c_{n+1}\}$.

Furthermore, if $\{x, c_n, a_0, b_0\}$ is a cocircuit for some x in $\{a_n, b_n\}$, then (i) holds.

Proof. Let j be the greatest integer, i , in $\{0, 1, \dots, n\}$ such that $T_{n+1} \cap T_i \neq \emptyset$. We show first that (i) or (ii) holds. Since (T_n, T_{n+1}, D) is a bowtie, $j \leq n-1$. If T_j meets $\{a_{n+1}, b_{n+1}\}$, then, by orthogonality, T_j contains $\{a_{n+1}, b_{n+1}\}$, so $T_j = T_{n+1}$. Also, if T_{n+1} meets $\{b_j, c_j\}$, then T_{n+1} contains $\{b_j, c_j\}$, so $T_j = T_{n+1}$. Thus either

- (a) $T_j = T_{n+1}$; or
- (b) $T_j \cap T_{n+1} = \{a_j\} = \{c_{n+1}\}$.

If (a) holds, then Lemma 4.2 implies that $j \leq n-2$, so (i) holds. Hence we may assume that (b) holds. Then $j > 0$, otherwise (ii) holds. Thus T_{n+1} meets D_{j-1} , so T_{n+1} meets $\{b_{j-1}, c_{j-1}\}$. Hence a_{n+1} or b_{n+1} is in T_{j-1} , that is, D meets T_{j-1} . But $T_n \cap T_{j-1} = \emptyset$, so $\{a_{n+1}, b_{n+1}\} \subseteq T_{j-1}$. Thus $T_{n+1} = T_{j-1}$. Hence $T_{n+1} \cap T_j = \emptyset$; a contradiction. We conclude that (i) or (ii) holds.

Finally, suppose that $\{x, c_n, a_0, b_0\}$ is a cocircuit for some $x \in \{a_n, b_n\}$ but that (i) does not hold. Then (ii) holds so orthogonality implies that T_{n+1} meets T_n ; a contradiction. \square

Lemma 5.5. *Let M be an internally 4-connected binary matroid containing at least thirteen elements. Suppose that M has a ring of bowties $(T_0, D_0, T_1, D_1, \dots, T_k, D_k)$. Then*

- (i) $M \setminus c_0, c_1, \dots, c_k$ is internally 4-connected; or
- (ii) $M \setminus c_0, c_1, \dots, c_k$ is sequentially 4-connected and every 4-fan of it has the form $(1, 2, 3, 4)$ where $\{1, 2, 3\}$ is disjoint from $T_0 \cup T_1 \cup \dots \cup T_k$, and M has a cocircuit $\{2, 3, 4, c_i\}$ for some i in $\{0, 1, \dots, k\}$ where $4 \in \{a_i, b_i\}$; or
- (iii) $M \setminus c_0, c_1, \dots, c_k$ has a 1- or 2-element cocircuit that meets $\{a_2, b_2, a_3, b_3, \dots, a_k, b_k\}$.

Proof. Assume that (iii) does not hold. Then Lemma 5.3 implies that $M \setminus c_0, c_1, \dots, c_k$ is 3-connected for all i in $\{0, 1, \dots, k\}$. Let $M' = M \setminus c_0, c_1, \dots, c_k$. We shall show next that

5.5.1. *M' is sequentially 4-connected.*

Assume that this fails. First we show that

5.5.2. *M' has a non-sequential 3-separation (X, Y) such that, for each i in $\{0, 1, \dots, k\}$, the pair $\{a_i, b_i\}$ is contained in X or Y .*

Certainly M' has a non-sequential 3-separation (X, Y) in which $\{a_0, b_0, b_k\}$ is contained in X or Y . Take the smallest index, i , such that $\{a_i, b_i\}$ meets both X and Y . We may assume that $a_i \in X$ and $b_i \in Y$. Now $\{a_{i-1}, b_{i-1}\}$ is contained in X or Y . Then $\{b_{i-1}, a_i\} \subseteq X$ or $\{b_{i-1}, b_i\} \subseteq Y$. Thus $b_i \in \text{cl}^*_{M'}(X)$ or $a_i \in \text{cl}^*_{M'}(Y)$, respectively. Hence $(X \cup b_i, Y - b_i)$ or $(X - a_i, Y \cup a_i)$ is a non-sequential 3-separation of M' in which $\{a_i, b_i\}$ is contained in one side. By repeating this process, we see that 5.5.2 holds.

By 5.5.2, each c_i is in the closure of X or Y , so M has a non-sequential 3-separation; a contradiction. We conclude that 5.5.1 holds.

We may assume that M' is not internally 4-connected otherwise the lemma holds. To complete the proof, we show the following.

5.5.3. *If $(1, 2, 3, 4)$ is a 4-fan of M' , then $\{1, 2, 3\}$ is disjoint from $T_0 \cup T_1 \cup \dots \cup T_k$, and M has $\{2, 3, 4, c_i\}$ as a cocircuit for some i in $\{0, 1, \dots, k\}$ where $4 \in \{a_i, b_i\}$.*

Evidently M has $\{1, 2, 3\}$ as a triangle and has a cocircuit C^* such that $\{2, 3, 4\} \subsetneq C^* \subseteq \{2, 3, 4, c_0, c_1, \dots, c_k\}$. Suppose $c_i \in C^*$. Then $\{2, 3, 4\}$ meets $\{a_i, b_i\}$. We shall show next that $\{2, 3\}$ avoids $\{a_i, b_i\}$. Assume the contrary. Then the cocircuit $\{b_{i-1}, a_i, b_i\}$ in M' implies that $\{1, 2, 3\}$ contains $\{b_{i-1}, a_i\}$ or $\{b_{i-1}, b_i\}$ where all subscripts are interpreted modulo $k+1$. The cocircuit $\{b_{i-2}, a_{i-1}, b_{i-1}\}$ implies that $\{1, 2, 3\}$ is $\{b_{i-2}, b_{i-1}, a_i\}$ or $\{b_{i-2}, b_{i-1}, b_i\}$. Orthogonality with the cocircuit $\{b_{i-3}, a_{i-2}, b_{i-2}\}$ implies that $b_{i-3} = b_i$, that is, $k = 2$. In that case, since $\{b_{i-2}, b_{i-1}, b_i\}$ is a triangle, we see that $\lambda(T_{i-2} \cup T_{i-1} \cup T_i) \leq 2$. This contradicts the fact that M is internally 4-connected. We conclude that $\{2, 3\} \cap \{a_i, b_i\} = \emptyset$. It follows that $4 \in \{a_i, b_i\}$. Hence $C_i^* = \{2, 3, 4, c_i\}$. Thus, by orthogonality, $\{1, 2, 3\}$ avoids $T_0 \cup T_1 \cup \dots \cup T_k$, so 5.5.3 holds. Hence so does the lemma. \square

When we trim the bowtie ring in Figure 6, we delete all of the dashed oblique edges. The next lemma shows that we obtain an isomorphic matroid by deleting, instead, all of the solid oblique edges.

Lemma 5.6. *Let $(T_0, D_0, T_1, D_1, \dots, T_n, D_n)$ be a ring of bowties in an internally 4-connected binary matroid M . Then*

- (i) $\{b_0, b_1, \dots, b_n\}$ is either a circuit or an independent set of M ; and
- (ii) $M \setminus c_0, c_1, \dots, c_n \cong M \setminus a_0, a_1, \dots, a_n$.

Proof. If $\{b_0, b_1, \dots, b_n\}$ is dependent, then it contains a circuit, C . By orthogonality between C and the cocircuit D_i , we see that if $b_i \in C$ for some i , then $b_{i+1} \in C$. Hence $C = \{b_0, b_1, \dots, b_n\}$, and (i) holds.

To prove (ii), let $E = E(M)$ and, for each j in $\{0, 1, \dots, n\}$, define the function $\varphi_j : E - \{c_0, c_1, \dots, c_n, b_j\} \rightarrow E - \{a_0, a_1, \dots, a_n, b_{j+1}\}$ by

$$\varphi_j(x) = \begin{cases} c_i & \text{if } x = a_i \text{ and } i \in \{0, 1, \dots, n\}; \\ b_{k+1} & \text{if } x = b_k \text{ and } k \neq j; \text{ and} \\ x & \text{otherwise.} \end{cases}$$

We show next that

5.6.1. φ_j is an isomorphism between $M \setminus c_0, c_1, \dots, c_n / b_j$ and $M \setminus a_0, a_1, \dots, a_n / b_{j+1}$.

By the symmetry of the ring of bowties, it suffices to prove this when $j = n$. Here we will exploit the isomorphisms noted in Lemma 5.2. For each t in $\{1, 2, \dots, n\}$, the isomorphism between $M \setminus c_0, c_1, \dots, c_{t-1}, c_t / b_t \setminus a_{t+1}, \dots, a_n$ and $M \setminus c_0, c_1, \dots, c_{t-1}, a_t / b_{t-1} \setminus a_{t+1}, \dots, a_n$ can be achieved by the mapping ψ_t that takes a_t to c_t and takes b_{t-1} to b_t while fixing every other element. In addition, let ω_0 be the isomorphism between $M \setminus c_0 / b_0 \setminus a_1, \dots, a_n$ and $M \setminus a_0 / b_0 \setminus a_1, \dots, a_n$ obtained by mapping a_0 to c_0 and fixing every other element. The composition $\omega_0 \circ \psi_1 \circ \psi_2 \circ \dots \circ \psi_n$ equals φ_n . Hence 5.6.1 holds.

Now define $\varphi : E - \{c_0, c_1, \dots, c_n\} \rightarrow E - \{a_0, a_1, \dots, a_n\}$ by

$$\varphi(x) = \begin{cases} c_i & \text{if } x = a_i \text{ and } i \in \{0, 1, \dots, n\}; \\ b_{i+1} & \text{if } x = b_i \text{ and } i \in \{0, 1, \dots, n\}; \text{ and} \\ x & \text{otherwise.} \end{cases}$$

Observe that, for all j in $\{0, 1, \dots, n\}$,

5.6.2. $\varphi(y) = \varphi_j(y)$ for all y in $E - \{c_0, c_1, \dots, c_n\} - b_j$.

To complete the proof of (ii), we shall show that

5.6.3. φ is an isomorphism between $M \setminus c_0, c_1, \dots, c_n$ and $M \setminus a_0, a_1, \dots, a_n$.

By 5.6.1 and 5.6.2, for all j in $\{0, 1, \dots, n\}$, the function φ induces an isomorphism between $M \setminus c_0, c_1, \dots, c_n/b_j$ and $M \setminus a_0, a_1, \dots, a_n/b_{j+1}$. To establish 5.6.3, we suffice to prove the following.

- (a) If X is a cocircuit of $M \setminus c_0, c_1, \dots, c_n$, then $\varphi(X)$ contains a cocircuit of $M \setminus a_0, a_1, \dots, a_n$.
- (b) If Y is a cocircuit of $M \setminus a_0, a_1, \dots, a_n$, then $\varphi^{-1}(Y)$ contains a cocircuit of $M \setminus c_0, c_1, \dots, c_n$.

We will show (a); a symmetric argument establishes (b). Suppose the cocircuit X of $M \setminus c_0, c_1, \dots, c_n$ avoids b_s for some s . Then X is a cocircuit of $M \setminus c_0, c_1, \dots, c_n/b_s$. Thus $\varphi(X)$ is a cocircuit of $M \setminus a_0, a_1, \dots, a_n/b_{s+1}$ and so is a cocircuit of $M \setminus a_0, a_1, \dots, a_n$. Then (a) holds unless X contains $\{b_0, b_1, \dots, b_n\}$. Consider the exceptional case. Since $M \setminus c_0, c_1, \dots, c_n$ has $\{b_0, a_1, b_1\}$ as a disjoint union of cocircuits and $\{b_0, b_1\} \subseteq X$, it follows that $a_1 \notin X$. Thus $X \triangle \{b_0, a_1, b_1\}$ contains a cocircuit D of $M \setminus c_0, c_1, \dots, c_n$ that contains a_1 but avoids $\{b_0, b_1\}$. Hence D is a cocircuit of $M \setminus c_0, c_1, \dots, c_n/b_0$ that avoids b_1 . Thus $\varphi(D)$ is a cocircuit of $M \setminus a_0, a_1, \dots, a_n/b_1$ that contains c_1 and avoids b_2 . Hence $\varphi(D)$ is a cocircuit of $M \setminus a_0, a_1, \dots, a_n$ that contains c_1 and avoids $\{b_1, b_2\}$. But $\{b_1, c_1, b_2\}$ is a disjoint union of cocircuits of $M \setminus a_0, a_1, \dots, a_n$, so $\varphi(D) \triangle \{b_1, c_1, b_2\}$ contains a cocircuit of $M \setminus a_0, a_1, \dots, a_n$ that avoids c_1 and so is contained in $\varphi(X)$. We conclude that (a) holds. Hence 5.6.3 holds and the lemma is proved. \square

We conclude this section with another property of strings of bowties that we will use often.

Lemma 5.7. *Let M be a binary matroid and N be an internally 4-connected binary matroid having at least seven elements. Let $T_0, D_0, T_1, D_1, \dots, T_n$ be a string of bowties in M . Suppose $M \setminus c_0, c_1$ has an N -minor but $M \setminus c_0, c_1/b_1$ does not. Then $M \setminus c_0, c_1, \dots, c_n$ has an N -minor, but $M \setminus c_0, c_1, \dots, c_i/b_i$ has no N -minor for all i in $\{1, 2, \dots, n\}$, and $M \setminus c_0, c_1, \dots, c_j/a_j$ has no N -minor for all j in $\{2, 3, \dots, n\}$.*

Proof. We may assume that $n \geq 2$ otherwise there is nothing to prove. For i in $\{1, 2, \dots, n\}$, it follows by Lemma 5.2 that $M \setminus c_0, c_1, \dots, c_i/b_i \cong M \setminus c_0, c_1/b_1 \setminus a_2, \dots, a_i$. As $M \setminus c_0, c_1/b_1$ has no N -minor, we deduce that $M \setminus c_0, c_1, \dots, c_i/b_i$ has no N -minor. If $M \setminus c_0, c_1, \dots, c_j/a_j$ has an N -minor for some $j \in \{2, 3, \dots, n\}$, then so do $M \setminus c_0, c_1, \dots, c_{j-1}, b_j/a_j$ and $M \setminus c_0, c_1, \dots, c_{j-1}, b_j/b_{j-1}$; a contradiction. Thus the second part of the lemma holds. For the first part, suppose that $M \setminus c_0, c_1, \dots, c_k$ has no N -minor for some k in $\{2, 3, \dots, n\}$ but that $M \setminus c_0, c_1, \dots, c_{k-1}$ does have an N -minor. As $M \setminus c_0, c_1, \dots, c_{k-1}$ has (c_k, b_k, a_k, b_{k-1}) as a 4-fan, we know by Lemma 4.1 that $M \setminus c_0, c_1, \dots, c_{k-1}/b_{k-1}$ has an N -minor; a contradiction. We conclude that $M \setminus c_0, c_1, \dots, c_n$ has an N -minor. \square

6. RESULTS FOR LADDER SEGMENTS

A string of bowties may be part of a quartic ladder segment within a binary matroid. In this section, we consider the ramifications of such an occurrence.

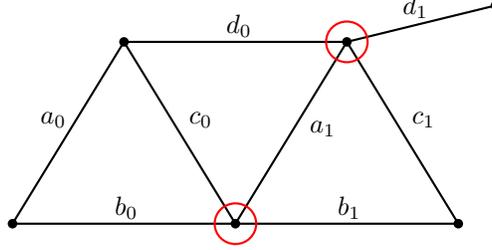


FIGURE 12

Lemma 6.1. *Assume that M is an internally 4-connected binary matroid that contains the configuration shown in Figure 12 and has at least thirteen elements. Then all of the elements in the figure are distinct, $M \setminus c_0, c_1$ is sequentially 4-connected, and a_1 is in no triangle of $M \setminus c_0, c_1$. Moreover, one of the following holds.*

- (i) $M \setminus c_0, c_1$ is internally 4-connected;
- (ii) $\{d_0, d_1\}$ is in a triangle of M ;
- (iii) $\{b_0, b_1\}$ is in a triangle of M ;
- (iv) M has a triangle $\{\alpha, \beta, a_0\}$ and has $\{\beta, a_0, d_0, c_0\}$ or $\{\beta, a_0, a_1, c_0, c_1\}$ as a cocircuit, where α and β are elements not shown in the figure, and $M \setminus c_0, c_1$ is $(4, 4, S)$ -connected with every 4-fan of it having α as its guts element or b_1 as its coguts element; or
- (v) $M \setminus c_0, c_1$ is $(4, 4, S)$ -connected and b_1 is the coguts element of all its 4-fans.

Proof. Recall that $T_i = \{a_i, b_i, c_i\}$ for each i in $\{0, 1\}$. First we show that

6.1.1. $|T_0 \cup T_1 \cup \{d_0, d_1\}| = 8$.

As M is internally 4-connected, $d_0 \neq d_1$. Evidently $|T_0 \cup T_1| = 6$ otherwise $a_0 = c_1$ and $\lambda(T_0 \cup T_1) \leq 2$; a contradiction. If $\{d_0, d_1\}$ meets T_1 , then M has a 4-fan or has a triangle contained in a cocircuit. Each possibility gives a contradiction. Finally, if $\{d_0, d_1\}$ meets T_0 , then, by orthogonality, $\{d_0, d_1\} \subseteq T_0$, so $\lambda(T_0 \cup T_1) \leq 2$; a contradiction. Thus 6.1.1 holds.

To see that a_1 is in no triangle of $M \setminus c_0, c_1$, we observe that such a triangle would have to be $\{b_0, a_1, d_1\}$. Then $\lambda(T_0 \cup T_1 \cup \{d_0, d_1\}) \leq 2$; a contradiction.

Next we show that

6.1.2. $M \setminus c_0, c_1$ is sequentially 4-connected.

Let (U, V) be a non-sequential 3-separation of $M \setminus c_0, c_1$. Then we may assume that $\{a_1, d_0, d_1\} \subseteq U$ and U is fully closed in $M \setminus c_0, c_1$. If a_0, b_0 , or b_1 is in U , then $\{a_0, b_0, b_1\} \subseteq U$ and $(U \cup \{c_0, c_1\}, V)$ is a non-sequential 3-separation of M ; a contradiction. Thus $\{a_0, b_0, b_1\} \subseteq V$. Then $a_1 \in \text{cl}_{M \setminus c_0, c_1}^*(V)$, and $(U - a_1, V \cup a_1 \cup c_0 \cup c_1)$ is a non-sequential 3-separation of M ; a contradiction. Thus 6.1.2 holds.

Now assume that (i) does not hold. Let $(\alpha, \beta, \gamma, \delta)$ be a 4-fan in $M \setminus c_0, c_1$. Then M has a cocircuit C^* such that $\{\beta, \gamma, \delta\} \subsetneq C^* \subseteq \{\beta, \gamma, \delta, c_0, c_1\}$. The rest of the argument will split into the cases when c_0 is and is not in C^* with the first of these occupying all but the last paragraph of the proof.

Suppose that $c_0 \in C^*$. Then C^* meets each of $\{a_1, d_0\}$ and $\{a_0, b_0\}$ in exactly one element. If $d_0 \in \{\beta, \gamma\}$, then, by orthogonality, $d_1 \in \{\alpha, \beta, \gamma\}$ so (ii) holds.

Hence we may assume that $\delta \in \{a_1, d_0\}$ since, as noted above, a_1 is in no triangle of $M \setminus c_0, c_1$. Moreover, without loss of generality, $\gamma \in \{a_0, b_0\}$.

Suppose $b_0 = \gamma$. Then orthogonality implies that $\{b_0, b_1\}$ or $\{b_0, a_1\}$ is contained in a triangle. In the former case, (iii) holds; the latter does not arise since $M \setminus c_0, c_1$ has no triangle containing a_1 .

We may now assume that $a_0 = \gamma$. It follows that M has a triangle containing a_0 , and C^* is one of $\{\beta, a_0, a_1, c_0\}$, $\{\beta, a_0, a_1, c_0, c_1\}$, $\{\beta, a_0, d_0, c_0\}$, or $\{\beta, a_0, d_0, c_0, c_1\}$. If C^* is $\{\beta, a_0, a_1, c_0\}$ or $\{\beta, a_0, d_0, c_0, c_1\}$, then, by orthogonality, $\beta = b_1$, so $\lambda(T_0 \cup T_1 \cup d_0) \leq 2$; a contradiction. Thus C^* is $\{\beta, a_0, a_1, c_0, c_1\}$ or $\{\beta, a_0, d_0, c_0\}$. Therefore (iv) holds provided α and β are new elements and every (4, 3)-violator of $M \setminus c_0, c_1$ is a 4-fan with α as its guts or b_1 as its coguts.

If β is an existing element, then $\lambda(T_0 \cup T_1 \cup \{d_0, d_1\}) \leq 2$; a contradiction. Suppose α is an existing element. Then $\beta \in \text{cl}(T_0 \cup T_1 \cup \{d_0, d_1\})$, so $\lambda(T_0 \cup T_1 \cup \{d_0, d_1, \beta\}) \leq 2$; a contradiction. Thus α and β are new elements. Let (u, v, w, x) be a 4-fan in $M \setminus c_0, c_1$ where $u \neq \alpha$. Since a_1 is in no triangle of $M \setminus c_0, c_1$, we know that if $\{u, v, w\}$ meets $\{d_0, d_1\}$ or $\{b_0, b_1\}$, then orthogonality implies that (ii) or (iii) holds, respectively. Hence we may assume that every triangle in $M \setminus c_0, c_1$ avoids $\{a_1, d_0, d_1, b_0, b_1\}$. We know that M has a cocircuit D^* such that $\{v, w, x\} \subsetneq D^* \subseteq \{v, w, x, c_0, c_1\}$.

Suppose $c_0 \in D^*$. Then orthogonality implies that $\{v, w, x\}$ meets $\{a_0, b_0\}$ and $\{a_1, d_0\}$, so $x \in \{a_1, d_0\}$ and, without loss of generality, $w = a_0$. Then orthogonality between the triangle $\{\alpha, \beta, a_0\}$ and the cocircuit D^* implies that $v \in \{\alpha, \beta\}$. Thus $\{u, v, w\} = \{\alpha, \beta, a_0\}$ so $v = \alpha$ and $u = \beta$. Hence the cocircuits C^* and D^* imply that $\lambda(T_0 \cup T_1 \cup \{\alpha, \beta, d_0\}) \leq 2$; a contradiction. We deduce that $c_0 \notin D^*$, so $D^* = \{v, w, x, c_1\}$. Thus Lemma 4.2 implies that $\{v, w, x\}$ avoids T_0 . Orthogonality implies that $\{v, w, x\}$ meets $\{a_1, b_1\}$, so $x \in \{a_1, b_1\}$. If $x = a_1$, then orthogonality implies that $\{v, w\}$ meets $\{a_0, b_0, d_0\}$, a contradiction. We conclude that every 4-fan of $M \setminus c_0, c_1$ has α as its guts or b_1 as its coguts, so (iv) holds provided $M \setminus c_0, c_1$ has no 5-fans and no 5-cofans. If $M \setminus c_0, c_1$ has (1, 2, 3, 4, 5) as a 5-fan, then, up to reversing the order of the fan, $1 = \alpha$ and $2 = b_1$, so b_1 is in a triangle of $M \setminus c_0, c_1$; a contradiction. Suppose then that (0, 1, 2, 3, 4) is a 5-cofan of $M \setminus c_0, c_1$. Then, up to reversing the order, $0 = b_1$ and $1 = \alpha$, so orthogonality implies that $2 \in \{\beta, a_0\}$. Thus M has a cocircuit containing $\{b_1, \alpha\}$ and contained in $\{b_1, \alpha, \beta, a_0, c_0, c_1\}$. Using this cocircuit together with the cocircuits C^* and $\{b_0, c_0, a_1, b_1\}$, we see that $\lambda(T_0 \cup T_1 \cup \{\alpha, \beta, d_0\}) \leq 2$; a contradiction. We deduce that (iv) holds.

We may now assume that $c_0 \notin C^*$. Then $c_1 \in C^*$, so $C^* = \{\beta, \gamma, \delta, c_1\}$. By orthogonality, a_1 or b_1 is in C^* . Suppose $a_1 \in C^*$. Then, as a_1 is not in a triangle of $M \setminus c_0, c_1$, we have $a_1 = \delta$, so $C^* = \{\beta, \gamma, a_1, c_1\}$. Orthogonality with $\{c_0, d_0, a_1\}$ implies that d_0 is in $\{\beta, \gamma\}$, so $C^* = \{d_0, d_1, a_1, c_1\}$. Hence $\{d_0, d_1\}$ is contained in a triangle and (ii) holds. We may now assume that $a_1 \notin C^*$. Thus $b_1 \in C^*$. If $b_1 \in \{\beta, \gamma\}$, then it follows that $\{b_0, b_1\}$ is in a triangle so (iii) holds. Hence we may assume that $b_1 = \delta$. Then $C^* = \{\beta, \gamma, b_1, c_1\}$. Thus b_1 is the coguts element of the 4-fan $(\alpha, \beta, \gamma, \delta)$, so $M \setminus c_0, c_1$ is (4, 4, S)-connected and (v) holds. \square

Lemma 6.2. *Let M be an internally 4-connected binary matroid. Assume that M contains the configuration shown in Figure 13 and that $|E(M)| \geq 13$. Then the elements in the figure are distinct except that p may be d_1 . Moreover, one of the following holds.*

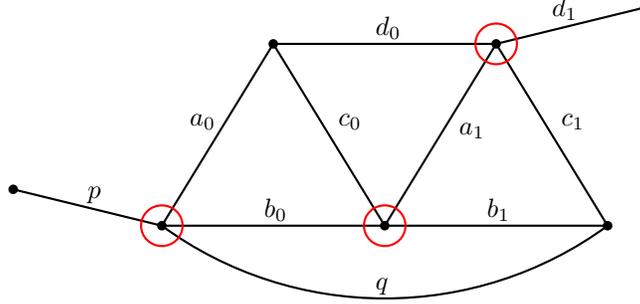


FIGURE 13

- (i) $M \setminus c_0, c_1, q$ is internally 4-connected; or
- (ii) $\{d_0, d_1\}$ is in a triangle of M ; or
- (iii) M has a triangle $\{s_1, s_2, s_3\}$ and a cocircuit $\{q, c_1, b_1, s_2, s_3\}$ where $\{s_1, s_2, s_3\}$ avoids $\{b_0, c_0, q, a_1, b_1, c_1\}$; or
- (iv) M has a triangle containing $\{a_0, p\}$ and some element that is not shown in the configuration; or
- (v) M has a 4-cocircuit that contains $\{q, b_1, c_1\}$.

Proof. First note that Lemma 6.1 implies that the elements in Figure 12 are distinct. Orthogonality between $\{b_0, b_1, q\}$ and $\{d_0, a_1, c_1, d_1\}$ implies that $q \notin \{d_0, a_1, c_1, d_1\}$. Moreover, $q \neq c_0$. If $q = a_0$, then $c_0 = b_1$; a contradiction. Thus $q \notin T_0 \cup T_1 \cup \{d_0, d_1\}$, and so orthogonality implies that $p \notin T_0 \cup T_1 \cup \{d_0, q\}$.

Next we show that

6.2.1. $M \setminus c_0, c_1, q$ is 3-connected or (v) holds.

By Lemma 5.3, $M \setminus c_0, c_1$ is 3-connected. Since $\text{si}(M \setminus c_0, c_1 / q)$ has a 2-element cocircuit, it is not 3-connected. Thus, by Bixby's Lemma [1], $\text{co}(M \setminus c_0, c_1, q)$ is 3-connected. Suppose that $M \setminus c_0, c_1, q$ is not 3-connected. Then $M \setminus c_0, c_1$ has a triad C^* containing q . By orthogonality, C^* meets $\{b_0, b_1\}$ and $\{b_1, a_1, d_0, a_0\}$. Thus $b_1 \in C^*$ otherwise $\lambda(T_0 \cup T_1 \cup \{q, d_0\}) \leq 2$; a contradiction.

Now M has a cocircuit D^* such that $C^* \subsetneq D^* \subseteq C^* \cup \{c_0, c_1\}$. Assume $c_0 \in D^*$. Then, by orthogonality, C^* is $\{q, b_1, a_0\}$ or $\{q, b_1, b_0\}$. Thus $\lambda(T_0 \cup T_1 \cup \{q, d_0\}) \leq 2$; a contradiction. We conclude that $c_0 \notin D^*$, so $D^* = C^* \cup c_1$ and (v) holds. Thus 6.2.1 holds.

Now we show:

6.2.2. If $\{a_0, p\}$ is contained in a triangle of M , then (iv) holds.

Assume that $\{a_0, p\}$ is contained in a triangle whose third element is already in the configuration. By orthogonality, this third element is not d_1 , so $p \in \text{cl}(T_0 \cup T_1 \cup \{d_0, q\})$. Thus $\lambda(T_0 \cup T_1 \cup \{d_0, q, p\}) \leq 2$. This is a contradiction as $|E(M)| \geq 13$. Hence 6.2.2 holds.

Next, suppose that (U, V) is a non-sequential 3-separation of $M \setminus c_0, c_1, q$. Then we may assume that $\{b_0, b_1, a_1\} \subseteq U$. Thus $(U \cup q, V)$ is a non-sequential 3-separation of $M \setminus c_0, c_1$; a contradiction to Lemma 6.1. We deduce that $M \setminus c_0, c_1, q$ is sequentially 4-connected.

Now let (s_1, s_2, s_3, s_4) be a 4-fan of $M \setminus c_0, c_1, q$. Then M has a cocircuit C^* such that $\{s_2, s_3, s_4\} \subsetneq C^* \subseteq \{s_2, s_3, s_4, c_0, c_1, q\}$, and M has $\{s_1, s_2, s_3\}$ as a triangle.

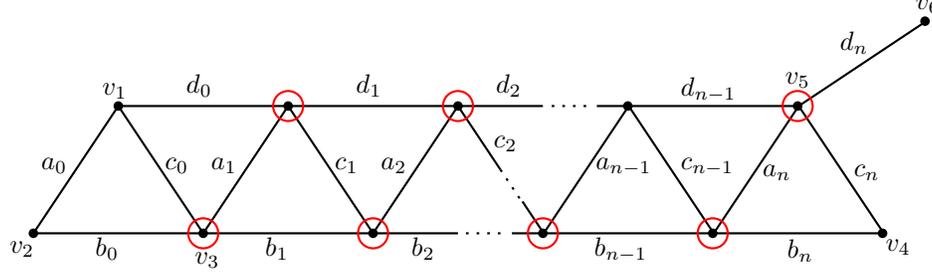


FIGURE 14. $\{d_{n-2}, a_{n-1}, c_{n-1}, d_{n-1}\}$ or $\{d_{n-2}, a_{n-1}, c_{n-1}, a_n, c_n\}$ is a cocircuit.

By Lemma 6.1, a_1 is not in a triangle of $M \setminus \{c_0, c_1, q\}$. It follows that

6.2.3. $\{a_1, b_0, b_1\} \cap \{s_1, s_2, s_3\} = \emptyset$.

Suppose that $a_1 \in C^*$. Then $a_1 = s_4$. Thus $q \notin C^*$ otherwise, by orthogonality, $\{s_2, s_3\}$ meets $\{b_0, b_1\}$. Moreover, by orthogonality again, $c_1 \in C^*$ and exactly one of d_0 and c_0 is in C^* . If $d_0 \in C^*$, then $\{d_0, d_1\} \subseteq \{s_1, s_2, s_3\}$, so (ii) holds. Hence we may assume that $c_0 \in C^*$. Thus $C^* = \{s_2, s_3, a_1, c_0, c_1\}$. As $b_0 \notin \{s_1, s_2, s_3\}$, we may assume, by orthogonality and symmetry, that $a_0 = s_3$. By orthogonality, $p \in \{s_1, s_2\}$, so $\{a_0, p\}$ is contained in a triangle. Thus, by 6.2.2, (iv) holds.

We may now assume that $a_1 \notin C^*$. Then C^* contains $\{b_1, c_1\}$ or avoids $\{b_1, c_1\}$. Consider the first case. Then 6.2.3 implies that $b_1 = s_4$ and $b_0 \notin C^*$. Then $q \in C^*$. Moreover, $\{a_0, c_0\} \subseteq C^*$ or $\{a_0, c_0\}$ avoids C^* . Suppose $\{a_0, c_0\} \subseteq C^*$. Then $C^* = \{s_2, s_3, b_1, c_1, c_0, q\}$, where $a_0 \in \{s_2, s_3\}$. Orthogonality implies that $d_0 \in \{s_2, s_3\}$, so $\lambda(T_0 \cup T_1 \cup \{d_0, q\}) \leq 2$; a contradiction. Now suppose C^* avoids $\{a_0, c_0\}$. Then $C^* = \{s_2, s_3, b_1, q, c_1\}$, so (iii) holds.

Finally, assume that C^* avoids $\{b_1, c_1\}$. Suppose first that $q \notin C^*$. Then, as C^* meets $\{c_0, c_1, q\}$, we deduce that $C^* = \{s_2, s_3, s_4, c_0\}$. By 6.2.3 and orthogonality with the circuit $\{b_0, b_1, q\}$, we deduce that $b_0 \notin C^*$. Hence, by orthogonality with the triangle $\{a_0, b_0, c_0\}$, we see that $a_0 \in C^*$. Hence $d_0 \in C^*$. Then $d_0 = s_4$ otherwise $\{d_0, d_1\}$ is contained in a triangle and (ii) holds. Thus $a_0 \in \{s_2, s_3\}$, so $p \in \{s_1, s_2, s_3\}$ and 6.2.2 implies that (iv) holds.

It remains to consider when C^* avoids $\{b_1, c_1\}$ but contains q . Then $b_0 \in C^*$ so 6.2.3 implies that $b_0 = s_4$. If $c_0 \in C^*$, then $d_0 \in \{s_1, s_2, s_3\}$, so $\{d_0, d_1\} \subseteq \{s_1, s_2, s_3\}$, and the lemma holds. Thus we may assume that $c_0 \notin C^*$. Then $C^* = \{s_2, s_3, b_0, q\}$, so $a_0 \in \{s_2, s_3\}$. As M has $\{a_0, p, b_0, q\}$ as a cocircuit, we must have that $C^* = \{a_0, p, b_0, q\}$. Thus M has a triangle containing $\{a_0, p\}$ and 6.2.2 implies that (iv) holds. \square

The next two lemmas concern the configuration shown in Figure 14.

Lemma 6.3. *Let M be an internally 4-connected binary matroid that has at least thirteen elements and contains the configuration shown in Figure 14 where $n \geq 2$, all of the elements shown are distinct, and, in addition to the cocircuits shown, exactly one of $\{d_{n-2}, a_{n-1}, c_{n-1}, d_{n-1}\}$ or $\{d_{n-2}, a_{n-1}, c_{n-1}, a_n, c_n\}$ is a cocircuit. Then a triangle T of $M \setminus \{c_0, c_1, \dots, c_n\}$ that meets $\{a_0, b_0, d_0, a_1, b_1, d_1, \dots, a_n, b_n, d_n\}$ does so in $\{a_0\}$, $\{d_{n-1}, d_n\}$, or $\{a_0, d_{n-1}, d_n\}$.*

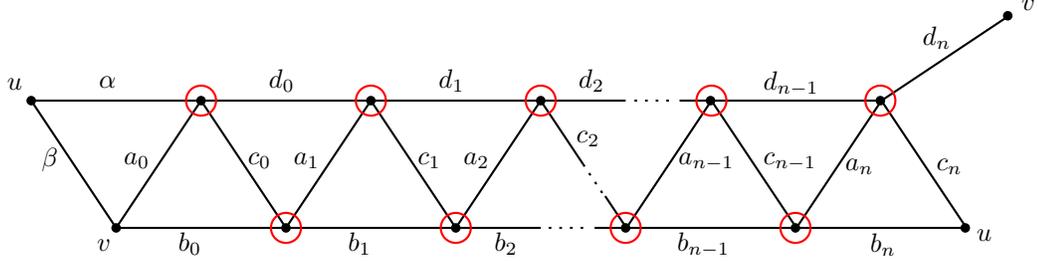


FIGURE 15. A quartic Möbius ladder.

Proof. Assume first that T meets $\{a_i, b_i\}$ for some i with $1 \leq i \leq n-1$. Then, as $\{b_{i-1}, a_i, b_i\}$ is a cocircuit of $M \setminus \{c_0, c_1, \dots, c_n\}$ and T does not contain $\{a_i, b_i\}$, we see, by orthogonality, that T contains $\{b_{i-1}, b_i\}$ or $\{b_{i-1}, a_i\}$.

Suppose $\{d_{i-1}, a_i, c_i, d_i\}$ is a cocircuit. Orthogonality implies that T is $\{b_{i-1}, b_i, a_{i+1}\}$, $\{b_{i-1}, b_i, b_{i+1}\}$, or $\{b_{i-1}, a_i, d_i\}$. If $T = \{b_{i-1}, b_i, b_{i+1}\}$, then $\lambda(\{b_{i-1}, c_{i-1}, d_{i-1}, a_i, b_i, c_i, d_i, a_{i+1}, b_{i+1}\}) \leq 2$, so $|E(M)| \leq 12$; a contradiction. If $T = \{b_{i-1}, a_i, d_i\}$, then $\lambda(\{b_{i-1}, c_{i-1}, d_{i-1}, a_i, b_i, c_i, d_i\}) \leq 2$; a contradiction. Thus $T = \{b_{i-1}, b_i, a_{i+1}\}$. But $\{d_i, a_{i+1}, c_{i+1}, d_{i+1}\}$ or $\{d_i, a_{i+1}, c_{i+1}, a_{i+2}, c_{i+2}\}$ is a cocircuit, and so orthogonality with T gives a contradiction. We conclude that $\{d_{i-1}, a_i, c_i, d_i\}$ is not a cocircuit. Hence $i = n-1$ and $\{d_{n-2}, a_{n-1}, c_{n-1}, a_n, c_n\}$ is a cocircuit. As T contains $\{b_{n-2}, b_{n-1}\}$ or $\{b_{n-2}, a_{n-1}\}$, orthogonality now implies that T is $\{b_{n-2}, b_{n-1}, b_n\}$. Then $\lambda(\{b_{n-2}, c_{n-2}, d_{n-2}\} \cup T_{n-1} \cup T_n) \leq 2$, so $|E(M)| \leq 12$; a contradiction. We conclude that T avoids $\{a_i, b_i\}$ for all i with $1 \leq i \leq n-1$. It follows that $b_0 \notin T$ otherwise $\{a_1, b_1\}$ meets T . Moreover, T avoids $\{a_n, b_n\}$ otherwise T contains b_{n-1} ; a contradiction. We conclude that

6.3.1. $T \cap \{a_0, b_0, a_1, b_1, \dots, a_n, b_n\} \subseteq \{a_0\}$.

Next we note the following.

6.3.2. If $d_0 \in T$, then $d_1 \in T$.

This follows by orthogonality and 6.3.1 since T must meet $\{a_1, c_1, d_1\}$ or $\{a_1, c_1, a_2, c_2\}$.

6.3.3. If $d_i \in T$ for some i with $1 \leq i \leq n$, then $i \in \{n-1, n\}$ and $\{d_{n-1}, d_n\} \subseteq T$.

To see this, observe that, by orthogonality, T meets $\{d_{i-1}, a_i\}$ if $i \neq n-1$, and T meets $\{a_n, d_n\}$ if $i = n-1$. Thus $\{d_{i-1}, d_i\} \subseteq T$ if $i \neq n-1$, and $\{d_{n-1}, d_n\} \subseteq T$ if $i = n-1$. Assume $i \leq n-2$. Then $\{d_{i-1}, d_i\} \subseteq T$ and, by orthogonality, either $d_{i+1} \in T$ and $\{d_i, a_{i+1}, c_{i+1}, d_{i+1}\}$ is a cocircuit, or $i = n-2$ and $\{d_{n-2}, a_{n-1}, c_{n-1}, a_n, c_n\}$ is a cocircuit. In the latter case, $\{a_{n-1}, c_{n-1}, a_n, c_n\}$ meets T ; a contradiction to 6.3.1. Hence $d_{i+1} \in T$, so $T = \{d_{i-1}, d_i, d_{i+1}\}$. But orthogonality implies that either T meets $\{a_{i+2}, c_{i+2}, d_{i+2}\}$; or $i+1 = n-2$ and T meets $\{a_{n-1}, c_{n-1}, a_n, c_n\}$. Both possibilities yield a contradiction since all the elements in the figure are distinct. We conclude that $i > n-2$, so 6.3.3 holds.

By combining 6.3.2 and 6.3.3, we deduce that the lemma holds unless $n = 2$ and $T = \{d_0, d_1, d_2\}$. In the exceptional case, $\lambda(T_0 \cup T_1 \cup \{d_0, d_1, d_2\}) \leq 2$; a contradiction. \square

Lemma 6.4. *Let M be an internally 4-connected binary matroid such that $|E(M)| \geq 13$. Suppose that M contains the structure in Figure 14, where $T_0, D_0, T_1, D_1, \dots, T_n$ is a string of bowties and, when $n \geq 2$, either $\{d_{n-2}, a_{n-1}, c_{n-1}, d_{n-1}\}$ or $\{d_{n-2}, a_{n-1}, c_{n-1}, a_n, c_n\}$ is a cocircuit. Then*

- (i) *all of the elements shown in Figure 14 are distinct; or*
 - (ii) *$(a_0, b_0) = (c_n, d_n)$ but all the other elements in the figure are distinct, M has both $\{d_{n-2}, a_{n-1}, c_{n-1}, d_{n-1}\}$ and $\{b_n, a_0, c_0, d_0\}$ as cocircuits and either*
 - (a) *all of the elements of M are shown in Figure 14, and M is the cycle matroid of the quartic Möbius ladder that is obtained from the structure in Figure 14 by identifying the vertices v_1, v_2 , and v_3 with the vertices v_4, v_5 , and v_6 , respectively; or*
 - (b) *M has exactly one element, γ , that is not shown in Figure 14, and M is the matroid for which $M \setminus \gamma$ is a wheel whose spokes, in cyclic order, are $c_0, a_1, c_1, a_2, c_2, \dots, a_n, c_n$ such that the fundamental circuit of γ with respect to the basis, B , consisting of this set of spokes is $B \cup \gamma$; and the rim of $M \setminus \gamma$, in cyclic order, is $d_0, b_1, d_1, b_2, d_2, \dots, b_n, d_n$;*
- or*
- (iii) *M has $\{d_{n-2}, a_{n-1}, c_{n-1}, a_n, c_n\}$ as a cocircuit, $(a_0, b_0) = (d_{n-1}, d_n)$ but all the other elements in Figure 14 are distinct, and M has at most one element that is absent from that figure.*

Proof. First we note that, by Lemma 6.1, if $n = 1$, then the elements in Figure 14 are distinct. Thus

6.4.1. *$n \geq 2$ or (i) holds.*

As $T_0, D_0, T_1, D_1, \dots, T_n$ is a bowtie string, we know that the elements in $T_0 \cup T_1 \cup \dots \cup T_n$ are all distinct except that a_0 may be c_n . We will begin by treating the case when $a_0 \neq c_n$, first showing the following.

6.4.2. *When $a_0 \neq c_n$, if some d_i is in $T_0 \cup T_1 \cup \dots \cup T_n$, then $\{d_i, a_{i+1}, c_{i+1}, a_{i+2}, c_{i+2}\}$ is not a cocircuit.*

Suppose $\{d_i, a_{i+1}, c_{i+1}, a_{i+2}, c_{i+2}\}$ is a cocircuit. Then, by orthogonality, $d_i \in T_{i+1} \cup T_{i+2}$. Hence, as M is binary, $d_i \in \{a_{i+1}, c_{i+1}, a_{i+2}, c_{i+2}\}$; a contradiction to Lemma 4.2. Thus 6.4.2 holds.

We now show:

6.4.3. *When $a_0 \neq c_n$, either $\{d_0, d_1, \dots, d_n\}$ avoids $T_0 \cup T_1 \cup \dots \cup T_n$; or M has at most one element that is not shown in Figure 14, M has $\{d_{n-2}, a_{n-1}, c_{n-1}, a_n, c_n\}$ as a cocircuit, $(a_0, b_0) = (d_{n-1}, d_n)$, and $\{d_0, d_1, \dots, d_{n-2}\}$ avoids $T_0 \cup T_1 \cup \dots \cup T_n$.*

This is certainly true if $n = 1$. Thus we may assume that $n \geq 2$. Suppose that some d_i is in $T_0 \cup T_1 \cup \dots \cup T_n$, choosing the least such i . Then $d_i \in T_j$, say. Suppose $\{d_{i-1}, a_i, c_i, d_i\}$ is a cocircuit. By orthogonality, T_j meets $\{d_{i-1}, a_i, c_i\}$. The minimality of i implies that $d_{i-1} \notin T_j$. Thus $j = i$. As M does not have a 4-fan, $d_i \notin \{a_i, c_i\}$ and, as M is binary, $d_i \neq b_i$. This contradiction implies that $\{d_{i-1}, a_i, c_i, d_i\}$ is not a cocircuit. Therefore either $i = 0$; or $i = n - 1$ and $\{d_{n-2}, a_{n-1}, c_{n-1}, d_{n-1}\}$ is not a cocircuit. It follows using 6.4.2 that, in both cases, $\{d_i, a_{i+1}, c_{i+1}, d_{i+1}\}$ is a cocircuit. Now orthogonality implies that either $d_{i+1} \in T_j$, or $j = i + 1$. The latter implies that T_j is contained in a cocircuit. Hence the former holds. Then 6.4.2 implies that $\{d_{i+1}, a_{i+2}, c_{i+2}, a_{i+3}, c_{i+3}\}$ is not a cocircuit. Thus

if $i = 0$, then $\{d_{i+1}, a_{i+2}, c_{i+2}, d_{i+2}\}$ is a cocircuit and orthogonality implies that $T_j = \{d_0, d_1, d_2\}$. Hence $\lambda(T_1 \cup T_2 \cup \{d_0, d_1, d_2\}) \leq 2$; a contradiction. We deduce that $i \neq 0$, so $i = n - 1$ and $\{d_{n-2}, a_{n-1}, c_{n-1}, d_{n-1}\}$ is not a cocircuit. Thus $\{d_{n-2}, a_{n-1}, c_{n-1}, a_n, c_n\}$ is a cocircuit. Moreover, since $\{d_i, d_{i+1}\} \subseteq T_j$, we see that $\{d_{n-1}, d_n\} \subseteq T_j$. Thus $\{d_0, d_1, \dots, d_n\}$ avoids $T_0 \cup T_1 \cup \dots \cup T_{j-1}$. Now $j \leq n - 2$ otherwise $\lambda(T_{n-1} \cup T_n \cup \{d_{n-1}, d_n\}) \leq 2$; a contradiction.

Let $Z = T_j \cup T_{j+1} \cup \dots \cup T_n \cup \{d_j, d_{j+1}, \dots, d_n\}$. Since $\{d_{n-1}, d_n\} \subseteq T_j$, we deduce that Z is spanned by $\{d_n\} \cup \{a_k, b_k : j + 1 \leq k \leq n\}$. Thus $r(Z) \leq 2(n - j) + 1$. Since Z contains at least $2(n - j)$ cocircuits of M none a symmetric difference of the others, we deduce that $\lambda(Z) \leq 1$. Thus Z contains at least $|E(M)| - 1$ elements, so $j = 0$. Furthermore, we know that T_0 is the only triangle in the bowtie string to meet $\{d_0, d_1, \dots, d_n\}$ and, by the minimality of i , it follows that $\{d_0, d_1, \dots, d_{n-2}\}$ avoids $T_0 \cup T_1 \cup \dots \cup T_n$. By orthogonality between the triangle $\{c_{n-1}, d_{n-1}, a_n\}$ and the cocircuit D_0 , we deduce that $d_{n-1} = a_0$. Since T_0 also contains d_n , we see that $d_n \in \{b_0, c_0\}$. If $d_n = c_0$, then orthogonality between the triangle $\{c_0, d_0, a_1\}$ and the cocircuit $\{d_{n-1}, a_n, c_n, d_n\}$ implies that $d_0 \in \{a_n, c_n\}$; a contradiction. Thus $d_n = b_0$ and 6.4.3 holds.

Next, we complete the proof of the lemma when $a_0 \neq c_n$ by proving the following.

6.4.4. *When $a_0 \neq c_n$, if $\{d_0, d_1, \dots, d_n\}$ avoids $T_0 \cup T_1 \cup \dots \cup T_n$, then the elements in $\{d_0, d_1, \dots, d_n\}$ are distinct.*

Assume that this is not so, and choose j to be the maximum member of $\{0, 1, \dots, n\}$ such that $d_j \in \{d_0, d_1, \dots, d_n\} - \{d_j\}$. Then $d_j = d_i$ for some i with $0 \leq i < j \leq n$. By orthogonality with the triangle $\{c_i, d_i, a_{i+1}\}$ and the maximality of j , we see that neither $\{d_j, a_{j+1}, c_{j+1}, a_{j+2}, c_{j+2}\}$ nor $\{d_j, a_{j+1}, c_{j+1}, d_{j+1}\}$ is a cocircuit. Hence $j = n$, and $\{d_{j-1}, a_j, c_j, d_j\}$ is a cocircuit. Then $i \neq j - 1$. But orthogonality implies that $\{c_i, a_{i+1}\}$ meets $\{d_{j-1}, a_j, c_j\}$; a contradiction. Hence 6.4.4 holds.

We may now assume that $a_0 = c_n$. Moreover, by 6.4.1, we may assume that $n \geq 2$. Next we show the following.

6.4.5. *The elements in $T_1 \cup T_2 \cup \dots \cup T_n \cup \{d_1, d_2, \dots, d_n\}$ are distinct.*

Clearly $a_1 \neq c_n$. Thus, by applying Lemma 6.1, 6.4.3, and 6.4.4 to the structure we get from Figure 14 by deleting $T_0 \cup d_0$, we deduce that the elements of $T_1 \cup T_2 \cup \dots \cup T_n \cup \{d_1, d_2, \dots, d_n\}$ are distinct unless $a_1 = d_{n-1}$. In the exceptional case, orthogonality between $\{c_{n-1}, a_1, a_n\}$ and D_0 implies that $\{c_{n-1}, a_n\}$ meets $\{b_0, c_0, b_1\}$; a contradiction. Thus 6.4.5 holds.

Since $a_0 = c_n$, orthogonality between T_0 and $\{d_{n-1}, a_n, c_n, d_n\}$ implies that $\{b_0, c_0\}$ meets $\{d_{n-1}, d_n\}$. If $d_{n-1} \in \{b_0, c_0\}$, then orthogonality between $\{c_{n-1}, d_{n-1}, a_n\}$ and D_0 implies that $\{c_{n-1}, a_n\}$ meets D_0 ; a contradiction. Therefore $d_n \in \{b_0, c_0\}$.

6.4.6. $d_n = b_0$.

Assume that this fails. Then $d_n = c_0$. Now orthogonality between $\{c_0, d_0, a_1\}$ and $\{d_{n-1}, a_n, c_n, d_n\}$ implies, using 6.4.5, that $d_0 \in \{d_{n-1}, a_n\}$. If $d_0 = a_n$, then orthogonality between $\{c_0, d_0, a_1\}$ and D_{n-1} gives a contradiction. Thus $d_0 = d_{n-1}$ so $n > 2$ otherwise $\lambda(T_1 \cup T_2 \cup \{d_1, d_2\}) \leq 2$; a contradiction. Hence $\{d_0, a_1, c_1, d_1\}$ is a cocircuit. By 6.4.5, this cocircuit meets the triangle $\{d_{n-1}, c_{n-1}, a_n\}$ in a single element; a contradiction. We conclude that 6.4.6 holds.

If $\{d_{n-2}, a_{n-1}, c_{n-1}, a_n, a_0\}$ is a cocircuit, then orthogonality with T_0 implies that $c_0 = d_{n-2}$, and so $\lambda(T_{n-1} \cup T_n \cup \{d_{n-1}, b_0, c_0\}) \leq 2$; a contradiction. Thus $\{d_{n-2}, a_{n-1}, c_{n-1}, d_{n-1}\}$ is a cocircuit. By orthogonality between D_0 and the triangles in Figure 14, we can easily check that $c_0 \notin T_1 \cup T_2 \cup \dots \cup T_n \cup \{d_1, d_2, \dots, d_n\}$. Certainly $c_0 \neq d_0$. Now, by orthogonality between $\{c_0, d_0, a_1\}$ and each of the indicated cocircuits in Figure 14 as well as $\{d_{n-2}, a_{n-1}, c_{n-1}, d_{n-1}\}$, we deduce that $d_0 \notin T_1 \cup T_2 \cup \dots \cup T_n \cup \{d_1, d_2, \dots, d_n\}$. Moreover, by taking the symmetric difference of this same set of cocircuits, we get the set $\{a_0, c_0, d_0, b_n\}$, so this set must also be a cocircuit of M .

Letting $Z = T_1 \cup T_2 \cup \dots \cup T_n \cup \{d_1, d_2, \dots, d_n\} \cup \{c_0, d_0\}$, we see that $|Z| = 4n + 2$. Moreover, we can easily check that Z is spanned by $\{a_1, b_1, a_2, b_2, \dots, a_n, b_n, d_n\}$ in M and by $\{d_0, d_1, \dots, d_n, b_1, b_2, \dots, b_n, c_n\}$ in M^* . Thus $r(Z) \leq 2n + 1$ and $r^*(Z) \leq 2n + 2$.

Suppose that $r(Z) \leq 2n$. Then $\lambda(Z) = 0$ and $Z = E(M)$. Thus the elements in Figure 14 are all of the elements in M , where $(a_0, b_0) = (c_n, d_n)$. Moreover, $r(M) = 2n$ and $r^*(M) = 2n + 2$. Since $\{a_1, b_1, a_2, b_2, \dots, a_n, b_n, d_n\}$ has $2n + 1$ elements, this set must contain a circuit C . By orthogonality with the known 4-cocircuits, including $\{a_0, c_0, d_0, b_n\}$, we deduce that C avoids $\{a_1, a_2, \dots, a_{n-1}, b_n\}$. Hence $C \subseteq \{b_1, b_2, \dots, b_{n-1}, a_n, d_n\}$. Again orthogonality with the known 4-cocircuits implies that $C = \{b_1, b_2, \dots, b_{n-1}, a_n, d_n\}$. Hence $\{a_1, b_1, a_2, b_2, \dots, a_n, b_n\}$ is a basis of M . Since M is binary, we deduce that M must be the cycle matroid of the quartic Möbius ladder that is obtained from Figure 14 by identifying the vertices v_1, v_2 , and v_3 with v_4, v_5 , and v_6 , respectively; that is, (ii)(a) holds.

We may now assume that $r(Z) = 2n + 1$. Then $\lambda(Z) \leq 1$. Suppose $\lambda(Z) = 0$. Then $E(M) = Z$ and $r^*(M) = 2n + 1$. Hence $\{d_0, d_1, \dots, d_n, b_1, b_2, \dots, b_n, c_n\}$ contains a cocircuit C^* . By orthogonality with the triangles in M , we deduce that C^* avoids $\{d_0, b_1, d_1, b_2, \dots, b_{n-1}, d_{n-1}\}$. Hence $C^* \subseteq \{d_n, b_n, c_n\}$. This contradicts the fact that M is internally 4-connected. It follows that $\lambda(Z) = 1$, so $E(M) - Z$ has a unique element, say γ . Now $\text{cl}(T_1 \cup T_2 \cup \dots \cup T_n)$ has rank $2n$ and avoids $\{b_0, c_0, d_0, \gamma\}$. Hence the last set is a cocircuit of M . In addition, by symmetry, $\{b_i, c_i, d_i, \gamma\}$ and $\{d_{i-1}, a_i, b_i, \gamma\}$ are cocircuits of M for all i in $\{1, 2, \dots, n\}$. It follows that every element of $M \setminus \gamma$ is in both a triangle and a triad of that matroid. But γ itself cannot be in a triad of M since every element of $E(M) - \gamma$ is in a triangle of $M \setminus \gamma$. Thus $M \setminus \gamma$ is 3-connected and so is a wheel. The rank of this wheel is $\frac{1}{2}(|E(M)| - 1) = 2n + 1$. From the set of triangles of $M \setminus \gamma$, we see that the spokes of this wheel, in cyclic order are $c_0, a_1, c_1, a_2, c_2, \dots, a_n, c_n$. These spokes form a basis, B , of M . From the 4-cocircuits of M containing γ , we see that the fundamental circuit of γ with respect to B is $B \cup \gamma$. Finally, the cyclic order on the spokes determines that on the rim, namely, $d_0, b_1, d_1, b_2, d_2, \dots, b_n, d_n$. Hence (ii)(b) holds. \square

Lemma 6.5. *Let M be an internally 4-connected binary matroid that has at least thirteen elements. Assume that M contains the configuration shown in Figure 14 where $n \geq 2$, all of the elements shown are distinct, and, in addition to the cocircuits shown, exactly one of $\{d_{n-2}, d_{n-1}, a_{n-1}, c_{n-1}\}$ or $\{d_{n-2}, a_{n-1}, c_{n-1}, a_n, c_n\}$ is a cocircuit. Then either*

- (i) $M \setminus c_0, c_1, \dots, c_n$ is internally 4-connected; or
- (ii) M has a triangle containing $\{d_{n-1}, d_n\}$, the matroid $M \setminus c_n$ is not $(4, 4, S)$ -connected, and M has $\{d_{n-2}, a_{n-1}, c_{n-1}, a_n, c_n\}$ as a cocircuit; or

- (iii) $M \setminus c_0, c_1, \dots, c_n$ is $(4, 4, S)$ -connected but not internally 4-connected, and one side of every $(4, 3)$ -violator of $M \setminus c_0, c_1, \dots, c_n$ is a 4-fan $F = (u_1, u_2, u_3, u_4)$ where either $u_4 = d_0$ and $a_0 \in \{u_2, u_3\}$, and F is a 4-fan of $M \setminus c_0$; or $u_4 = b_n$ and F is a 4-fan of $M \setminus c_n$; or
- (iv) M is the cycle matroid of a quartic Möbius ladder labelled as in Figure 15 where the two vertices labelled v are identified and the two vertices labelled u are identified.

Proof. It follows easily from Lemma 6.3 that if $M \setminus c_0, c_1, \dots, c_n$ has a triangle meeting $\{d_{n-1}, d_n\}$, then orthogonality implies that $\{d_{n-2}, a_{n-1}, c_{n-1}, d_{n-1}\}$ is not a cocircuit, so $\{d_{n-2}, a_{n-1}, c_{n-1}, a_n, c_n\}$ is a cocircuit and (ii) holds since $M \setminus c_n$ has a 5-fan. Thus we shall assume that M has no triangle meeting $\{d_{n-1}, d_n\}$. Hence a triangle of $M \setminus c_0, c_1, \dots, c_n$ that meets $\{a_0, b_0, d_0, a_1, b_1, d_1, \dots, a_n, b_n, d_n\}$ does so in $\{a_0\}$.

Let $S = \{c_0, c_1, \dots, c_n\}$. Lemma 5.3 implies that either $M \setminus S$ is 3-connected, or $M \setminus S$ has a_i or b_i in a 1- or 2-element cocircuit for some i in $\{2, 3, \dots, n\}$. Suppose the latter. Then $\{x, y\}$ is a cocircuit of $M \setminus S$ for $x \in \{a_i, b_i\}$, so M has a cocircuit C^* such that $\{x, y\} \subseteq C^* \subseteq \{x, y\} \cup S$. By orthogonality, if $c_j \in C^*$, then C^* meets $\{a_j, b_j\}$. Hence C^* contains at most two elements in S . Furthermore, if C^* contains only one element in S , then C^* is a triad that meets a triangle of M ; a contradiction. Thus $C^* = \{x, y, c_j, c_k\}$ for some $k \neq j$. Then orthogonality between C^* and the triangles T_j and T_k implies, without loss of generality, that $y \in \{a_j, b_j\}$, that $i = k$, and that $j < k$. Orthogonality with the triangle $\{c_j, d_j, a_{j+1}\}$ implies that $a_{j+1} = a_k = x$. If $\{c_k, d_k, a_{k+1}\}$ is a triangle, then it contains exactly one element of C^* ; a contradiction. Thus $j + 1 = n$. Orthogonality with $\{c_{j-1}, d_{j-1}, a_j\}$ implies that $a_j \notin C^*$. Hence $b_j \in C^*$, and $C^* \triangle \{b_j, c_j, a_k, b_k\}$, which equals $\{b_k, c_k\}$, is a cocircuit in M ; a contradiction. We conclude that $M \setminus S$ is 3-connected.

Next we show that

6.5.1. $M \setminus S$ is sequentially 4-connected.

Let (U, V) be a non-sequential 3-separation of $M \setminus S$. Then we may assume that $\{b_0, a_1, b_1\} \subseteq U$. If $\{a_0, d_0, d_1, a_2\}$ meets U , then $\text{fcl}_{M \setminus S}(U) \supseteq \{a_0, d_0, d_1, a_2\}$ and it is straightforward to check that $\text{fcl}_{M \setminus S}(U)$ contains $\{a_0, b_0, d_0, a_1, b_1, d_1, \dots, a_n, b_n, d_n\}$. Hence $(\text{fcl}_{M \setminus S}(U) \cup S, V - \text{fcl}(U))$ is a non-sequential 3-separation of M ; a contradiction. Thus we may assume that $\{a_0, d_0, d_1, a_2\} \subseteq V$. Then $\text{fcl}_{M \setminus S}(V)$ contains $\{a_0, b_0, d_0, a_1, b_1, d_1, \dots, a_n, b_n, d_n\}$, so $(U - \text{fcl}(V), \text{fcl}(V) \cup S)$ is a non-sequential 3-separation of M ; a contradiction. Hence 6.5.1 holds.

Next we show the following.

6.5.2. Each 4-fan of $M \setminus S$ is either a fan of $M \setminus c_0$ and has d_0 as its coguts element and a_0 as an interior element, or is a fan of $M \setminus c_n$ and has b_n as its coguts element.

Let (u_1, u_2, u_3, u_4) be a 4-fan in $M \setminus S$. Then M has a cocircuit C^* such that $\{u_2, u_3, u_4\} \subsetneq C^* \subseteq \{u_2, u_3, u_4\} \cup S$. We now show that

6.5.3. $|S \cap C^*| = 1$.

Assume that $|S \cap C^*| \geq 2$. Then $c_i \in C^*$ for some $i > 0$. Hence C^* meets $\{a_i, b_i\}$, and it follows, by Lemma 6.3, that $u_4 \in \{a_i, b_i\}$. Thus c_i is the unique element of $S - c_0$ that is in C^* , so $c_0 \in C^*$. Therefore, by orthogonality, $\{u_2, u_3\}$

meets both $\{a_0, b_0\}$ and $\{d_0, a_1\}$; a contradiction to Lemma 6.3. We conclude that 6.5.3 holds.

Using 6.5.3, suppose first that $c_0 \in C^*$. Then $C^* = \{u_2, u_3, u_4, c_0\}$ and $u_4 \in \{d_0, a_1\}$. Moreover, $a_0 \in \{u_2, u_3\}$. If $u_4 = a_1$, then $b_1 \in \{u_2, u_3\}$; a contradiction. Thus $u_4 = d_0$. We deduce that the 4-fan (u_1, u_2, u_3, u_4) has d_0 as its coguts element and has a_0 as an interior element.

Next suppose that $c_n \in C^*$. Then $u_4 \in \{a_n, b_n\}$. If $u_4 = a_n$, then $\{u_2, u_3\}$ meets $\{d_{n-1}, c_{n-1}\}$. Thus $d_{n-1} \in \{u_2, u_3\}$ so, by orthogonality, $\{d_{n-1}, d_n\} \subseteq \{u_1, u_2, u_3\}$; a contradiction. Thus $u_4 = b_n$.

Finally, suppose that $c_i \in C^*$ for some i with $0 < i < n$. Then $u_4 \in \{a_i, b_i\} \cap \{d_i, a_{i+1}\}$; a contradiction. Thus 6.5.2 holds.

We shall now assume that $M \setminus S$ is not $(4, 4, S)$ -connected, otherwise (i) or (iii) holds. Next we show the following.

6.5.4. *If (U, V) is a $(4, 4, S)$ -violator of $M \setminus S$, then U or V is a 5-cofan of the form $(d_0, a_0, \alpha, \beta, b_n)$. Moreover, $\{\alpha, \beta\} \cap \{a_i, b_i, c_i, d_i : 0 \leq i \leq n\} = \emptyset$.*

To see this, first suppose that $M \setminus S$ has a 5-fan $(w_1, w_2, w_3, w_4, w_5)$. Then, by 6.5.2, we may assume that $(w_2, w_4) = (b_n, d_0)$. Thus $M \setminus S$ has a triangle containing d_0 ; a contradiction to Lemma 6.3. Therefore M has no 5-fans.

Next let $(w_1, w_2, w_3, w_4, w_5)$ be a 5-cofan in $M \setminus S$. Then, by 6.5.2, we may assume that $(w_1, w_5) = (d_0, b_n)$ and $a_0 \in \{w_2, w_3\}$, and $\{d_0, w_2, w_3, c_0\}$ and $\{w_3, w_4, b_n, c_n\}$ are cocircuits of M . If $a_0 = w_3$, then M has $\{a_0, w_4, b_n, c_n\}$ as a cocircuit and has $\{a_0, b_0, c_0\}$ as a circuit. Using Lemma 6.3, we see that $w_4 \notin \{b_0, c_0\}$. Thus we have a contradiction to orthogonality. We deduce that $a_0 = w_2$.

Now, by [5, Lemma 2.2(iv)], if the first sentence of 6.5.4 is false, then there is an element w_6 outside of the 5-cofan $(w_1, w_2, w_3, w_4, w_5)$ that is in its coguts or its guts. It is clear that w_6 is not in the coguts since adjoining such an element to the 5-cofan gives a 6-element set that contains three elements that are ends of 5-cofans. Yet each such end must be in $\{d_0, b_n\}$. Now assume that w_6 is in the guts of $(w_1, w_2, w_3, w_4, w_5)$. Then $M \setminus S$ has $\{d_0, w_3, b_n, w_6\}$ as a circuit C . By orthogonality, C meets $\{d_1, a_1\}$ and $\{b_{n-1}, a_n\}$ so $w_3 \in \{d_1, a_1, b_{n-1}, a_n\}$ and the triangle $\{w_2, w_3, w_4\}$ gives a contradiction to Lemma 6.3.

We conclude that if $M \setminus S$ has a $(4, 4, S)$ -violator, then it is the 5-cofan $(d_0, a_0, w_3, w_4, b_n)$. Writing α and β for w_3 and w_4 , respectively, we get that the first sentence of 6.5.4 holds. Moreover, M has $\{\alpha, \beta, a_0\}$ as a circuit.

Since $\{\alpha, \beta, a_0\}$ is a triangle of $M \setminus S$, Lemma 6.3 implies that either $\{\alpha, \beta\}$ avoids $\{a_i, b_i, c_i, d_i : 0 \leq i \leq n\}$ or $\{\alpha, \beta\} = \{d_{n-1}, d_n\}$. The latter contradicts our assumption. We deduce that the former must hold. Hence 6.5.4 holds.

We now know that M has $\{\alpha, \beta, a_0\}$ as a circuit and has $\{b_n, c_n, \alpha, \beta\}$ and $\{d_0, c_0, a_0, \alpha\}$ as cocircuits. Next we aim to show that (iv) of Lemma 6.5 holds. Let $Y = \{\beta, c_0, \dots, c_n, a_0, a_1, \dots, a_n\}$ and $Z = Y \cup \{\alpha, d_0, \dots, d_{n-1}\} \cup \{b_0, b_1, \dots, b_n\}$. Evidently Z is spanned by Y . Thus $r(Z) \leq 2n + 3$. Since we know of $2n + 1$ cocircuits that are contained in Z , none of which is the symmetric difference of any others, we deduce that

6.5.5. $\lambda(Z) \leq 2$.

Next we show that

6.5.6. *M has cocircuits that are contained in $Z \cup d_n$ and meet Y in each of $\{a_0, c_0\}, \{c_0, a_1\}, \{a_1, c_1\}, \dots, \{a_{n-1}, c_{n-1}\}, \{c_{n-1}, a_n\}, \{a_n, c_n\}$, and $\{c_n, \beta\}$. Thus*

a circuit of M whose intersection with $Z \cup d_n$ is a non-empty subset of Y must contain Y .

This is immediate for all the indicated pairs except $\{a_{n-1}, c_{n-1}\}$. It is also true for the last pair unless $\{d_{n-2}, a_{n-1}, c_{n-1}, d_{n-1}\}$ is not a cocircuit of M . In the exceptional case, $\{d_{n-2}, a_{n-1}, c_{n-1}, a_n, c_n\}$ is a cocircuit. Since $\{d_{n-1}, a_n, c_n, d_n\}$ is also a cocircuit, the symmetric difference of the last two sets, which equals $\{d_{n-2}, d_{n-1}, a_{n-1}, c_{n-1}, d_n\}$, is also a cocircuit. In this case, we again get that M has a cocircuit meeting Y in $\{a_{n-1}, c_{n-1}\}$. Hence 6.5.6 holds.

Now consider the sets $\{b_0, b_1, \dots, b_n, \beta\}$ and $\{\alpha, d_0, d_1, \dots, d_{n-1}, c_n\}$, which we denote by B and D , respectively. Next we show the following.

6.5.7. *Either both B and D are circuits of $M|(B \cup D)$; or $B \cup D$ is a circuit of $M|(B \cup D)$.*

Since $B \cup D$ is the symmetric difference of a set of triangles of M , it is a disjoint union of circuits. Assume that $B \cup D$ is not a circuit. For each i in $\{0, 1, \dots, n-1\}$, since M has a cocircuit that meets $B \cup D$ in $\{b_i, b_{i+1}\}$, it follows that a circuit of $M|(B \cup D)$ that meets $B - \beta$ must contain $B - \beta$. Similarly, the cocircuits of M shown in Figure 15 imply that a circuit of $M|(B \cup D)$ that meets $\{\alpha, d_0, \dots, d_{n-2}\}$ must contain $\{\alpha, d_0, d_1, \dots, d_{n-2}\}$. Moreover, M has a cocircuit that meets $B \cup D$ in $\{d_{n-1}, c_n\}$ and has a cocircuit that meets $B \cup D$ in either $\{d_{n-2}, d_{n-1}\}$ or $\{d_{n-2}, c_n\}$. Hence every circuit of $M|(B \cup D)$ that meets $\{\alpha, d_0, d_1, \dots, d_{n-2}\}$ must contain D . Let C be a circuit of $M|(B \cup D)$ that contains D . If 6.5.7 fails, then $\beta \in C$ and $B - \beta$ is a circuit of M . But the last circuit contradicts orthogonality with the cocircuit $\{b_n, c_n, \alpha, \beta\}$. Hence 6.5.7 holds.

Next we show that

6.5.8. *$B \cup D$ is not a circuit of M .*

Assume that $B \cup D$ is a circuit of M . Then, as $|B \cup D| = 2n + 4$, we deduce that $r(Z) = 2n + 3$ so Y is a basis of Z . Suppose first that $\lambda(Z) = 2$. Then $|E(M) - Z| \in \{2, 3\}$. Suppose $E(M) - Z = \{x_1, x_2\}$. Then Y is a basis of M . By 6.5.6, the fundamental circuits $C(x_1, Y)$ and $C(x_2, Y)$ are $\{x_1\} \cup Y$ and $\{x_2\} \cup Y$, respectively. Thus M has $\{x_1, x_2\}$ as a circuit; a contradiction. It follows that $|E(M) - Z| = 3$ so $E(M) - Z$ is a triangle or a triad containing d_n . Since M has a cocircuit that contains d_n and is contained in $Z \cup d_n$, it follows that $E(M) - Z$ is a triad, say $\{d_n, x, y\}$. Now $\{d_n, a_0, c_0, a_1, c_1, \dots, a_n, c_n\}$ spans a hyperplane of M whose complementary cocircuit, C^* , is contained in $\{\beta, \alpha, x, y\}$. As M has no 4-fans, it follows that $C^* = \{\beta, \alpha, x, y\}$. But the symmetric difference of C^* and $\{d_n, x, y\}$ is $\{\beta, \alpha, d_n\}$; a contradiction.

We may now assume that $\lambda(Z) \leq 1$. Since $E(M) - Z$ contains d_n , it follows that $E(M) - Z = \{d_n\}$ and Y is a basis of M . Then $Y - c_n$ spans a hyperplane of M whose complementary cocircuit is contained in $\{b_n, c_n, d_n\}$. Thus M has a 4-fan; a contradiction. We conclude that 6.5.8 holds.

By 6.5.7 and 6.5.8, both B and D are circuits of M , so $r(Z) \leq 2n + 2$. Thus $\lambda(Z) \leq 1$, so $E(M) - Z = \{d_n\}$. Since $E(M) - Z$ contains d_n , it follows that $\lambda(Z) = 1$ and $r(Z) = 2n + 2 = r(M)$. Now Y spans M and has $r(M) + 3$ elements, so it contains a circuit. By 6.5.6, Y is a circuit of M . Thus $Y - \beta$ is a basis of M .

To complete the proof that M is the cycle matroid of the quartic Möbius ladder labelled as in Figure 15, we first observe that both matroids have $Y - \beta$ as a

basis. Since both matroids are binary, it suffices to show that they have the same fundamental circuits with respect to this basis. Evidently the fundamental circuits of each of $\beta, b_0, b_1, \dots, b_n, d_0, d_1, \dots, d_{n-1}$ are the same. Moreover, the cocircuit $\{\alpha, \beta, b_n, c_n\}$ implies that the fundamental circuit $C_M(\alpha, Y - \beta)$ must contain c_n and so, by 6.5.6, $C_M(\alpha, Y - \beta)$ contains $a_n, c_{n-1}, a_{n-1}, \dots, a_1, c_0$. Since M is binary, $C_M(\alpha, Y - \beta)$ does not contain a_0 and so is $\alpha \cup (Y - \{\beta, a_0\})$, which is also a circuit in the cycle matroid of the quartic Möbius ladder.

Finally, let $C' = C_M(d_n, Y - \beta)$. By orthogonality, exactly one of a_n and c_n is in C' . As $\beta \notin C'$, it follows by orthogonality that $c_n \notin C'$. Thus $a_n \in C'$, so $c_{n-1} \in C'$. Suppose that $a_{n-1} \notin C'$. Then, by orthogonality, none of $c_{n-2}, a_{n-2}, c_{n-3}, a_{n-3}, \dots, c_0, a_0$ is in C' , so $C' = \{d_n, a_n, c_{n-1}\}$. This is a contradiction as M has $\{d_{n-1}, a_n, c_{n-1}\}$ as a circuit. We deduce that $a_{n-1} \in C'$, so all of $c_{n-2}, a_{n-2}, c_{n-3}, a_{n-3}, \dots, c_0, a_0$ are in C' . Thus $C' = \{a_0, c_0, a_1, c_1, \dots, a_{n-1}, c_{n-1}, a_n, d_n\}$. It now follows that M is indeed the cycle matroid of the quartic Möbius ladder in Figure 15. This completes the proof of the lemma. \square

7. A QUICK WRAP

In this section, we deal with a situation when a short string of bowties wraps around on itself as in Figure 16. Observe that T_0, D_0, T_1, D_1, T_2 is not contained in a ring of bowties since $\{b_2, c_2, a_0, c_0\}$ is properly contained in a cocircuit, while $\{a_2, c_2, a_0, b_0\}$ is not a cocircuit otherwise we obtain the contradiction that $\lambda(T_0 \cup T_1 \cup T_2) \leq 2$. Lemma 7.2 deals with this situation but the argument is long and technical. The following preliminary lemma will be used several times in this proof.

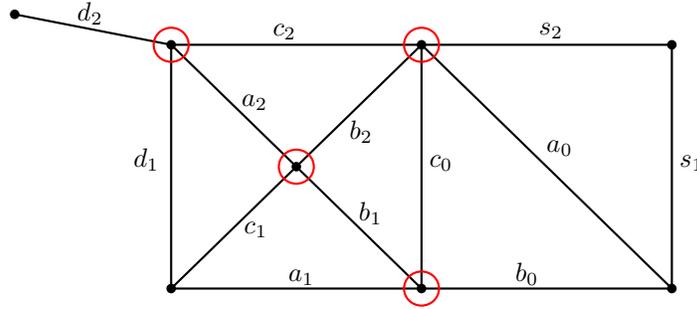


FIGURE 16

Lemma 7.1. *Let M be a binary internally 4-connected matroid having at least thirteen elements and let N be an internally 4-connected proper minor of M that has at least seven elements. Suppose M has a rotor chain $((a_0, b_0, c_0), (a_1, b_1, c_1), \dots, (a_n, b_n, c_n))$ and that $M \setminus c_0, c_1$ has an N -minor. Then either $M \setminus c_0, c_1, \dots, c_n$ is sequentially 4-connected with an N -minor, or M has a minor M' such that M' is internally 4-connected with an N -minor and $1 \leq |E(M)| - |E(M')| \leq 3$.*

Proof. Let $S = \{c_0, c_1, \dots, c_n\}$. We may assume that M/b_1 has no N -minor, otherwise, by Lemma 4.5, the lemma holds. Thus $M \setminus c_0, c_1/b_1$ has no N -minor, but $M \setminus c_0, c_1$ has an N -minor. By Lemma 5.7, we know that $M \setminus c_0, c_1, \dots, c_i/b_i$ has no

N -minor for all $i \in \{1, 2, \dots, n\}$, that $M \setminus c_0, c_1, \dots, c_j/a_j$ has no N -minor for all $j \in \{2, 3, \dots, n\}$, and that $M \setminus S$ has an N -minor. Lemma 5.3 implies that either $M \setminus S$ is 3-connected, or $M \setminus S$ has a_i or b_i in a cocircuit of size at most two for some i in $\{2, 3, \dots, n\}$. The latter implies that $M \setminus S/a_i$ or $M \setminus S/b_i$ has an N -minor; a contradiction. Hence $M \setminus S$ is 3-connected.

Let (X, Y) be a non-sequential 3-separation of $M \setminus S$. Without loss of generality, we may assume that the triad $\{b_0, a_1, b_1\} \subseteq X$. If a_0, a_2 , or b_2 is in X , then all of them are in the full closure of X , and we may assume that $\{a_0, a_2, b_2\} \subseteq X$. Then the full closure of X in $M \setminus S$ contains $\{a_0, b_0, a_1, b_1, \dots, a_n, b_n\}$, and we see that $(\text{fcl}_{M \setminus S}(X) \cup S, Y - \text{fcl}_{M \setminus S}(X))$ is a non-sequential 3-separation of M ; a contradiction. We may now assume that $\{a_0, a_2, b_2\} \subseteq Y$. Then the full closure of Y in $M \setminus S$ contains b_0, a_1 , and b_1 , and we obtain the same contradiction as before. \square

Beginning with the next lemma and for the rest of the paper, we shall start abbreviating how we refer to the following three outcomes in the main theorem.

- (i) M has a proper minor M' such that $|E(M)| - |E(M')| \leq 3$ and M' is internally 4-connected with an N -minor;
- (ii) M contains an open rotor chain, a ladder structure, or a ring of bowties that can be trimmed to obtain an internally 4-connected matroid with an N -minor;
- (iii) M contains an enhanced quartic ladder from which an internally 4-connected minor of M with an N -minor can be obtained by an enhanced-ladder move.

When (i) or (iii) holds, we say, respectively, that M has a *quick win* or an *enhanced-ladder win*. When trimming an open rotor chain, a ladder structure, or a ring of bowties in M produces an internally 4-connected matroid with an N -minor, we say, respectively, that M has an *open-rotor-chain win*, a *ladder win*, or a *bowtie-ring win*.

Lemma 7.2. *Let M and N be internally 4-connected binary matroids with $|E(M)| \geq 13$ and $|E(N)| \geq 7$. Assume that M contains the structure in Figure 16 and that $M \setminus c_0, c_1, c_2, s_1$ has an N -minor. Then*

- (i) M has a quick win; or
- (ii) $\{d_1, d_2\}$ is contained in a triangle of M ; or
- (iii) $\{b_0, a_1\}$ is contained in a triangle of M ; or
- (iv) M has an open-rotor-chain win or a ladder win; or
- (v) M has an enhanced-ladder win.

Proof. We assume that none of (i)–(v) holds. First we show the following.

7.2.1. *All the elements in Figure 16 are distinct.*

With $T_i = \{a_i, b_i, c_i\}$ for all i , we see that $\lambda(T_0 \cup T_1 \cup T_2) \leq 3$. Since $|E(M)| \geq 13$, we have that $|T_0 \cup T_1 \cup T_2| = 9$ otherwise $\lambda(T_0 \cup T_1 \cup T_2) \leq 2$. Clearly $s_2 \notin T_0 \cup T_1 \cup T_2$ otherwise $\lambda(T_0 \cup T_1 \cup T_2) \leq 2$. Thus $s_1 \neq a_0$, so $s_1 \notin T_0 \cup s_2$. Moreover, $s_1 \notin \{b_2, c_2\}$ otherwise a triangle is contained in a cocircuit. By orthogonality between the triangle $\{s_1, s_2, a_0\}$ and the cocircuits $\{a_1, b_1, b_0, c_0\}$ and $\{b_1, c_1, a_2, b_2\}$, we see that $s_1 \notin \{a_1, b_1\}$ and $s_1 \notin \{a_2, b_2\}$. Thus $s_1 \notin T_0 \cup T_1 \cup T_2$. Hence $\{s_1, s_2\}$ avoids $T_0 \cup T_1 \cup T_2$. If $\{d_1, d_2\}$ meets $T_0 \cup T_1 \cup T_2$, then, since M is binary, $\{d_1, d_2\}$ is contained in T_0 or T_1 and again we obtain the contradiction that $\lambda(T_0 \cup T_1 \cup T_2) \leq 2$.

Finally, if $\{d_1, d_2\}$ meets $\{s_1, s_2\}$, then, by orthogonality between the cocircuit $\{d_1, a_2, c_2, d_2\}$ and the triangle $\{a_0, s_1, s_2\}$, we deduce that $\{d_1, d_2\} = \{s_1, s_2\}$. Thus $\{d_1, d_2\}$ is contained in a triangle; a contradiction. We conclude that 7.2.1 holds.

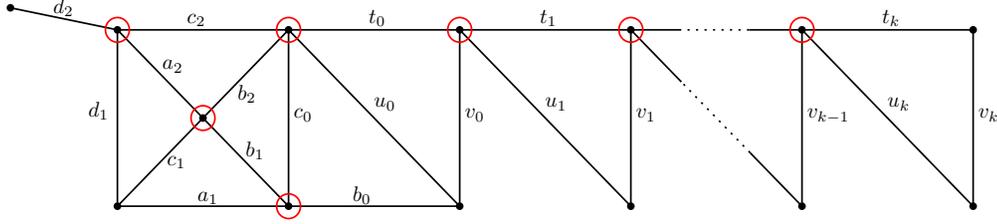


FIGURE 17. The elements are distinct.

We relabel (s_2, a_0, s_1) as (t_0, u_0, v_0) . Then T_0 becomes $\{u_0, b_0, c_0\}$. Take k to be maximal such that $\{t_0, u_0, v_0\}, \{t_0, v_0, t_1, u_1\}, \{t_1, u_1, v_1\}, \{t_1, v_1, t_2, u_2\}, \dots, \{t_k, u_k, v_k\}$ is a string of bowties and all of the elements in Figure 17 are distinct. As $v_0 = s_1$, it follows by assumption that $M \setminus c_0, c_1, c_2, v_0$ has an N -minor.

We now show that

7.2.2. $M \setminus c_0, c_1, c_2, v_0, v_1, \dots, v_i/t_i$ has no N -minor for every i in $\{0, 1, \dots, k\}$ and $M \setminus c_0, c_1, c_2, v_0, v_1, \dots, v_k$ has an N -minor.

Observe that M has $(T_0, T_1, T_2, D_0, D_1, \{c_0, b_1, b_2\})$ as a quasi rotor. Thus, by Lemma 4.5, M/b_2 has no N -minor. Note that $T_1, D_1, T_2, \{b_2, c_2, u_0, t_0\}, \{u_0, t_0, v_0\}, \{t_0, v_0, u_1, t_1\}, \{u_1, t_1, v_1\}, \{t_1, v_1, u_2, t_2\}, \dots, \{u_k, t_k, v_k\}$ is a string of bowties in $M \setminus c_0$, and $M \setminus c_0 \setminus c_1, c_2$ has an N -minor but $M \setminus c_0 \setminus c_1, c_2/b_2$ has no N -minor. Lemma 5.7 implies that 7.2.2 holds.

Next we show that

7.2.3. $M \setminus c_0, c_1, c_2, v_0, \dots, v_i$ is sequentially 4-connected for all i in $\{0, 1, \dots, k\}$.

Lemma 7.1 implies that $M \setminus c_0, c_1, c_2$ is sequentially 4-connected, so it is 3-connected. By Tutte's Triangle Lemma [13] (or see [11, Lemma 8.7.7]), $M \setminus c_0, c_1, c_2, v_0$ is 3-connected unless $M \setminus c_0, c_1, c_2$ has v_0 in a triad with an element x in $\{t_0, u_0\}$. In the exceptional case, $M \setminus c_0, c_1, c_2, v_0/x$ has an N -minor, so, by 7.2.2, we know that $x = u_0$. Then $M \setminus c_0, c_1, c_2, t_0/u_0$ and $M \setminus c_0, c_1, c_2, t_0/b_2$ also have N -minors, and Lemma 4.5 gives a contradiction. We conclude that $M \setminus c_0, c_1, c_2, v_0$ is 3-connected.

Suppose (X, Y) is a non-sequential 3-separation of $M \setminus c_0, c_1, c_2, v_0$. Without loss of generality, the triad $\{b_2, t_0, u_0\}$ is contained in X . Hence $(X \cup v_0, Y)$ is a non-sequential 3-separation of $M \setminus c_0, c_1, c_2$; a contradiction. We conclude that $M \setminus c_0, c_1, c_2, v_0$ is sequentially 4-connected, so 7.2.3 holds for $i = 0$.

Now, for some i in $\{1, 2, \dots, k\}$, suppose that $M \setminus c_0, c_1, c_2, v_0, \dots, v_j$ is sequentially 4-connected for all $j < i$ but that $M \setminus c_0, c_1, c_2, v_0, \dots, v_i$ is not 3-connected. Then, by Tutte's Triangle Lemma again, $M \setminus c_0, c_1, c_2, v_0, \dots, v_{i-1}$ has v_i in a triad with u_i or t_i . By 7.2.2, this triad contains u_i , and $M \setminus c_0, c_1, c_2, v_0, \dots, v_i/u_i$ has an N -minor. Now $M \setminus c_0, c_1, c_2, v_0, \dots, v_i/u_i \cong M \setminus c_0, c_1, c_2, v_0, \dots, v_{i-1}/u_i \setminus t_i \cong$

$M \setminus c_0, c_1, c_2, v_0, \dots, v_{i-1}, t_i/t_{i-1}$, so we obtain a contradiction to 7.2.2. We conclude that $M \setminus c_0, c_1, c_2, v_0, \dots, v_i$ is 3-connected. Now suppose that (X, Y) is a non-sequential 3-separation of $M \setminus c_0, c_1, c_2, v_0, \dots, v_i$. Without loss of generality, the triad $\{t_{i-1}, u_i, t_i\}$ is contained in X , and $(X \cup v_i, Y)$ is a non-sequential 3-separation of $M \setminus c_0, c_1, c_2, v_0, \dots, v_{i-1}$; a contradiction. We conclude, by induction, that 7.2.3 holds.

We show next that

7.2.4. *M has no triangle $\{t_{k+1}, u_{k+1}, v_{k+1}\}$ such that $(\{t_k, u_k, v_k\}, \{t_{k+1}, u_{k+1}, v_{k+1}\}, \{z, v_k, t_{k+1}, u_{k+1}\})$ is a bowtie in M for some z in $\{u_k, t_k\}$.*

Suppose instead that M has such a triangle $T = \{t_{k+1}, u_{k+1}, v_{k+1}\}$. Clearly, T avoids $\{t_k, u_k, v_k\}$. By the choice of k , it follows that T must contain some element in Figure 17.

As a step towards proving 7.2.4, we show next that

7.2.5. *T avoids $T_0 \cup T_1 \cup T_2 \cup \{d_1, d_2, t_0\}$.*

Suppose that T meets $T_0 \cup T_1 \cup T_2 \cup \{d_1, d_2, t_0\}$. Since the last set is a union of vertex cocircuits in Figure 17, T must contain at least two elements of $T_0 \cup T_1 \cup T_2 \cup \{d_1, d_2, t_0\}$. Orthogonality with these vertex cocircuits implies that T is contained in $T_0 \cup T_1 \cup T_2 \cup \{d_1, d_2, t_0, v_0\}$; otherwise T contains $\{d_1, d_2\}$ or $\{b_0, a_1\}$, a contradiction. Thus one easily checks that either T is one of the triangles shown in Figure 17, or T meets $\{t_0, u_0\}$.

Assume that the first possibility does not hold. Since $T \subseteq T_0 \cup T_1 \cup T_2 \cup \{d_1, d_2, t_0, v_0\}$, we see that $t_0 \notin T$, otherwise, by orthogonality, $k = 0$ and T meets T_k ; a contradiction. If $u_0 \in T$, then orthogonality implies that T is $\{u_0, c_2, d_2\}$ or $\{u_0, b_2, c_1\}$. The latter possibility is excluded because $\{u_0, b_2, c_1, a_1, b_0\}$ is a circuit. If $T = \{u_0, c_2, d_2\}$, then the cocircuit $\{z, v_k, t_{k+1}, u_{k+1}\}$ meets either T_2 or T_0 in a single element; a contradiction. We conclude that T is one of the triangles shown in Figure 17.

Suppose that T meets $\{b_1, c_1, a_2, b_2\}$. Then $\{b_1, c_1, a_2, b_2\}$ meets the cocircuit $\{z, v_k, t_{k+1}, u_{k+1}\}$ in a subset of $\{t_{k+1}, u_{k+1}\}$. But each of b_1, c_1, a_2 , and b_2 is in two triangles, and orthogonality implies that a second element of each of these triangles must also be in $\{t_{k+1}, u_{k+1}\}$; a contradiction. We conclude that T avoids $\{b_1, c_1, a_2, b_2\}$, so T is $\{u_0, b_0, c_0\}$ or $\{u_0, t_0, v_0\}$. But T avoids T_k so $k > 0$. If $\{u_0, c_0\}$ meets the cocircuit $\{z, v_k, t_{k+1}, u_{k+1}\}$, then, because each of u_0 and c_0 is in two triangles, we again obtain the contradiction that $\{t_{k+1}, u_{k+1}\}$ contains at least three elements. Thus $\{u_0, c_0\}$ avoids $\{z, v_k, t_{k+1}, u_{k+1}\}$ so $T = \{u_0, t_0, v_0\}$ and $\{t_{k+1}, u_{k+1}\} = \{t_0, v_0\}$. Hence $\{z, v_k, t_{k+1}, u_{k+1}\} = \{z, v_k, t_0, v_0\}$, so $M \setminus c_0, c_1, c_2, v_0, v_1, \dots, v_k$ has $\{t_0, z\}$ as a cocircuit. Thus $M \setminus c_0, c_1, c_2, v_0/t_0$ has an N -minor; a contradiction to 7.2.2. We conclude that 7.2.5 holds.

We now know that T meets $\{v_0, t_1, u_1, v_1, \dots, t_{k-1}, u_{k-1}, v_{k-1}\}$. If $k = 0$, then $v_0 \in \{t_1, u_1, v_1\}$; a contradiction. Hence $k \geq 1$. Thus, by Lemma 5.4, either $T = \{u_i, t_i, v_i\}$ for some i with $0 \leq i \leq k - 2$, or T meets $\{u_0, t_0, v_0\}$ in $\{u_0\}$. By 7.2.5, the latter does not occur and so the former occurs with $i \geq 1$. Hence $k \geq 3$. If $v_i \in \{t_{k+1}, u_{k+1}\}$, then $M \setminus c_0, c_1, c_2, v_0, v_1, \dots, v_k$ has z in a 2-element cocircuit, so $M \setminus c_0, c_1, c_2, v_0, v_1, \dots, v_k/z$ has an N -minor. Let y be the element in $\{t_k, u_k\}$ that is not z . Then $M \setminus c_0, c_1, c_2, v_0, v_1, \dots, v_k/z \cong$

$M \setminus c_0, c_1, c_2, v_0, v_1, \dots, v_{k-1}, y/z \cong M \setminus c_0, c_1, c_2, v_0, v_1, \dots, v_{k-1}, y/t_{k-1}$; a contradiction to 7.2.2. We may assume then that $\{t_i, u_i\} = \{t_{k+1}, u_{k+1}\}$. It follows that $\{z, v_k, t_{k+1}, u_{k+1}\} \triangle \{t_{i-1}, v_{i-1}, t_i, u_i\}$, which equals $\{z, v_k, t_{i-1}, v_{i-1}\}$, is a cocircuit of M . Thus $M \setminus c_0, c_1, c_2, v_0, v_1, \dots, v_k/t_{i-1}$ has an N -minor; a contradiction to 7.2.2. We conclude that 7.2.4 holds.

Next we show that

7.2.6. $M \setminus v_0$ is sequentially 4-connected.

By [6, Lemma 3.1], $M \setminus v_0$ is 3-connected. Suppose (U, V) is a non-sequential 3-separation of $M \setminus v_0$. By [6, Lemma 3.3], we may assume that $\{u_0, b_0, c_0\} \cup T_1 \cup b_2 \subseteq U$. It follows that we may assume that $\{u_0, b_0, c_0\} \cup T_1 \cup b_2 \cup a_2 \cup c_2 \cup t_0 \subseteq U$. Then $(U \cup v_0, V)$ is a non-sequential 3-separation of M ; a contradiction. We conclude that 7.2.6 holds.

We now show that

7.2.7. $k \geq 1$.

Suppose $k = 0$. We know by 7.2.3 that $M \setminus c_0, c_1, c_2, v_0$ is sequentially 4-connected. We show next that

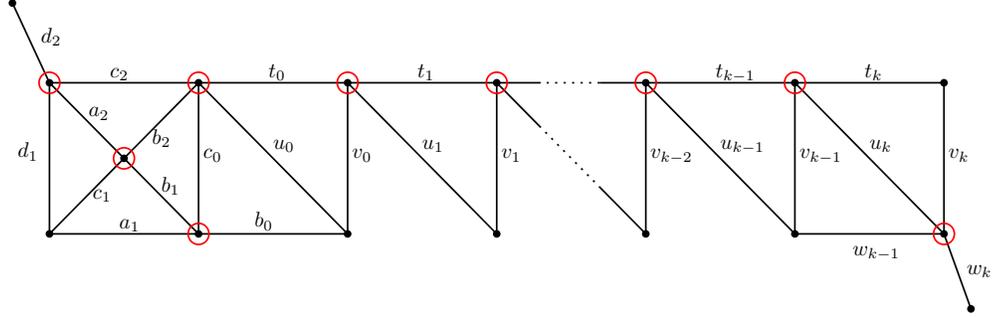
7.2.8. $M \setminus c_0, c_1, c_2, v_0$ is not internally 4-connected.

Assume the contrary. By 7.2.6, $M \setminus v_0$ is sequentially 4-connected. Since (i) does not hold, $M \setminus v_0$ has a 4-fan, say $(\alpha, \beta, \gamma, \delta)$. Then $\{\beta, \gamma, \delta, v_0\}$ is a cocircuit so orthogonality implies that $\{\beta, \gamma, \delta\}$ meets $\{t_0, u_0\}$. Observe that $c_0 \notin \{\beta, \gamma, \delta\}$, otherwise we get a contradiction to orthogonality with one of the circuits $\{c_0, b_1, b_2\}$ or $\{c_0, a_1, d_1, c_2\}$.

Suppose $u_0 \in \{\beta, \gamma, \delta\}$. Then, since $c_0 \notin \{\beta, \gamma, \delta\}$, orthogonality implies that $b_0 \in \{\beta, \gamma, \delta\}$. Thus we may relabel $\{\beta, \gamma, \delta, v_0\}$ as $\{b_0, u_0, v_0, w_0\}$, and M contains the configuration in Figure 3(a). Orthogonality between the cocircuit $\{b_0, u_0, v_0, w_0\}$, and the circuits in Figure 17 implies that the elements are distinct except that d_2 may be w_0 , so (v) holds; a contradiction. Hence $u_0 \notin \{\beta, \gamma, \delta\}$.

Since $\{t_0, u_0\}$ meets $\{\beta, \gamma, \delta\}$, we may now assume that $t_0 \in \{\beta, \gamma, \delta\}$. Then 7.2.4 implies that $t_0 \neq \delta$. Hence, by symmetry, we may assume that $t_0 = \gamma$. By orthogonality between $\{\alpha, \beta, t_0\}$ and the vertex cocircuits in Figure 17, we deduce that $\{\alpha, \beta\}$ is $\{c_0, a_1\}, \{b_2, c_1\}, \{c_2, d_1\}$, or $\{c_2, d_2\}$. By orthogonality between the cocircuit $\{\beta, t_0, \delta, v_0\}$ and the triangles in Figure 17 other than $\{t_0, v_0, u_0\}$, we deduce that if $\{\beta, \delta\}$ meets one of these triangles, then $\{\beta, \delta\}$ is contained in that triangle. But $\{\beta, \delta\}$ avoids $\{c_0, u_0\}$ and cannot meet $\{a_2, b_2, b_1, c_1\}$. We conclude that $\{\beta, \delta\}$ avoids $T_0 \cup T_1 \cup T_2 \cup d_1$. Thus $(\alpha, \beta) = (c_2, d_2)$ and $\delta \notin T_0 \cup T_1 \cup T_2 \cup \{d_1, d_2, t_0, v_0\}$. Then $\lambda(T_0 \cup T_1 \cup T_2 \cup \{d_1, d_2, t_0\}) \leq 2$. Now $T_0 \cup T_1 \cup T_2 \cup \{d_1, d_2, t_0\}$ contains twelve elements, and avoids the two-element set $\{\delta, v_0\}$. Thus, as M is internally 4-connected, we deduce that $|E(M)| \in \{14, 15\}$ and $\lambda(T_0 \cup T_1 \cup T_2 \cup \{d_1, d_2, t_0\}) = 2$. Hence $r(T_0 \cup T_1 \cup T_2 \cup \{d_1, d_2, t_0\}) = 6$. Thus $r(M) = 6$ and $|E(M)| = 14$ otherwise M has a triad containing $\{\delta, v_0\}$, which gives the contradiction that M has a 4-fan since v_0 is in a triangle. It follows that $\{b_2, u_0, b_0, a_1, d_1, d_2\}$ is a basis B of M . Now consider the fundamental circuits $C(v_0, B)$ and $C(\delta, B)$. By using orthogonality between these circuits and the cocircuit $\{v_0, d_2, t_0, \delta\}$ as well as the vertex cocircuits in Figure 17, we deduce that $C(v_0, B)$ and $C(\delta, B)$ are $\{v_0, a_1, b_0, d_1, d_2\}$ and $\{\delta, a_1, b_0, d_1, d_2\}$. The symmetric difference of $C(v_0, B)$ and $C(\delta, B)$ is $\{v_0, \delta\}$, which must contain a circuit of M ; a contradiction. We conclude that 7.2.8 holds.

By 7.2.3 and 7.2.8, we deduce that $M \setminus \{c_0, c_1, c_2, v_0\}$ has a 4-fan (y_1, y_2, y_3, y_4) . Suppose $\{y_1, y_2, y_3\}$ meets an element in Figure 17 where $k = 0$. Orthogonality and the fact that M contains no parallel pairs implies that either $\{y_1, y_2, y_3\}$ contains $\{d_1, d_2\}$, and (ii) holds; or $\{y_1, y_2, y_3\}$ contains $\{b_0, a_1\}$, and (iii) holds. Since neither (ii) nor (iii) holds by assumption, we deduce that $\{y_1, y_2, y_3\}$ avoids the elements in Figure 17. Now M has a cocircuit C^* such that $\{y_2, y_3, y_4\} \subsetneq C^* \subseteq \{y_2, y_3, y_4, c_0, c_1, c_2, v_0\}$. Orthogonality implies that C^* avoids $\{c_0, c_1\}$, since each of these elements is in two triangles in the figure but y_4 is the only element of C^* that can be in such a triangle. If $c_2 \in C^*$, then $\{y_2, y_3, y_4\}$ meets $\{a_2, b_2\}$ and $\{a_1, d_1, c_1\}$; a contradiction. Thus $C^* = \{v_0, y_2, y_3, y_4\}$ and $y_4 \in \{t_0, u_0\}$. Hence $(\{t_0, u_0, v_0\}, \{y_1, y_2, y_3\}, \{x, v_0, y_2, y_3\})$ is a bowtie in M for some x in $\{t_0, u_0\}$; a contradiction to 7.2.4. We conclude that 7.2.7 holds.

FIGURE 18. $k \geq 1$

Next we show:

7.2.9. *After possibly interchanging the labels on t_k and u_k , the matroid M contains the configuration in Figure 18, where $k \geq 1$.*

Consider $M \setminus v_k$. It certainly has an N -minor so it is not internally 4-connected. Thus, by 7.2.4, Lemma 4.3 implies that $M \setminus v_k$ is $(4, 4, S)$ -connected and that either $M \setminus v_k$ has a 4-fan (e, f, w_{k-1}, w_k) where $\{w_{k-1}, w_k\}$ avoids $\{u_{k-1}, v_{k-1}, t_{k-1}, u_k, v_k, t_k\}$, while $e \in \{t_{k-1}, v_{k-1}\}$ and $f \in \{t_k, u_k\}$; or M has a triangle $\{u_{k-1}, p, q\}$ and a cocircuit $\{a, v_k, p, q\}$ where $a \in \{u_k, t_k\}$, and $\{p, q\}$ avoids $\{u_{k-1}, v_{k-1}, t_{k-1}, u_k, v_k, t_k\}$. The latter implies that $(\{t_k, u_k, v_k\}, \{u_{k-1}, p, q\}, \{a, p, q, v_k\})$ is a bowtie, a contradiction to 7.2.4. Thus we may assume that the former holds. Suppose $e = t_{k-1}$. Then orthogonality between the triangle $\{w_{k-1}, t_{k-1}, f\}$ and the vertex cocircuit containing $\{t_{k-1}, u_{k-1}\}$ implies that w_{k-1} is in a triangle T' in Figure 17 that is disjoint from $\{u_{k-1}, v_{k-1}, t_{k-1}, u_k, v_k, t_k\}$. By orthogonality between T' and the cocircuit $\{v_k, f, w_{k-1}, w_k\}$ of M , we deduce that T' also contains w_k , so we get a contradiction to 7.2.4. We conclude that $e \neq t_{k-1}$. Thus $e = v_{k-1}$, so M contains the configuration shown in Figure 18, that is, 7.2.9 holds.

As the elements in Figure 17 are all distinct, the elements in Figure 18 are also distinct unless w_{k-1} or w_k is equal to another element. Suppose w_{k-1} is an element in Figure 17. As every element in that figure except v_k is in a vertex cocircuit, w_{k-1} is in such a cocircuit. By orthogonality, $\{v_{k-1}, u_k\}$ contains an element in this

cocircuit. Hence $w_{k-1} \in \{t_{k-1}, v_{k-1}, u_k, t_k\}$; a contradiction. Clearly $w_k \neq w_{k-1}$. Suppose w_k is an element in Figure 17. Then orthogonality implies that w_k is not in a circuit in Figure 17, so $w_k = d_2$. We conclude that the following holds.

7.2.10. *The elements in Figure 18 are distinct except that w_k may equal d_2 .*

Next we observe that

7.2.11. *M has no triangle containing $\{u_0, b_2\}$ or $\{u_0, c_2\}$.*

Suppose M has a triangle T' containing $\{u_0, b_2\}$. Then, by orthogonality with the cocircuit $\{b_2, b_1, c_1, a_2\}$, we deduce that $T' = \{u_0, b_2, c_1\}$. Thus $\{u_0, b_0\} \subseteq \text{cl}(T_1 \cup T_2)$, so $\lambda(T_0 \cup T_1 \cup T_2) \leq 2$; a contradiction. Hence M has no triangle containing $\{u_0, b_2\}$.

Now assume M has a triangle T'' containing $\{u_0, c_2\}$. Then, by orthogonality with the cocircuit $\{c_2, a_2, d_1, d_2\}$, we deduce that T'' is $\{u_0, c_2, d_1\}$ or $\{u_0, c_2, d_2\}$. In the first case, we again get the contradiction that $\{u_0, b_0\} \subseteq \text{cl}(T_1 \cup T_2)$. In the second case, $\lambda(T_0 \cup T_1 \cup T_2 \cup \{d_1, d_2\}) \leq 2$. This is a contradiction because $|E(M)| \geq 16$ since $k \geq 1$ and the elements in Figure 17 are distinct. Thus 7.2.11 holds.

The following will be useful not only in the proof of the subsequent assertion but also later in the proof of Lemma 7.2.

7.2.12. *If (y_1, y_2, y_3, y_4) is a 4-fan in $M \setminus v_0$, then either $\{y_1, y_2, y_3\} = T_0$, or $y_4 \in \{t_0, u_0\}$.*

Assume that this fails. As $\{y_2, y_3, y_4, v_0\}$ is a cocircuit, t_0 or u_0 is in $\{y_2, y_3, y_4\}$. By orthogonality between $\{y_1, y_2, y_3\}$ and the cocircuits displayed in Figure 18, we see that $t_0 \notin \{y_2, y_3\}$. Thus $u_0 \in \{y_2, y_3\}$. Then orthogonality between $\{y_1, y_2, y_3\}$ and the cocircuits in Figure 18 implies that $\{y_1, y_2, y_3\}$ contains $\{u_0, b_2\}$ or $\{u_0, c_2\}$; a contradiction to 7.2.11. We conclude that 7.2.12 holds.

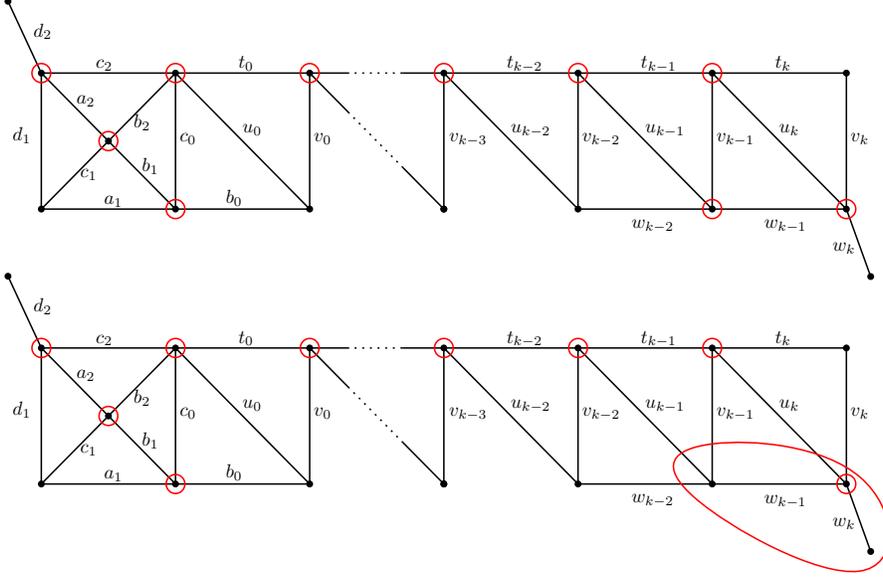
We will now show that

7.2.13. *$M \setminus v_0$ is $(4, 4, S)$ -connected.*

By 7.2.6, $M \setminus v_0$ is sequentially 4-connected. Assume that 7.2.13 fails. Then $M \setminus v_0$ has $(x_1, x_2, x_3, x_4, x_5)$ as a 5-fan or a 5-cofan. Now $\{x_1, x_2, x_3, x_4, x_5\}$ contains at most one element of $\{t_0, u_0\}$ otherwise v_0 is in the closure of $\{x_1, x_2, x_3, x_4, x_5\}$ and so $\{x_1, x_2, x_3, x_4, x_5, v_0\}$ is 3-separating in M ; a contradiction. Suppose that $(x_1, x_2, x_3, x_4, x_5)$ is a 5-fan. Then (x_1, x_2, x_3, x_4) and (x_5, x_4, x_3, x_2) are 4-fans. By 7.2.12, we may assume that $T_0 = \{x_1, x_2, x_3\}$ and $x_2 \in \{t_0, u_0\}$. As $T_0 = \{u_0, b_0, c_0\}$, we deduce that $x_2 = u_0$, so $\{x_1, x_3\} = \{b_0, c_0\}$. Now M has $\{x_2, x_3, x_4, v_0\}$ as a cocircuit. If $c_0 = x_3$, then, by orthogonality, $x_4 \in \{b_1, b_2\}$. Hence the cocircuit $\{u_0, c_0, x_4, v_0\}$ meets the circuit $\{b_1, b_2, a_1, d_1, c_2\}$ in a single element; a contradiction. Thus $(x_1, x_3) = (c_0, b_0)$. By orthogonality between the triangle $\{b_0, x_4, x_5\}$ and the cocircuits D_0 and D_1 , we deduce that this triangle also contains a_1 . This contradicts the assumption that (ii) does not hold.

We may now assume that $(x_1, x_2, x_3, x_4, x_5)$ is a 5-cofan. Then the 4-fans (x_2, x_3, x_4, x_5) and (x_4, x_3, x_2, x_1) imply, by 7.2.12, that $T_0 = \{x_2, x_3, x_4\}$. Since both $\{x_3, x_2, x_1, v_0\}$ and $\{x_3, x_4, x_5, v_0\}$ are cocircuits of M , it follows that $\{c_0, v_0\}$ is contained in one of these 4-cocircuit of M . But this cocircuit must meet each of $\{b_1, b_2\}$, $\{a_1, d_1, c_2\}$, and $\{u_0, t_0\}$; a contradiction. We conclude that 7.2.13 holds.

Next we show that

FIGURE 19. $k \geq 2$

7.2.14. $M \setminus v_{k-1}, v_k$ has no 4-fan having t_k as its coguts element.

Assume that $M \setminus v_{k-1}, v_k$ has (z_1, z_2, z_3, t_k) as a 4-fan. Then M has one of $\{z_2, z_3, t_k, v_k\}$, $\{z_2, z_3, t_k, v_{k-1}\}$, or $\{z_2, z_3, t_k, v_{k-1}, v_k\}$ as a cocircuit. The first possibility is excluded by 7.2.4. The second and third possibilities imply, by orthogonality, that $\{z_2, z_3\}$ meets both $\{t_{k-1}, u_{k-1}\}$ and $\{u_k, w_{k-1}\}$. Then orthogonality using the triangle $\{z_1, z_2, z_3\}$ gives a contradiction. Hence 7.2.14 holds.

The next assertion will take some time to prove.

7.2.15. $\{t_{k-1}, t_k\}$ is contained in a triangle of M .

Suppose that $\{t_{k-1}, t_k\}$ is not contained in a triangle of M . We can now apply Lemma 6.1 to the configuration induced by $\{t_{k-1}, u_{k-1}, v_{k-1}, t_k, u_k, v_k, w_{k-1}, w_k\}$ where (v_{k-1}, v_k) corresponds to (c_0, c_1) in the lemma. Then neither (i) nor (iii) of Lemma 6.1 holds. Moreover, by 7.2.4, (ii) of the lemma also does not hold.

Next we eliminate the possibility that (v) of Lemma 6.1 holds by showing that $M \setminus v_{k-1}, v_k$ has no 4-fan with t_k as its coguts element. Assume instead that $M \setminus v_{k-1}, v_k$ has such a 4-fan, (y_1, y_2, y_3, t_k) . Then M has a cocircuit C^* such that $\{y_2, y_3, t_k\} \subsetneq C^* \subseteq \{y_2, y_3, t_k, v_{k-1}, v_k\}$. By 7.2.4, we know that $v_{k-1} \in C^*$. Furthermore, Lemma 4.2 implies that $v_k \in C^*$ unless $\{y_2, y_3, t_k, v_{k-1}\} = \{t_{k-1}, u_k, t_k, v_{k-1}\}$. In the exceptional case, by orthogonality, $\{y_1, y_2, y_3\} = \{t_{k-1}, u_k, w_k\}$, so $\lambda(\{t_{k-1}, u_{k-1}, v_{k-1}, t_k, u_k, v_k, w_{k-1}, w_k\}) \leq 2$; a contradiction. We deduce that $C^* = \{y_2, y_3, t_k, v_{k-1}, v_k\}$. As M is binary and C^* contains three elements of the circuit $\{t_k, v_{k-1}, v_k, w_{k-1}\}$, it follows that $w_{k-1} \in C^*$. Orthogonality between the triangle $\{y_1, y_2, y_3\}$ and the cocircuit $\{w_{k-1}, u_k, v_k, w_k\}$ implies that $\{y_1, y_2, y_3\}$ meets $\{u_k, w_k\}$. As $\{v_{k-1}, w_{k-1}, u_k\}$ is a triangle, we deduce that $\{y_1, y_2, y_3\}$ does not contain $\{w_{k-1}, u_k\}$, so $\{w_{k-1}, w_k\} \subseteq \{y_1, y_2, y_3\}$; a

contradiction to 7.2.4. We conclude that $M \setminus v_{k-1}, v_k$ has no 4-fan with t_k as its cocuts element, so (v) of Lemma 6.1 does not hold.

We now know that (iv) of Lemma 6.1 holds. Combining this with 7.2.14, we see that

7.2.16. *M has elements α and β not in $\{t_{k-1}, u_{k-1}, v_{k-1}, t_k, u_k, v_k, w_{k-1}, w_k\}$ such that $\{\alpha, \beta, u_{k-1}\}$ is a triangle, $\{\beta, u_{k-1}, v_{k-1}, w_{k-1}\}$ or $\{\beta, u_{k-1}, v_{k-1}, u_k, v_k\}$ is a cocircuit, D^* , and every $(4, 3)$ -violator of $M \setminus v_{k-1}, v_k$ is a 4-fan with α as its guts.*

We show next that

7.2.17. $k \geq 2$.

Suppose that $k = 1$. By orthogonality between $\{\alpha, \beta, u_0\}$ and the cocircuit $\{c_0, b_2, c_2, t_0, u_0\}$, we deduce that $\{\alpha, \beta\}$ meets $\{c_0, b_2, c_2\}$. But, by 7.2.11, $\{\alpha, \beta\}$ avoids $\{b_2, c_2\}$. Thus $\{\alpha, \beta, u_0\} = \{c_0, b_0, u_0\}$. Orthogonality between D^* and the triangle $\{b_2, b_1, c_0\}$ implies that $\beta \neq c_0$. Thus $(\alpha, \beta) = (c_0, b_0)$.

To complete the proof of 7.2.17, we will show that (ii) holds thereby obtaining a contradiction. By 7.2.2 and 7.2.3, $M \setminus c_0, c_1, c_2, v_0, v_1$ has an N -minor and is sequentially 4-connected. As (v) does not hold, $M \setminus c_0, c_1, c_2, v_0, v_1$ is not internally 4-connected, so it has a 4-fan (z_1, z_2, z_3, z_4) . Thus M has a cocircuit D such that $\{z_2, z_3, z_4\} \subsetneq D \subseteq \{z_2, z_3, z_4, c_0, c_1, c_2, v_0, v_1\}$.

Suppose (z_1, z_2, z_3, z_4) is a 4-fan of $M \setminus v_0, v_1$. Then, by 7.2.16, since $(\alpha, \beta) = (c_0, b_0)$, we get the contradiction that $z_1 = c_0$. Hence (z_1, z_2, z_3, z_4) is not a 4-fan of $M \setminus v_0, v_1$, so D meets $\{c_0, c_1, c_2\}$. Then orthogonality between D and T_0, T_1 , and T_2 implies that $\{z_2, z_3, z_4\}$ meets one of these triangles, specifically it meets $\{u_0, b_0, a_1, b_1, a_2, b_2\}$. Suppose $\{z_2, z_3\}$ meets the last set. Then orthogonality with the cocircuits shown in Figure 18 implies that $\{z_1, z_2, z_3\}$ is $\{b_0, w_0, a_1\}$, a contradiction to orthogonality with $\{w_0, u_1, v_1, w_1\}$. Thus $z_4 \in \{u_0, b_0, a_1, b_1, a_2, b_2\}$. But orthogonality with the circuits $\{u_0, b_0, b_1, b_2\}$ and $\{a_1, b_1, d_1, a_2\}$ in $M \setminus c_0, c_1, c_2, v_0, v_1$ implies that $\{z_2, z_3\}$ also meets one of these circuits. As $\{z_2, z_3\}$ avoids $\{u_0, b_0, a_1, b_1, a_2, b_2\}$, we deduce that $d_1 \in \{z_2, z_3\}$. Since $\{z_1, z_2, z_3\} \neq \{c_1, d_1, a_2\}$, orthogonality between $\{z_1, z_2, z_3\}$ and the cocircuit $\{d_1, a_2, d_2\}$ in $M \setminus c_0, c_1, c_2, v_0, v_1$ implies that $d_2 \in \{z_1, z_2, z_3\}$. Hence (ii) holds. This contradiction completes the proof of 7.2.17.

Now we show that

7.2.18. *M contains one of the configurations shown in Figure 19 where the elements in each part are distinct except that d_2 may equal w_k .*

As M has $\{\alpha, \beta, u_{k-1}\}$ as a triangle, by orthogonality, $\{\alpha, \beta\}$ meets $\{t_{k-2}, v_{k-2}\}$ in a single element. Orthogonality between D^* and the triangle $\{t_{k-2}, u_{k-2}, v_{k-2}\}$ implies that β avoids this triangle. Thus $\alpha \in \{t_{k-2}, v_{k-2}\}$.

Suppose $\alpha = t_{k-2}$. We know that M has a 4- or 5-cocircuit containing $\{t_{k-2}, u_{k-2}\}$. Thus β also meets this cocircuit, so β is in a triangle that violates orthogonality with D^* . Hence $\alpha = v_{k-2}$. Relabelling β as w_{k-2} , we get that M contains one of the configurations in Figure 3. Moreover, by orthogonality, β is not an existing element of Figure 18. Thus 7.2.18 holds.

Next we show that

7.2.19. *M contains one of the configurations shown in Figure 3 where all the elements in the figure are distinct except that d_2 may equal w_k .*

To prove this, we shall apply Lemma 6.5 to the configuration in M induced by $\{t_{k-2}, u_{k-2}, v_{k-2}, t_{k-1}, u_{k-1}, v_{k-1}, t_k, u_k, v_k, w_{k-2}, w_{k-1}, w_k\}$. By 7.2.4, M is not a quartic Möbius ladder and $\{w_{k-1}, w_k\}$ is not contained in a triangle of M , so neither (iv) nor (ii) of Lemma 6.5 holds. Also (i) of that lemma does not hold as $N \preceq M \setminus v_{k-2}, v_{k-1}, v_k$, and M has no ladder win. Thus (iii) of Lemma 6.5 holds. But, by 7.2.14, $M \setminus v_k$ does not have a 4-fan that avoids v_{k-1} but has t_k as its coguts element. Thus $M \setminus v_{k-2}, v_{k-1}, v_k$ has a 4-fan $(y_1, y_2, u_{k-2}, w_{k-2})$ that is also a 4-fan of $M \setminus v_{k-2}$. Then $\{y_2, u_{k-2}, w_{k-2}, v_{k-2}\}$ is a cocircuit of M . Suppose first that $k = 2$. Then orthogonality between the last cocircuit and the circuit $\{u_0, b_0, c_0\}$ implies that $y_2 \in \{c_0, b_0\}$. Moreover, orthogonality between the specified cocircuit and the circuit $\{c_0, b_1, b_2\}$ implies that $y_2 \neq c_0$, so $y_2 = b_0$. Thus, when $k = 2$, we conclude using 7.2.18 that 7.2.19 holds.

Now suppose that $k \geq 3$. By orthogonality between the cocircuit $\{y_2, u_{k-2}, w_{k-2}, v_{k-2}\}$ and the triangle $\{u_{k-3}, v_{k-3}, t_{k-3}\}$, we deduce that $y_2 \notin \{v_{k-3}, t_{k-3}\}$. Moreover, by Lemma 6.3, $\{y_1, y_2\}$ avoids $\{t_{k-2}, u_{k-2}, v_{k-2}, t_{k-1}, u_{k-1}, v_{k-1}, t_k, u_k, v_k, w_{k-2}, w_{k-1}, w_k\}$. By orthogonality between the triangle $\{y_1, y_2, u_{k-2}\}$ and the cocircuit $\{u_{k-2}, t_{k-2}, v_{k-3}, t_{k-3}\}$, we deduce that $y_1 \in \{v_{k-3}, t_{k-3}\}$. Suppose $y_1 = t_{k-3}$. Then the triangle $\{t_{k-3}, y_2, u_{k-2}\}$ gives a contradiction to orthogonality unless $k \geq 4$ and $y_2 = v_{k-4}$. In the exceptional case, we get a contradiction to orthogonality between the triangle $\{t_{k-4}, u_{k-4}, v_{k-4}\}$ and the cocircuit $\{y_2, u_{k-2}, w_{k-2}, v_{k-2}\}$. We deduce that $y_1 = v_{k-3}$. We now relabel y_2 as w_{k-3} noting that, by orthogonality, it differs from the existing elements in Figure 19.

We now see that we have a configuration of the same form as the one to which we just applied Lemma 6.5, this new configuration being induced by $\{t_i, u_i, v_i, w_i : k-3 \leq i \leq k\}$. We apply Lemma 6.5 again and repeat this process until we find that M contains one of the configurations shown in Figure 3 where all of the elements $w_{k-3}, w_{k-4}, \dots, w_0$ added to Figure 19 are distinct and differ from the existing elements in the figure. We conclude that 7.2.19 holds.

By 7.2.3 and 7.2.2, we know that $M \setminus c_0, c_1, c_2, v_0, v_1, \dots, v_k$ is sequentially 4-connected with an N -minor. If $M \setminus c_0, c_1, c_2, v_0, v_1, \dots, v_k$ is internally 4-connected, then (v) of the lemma holds; a contradiction. Thus we may assume that $M \setminus c_0, c_1, c_2, v_0, v_1, \dots, v_k$ has a 4-fan (y_1, y_2, y_3, y_4) .

Suppose first that (y_1, y_2, y_3, y_4) is a 4-fan of $M \setminus v_0, v_1, \dots, v_k$. Then it follows by Lemma 6.5 and symmetry, either $(y_3, y_4) = (u_0, w_0)$ and (y_1, y_2, y_3, y_4) is a 4-fan of $M \setminus v_0$; or $y_4 = t_k$ and (y_1, y_2, y_3, y_4) is a 4-fan of $M \setminus v_k$. In the latter case, M has $\{y_2, y_3, t_k, v_k\}$ as a cocircuit, so M has $(\{t_k, u_k, v_k\}, \{y_1, y_2, y_3\}, \{y_2, y_3, t_k, v_k\})$ as a bowtie, a contradiction to 7.2.4. In the latter case, by 7.2.12, $\{y_1, y_2, y_3\} = \{c_0, b_0, u_0\}$, so (y_1, y_2, y_3, y_4) is not a 4-fan of $M \setminus c_0, c_1, c_2, v_0, v_1, \dots, v_k$; a contradiction. We conclude that (y_1, y_2, y_3, y_4) is not a 4-fan of $M \setminus v_0, v_1, \dots, v_k$.

Next we observe that

7.2.20. $\{y_1, y_2, y_3\}$ avoids $\{b_0, a_1, b_1, a_2, b_2, d_1, u_0\}$.

To see this, we observe that $b_0 \notin \{y_1, y_2, y_3\}$ otherwise, by orthogonality with the vertex cocircuits in $M \setminus c_0, c_1, c_2, v_0, v_1, \dots, v_k$ shown in Figure 3, we get the contradiction that $\{y_1, y_2, y_3\}$ has at least four elements. Next, orthogonality with the cocircuit $\{a_1, b_1, b_0\}$ implies that $\{a_1, b_1\}$ avoids $\{y_1, y_2, y_3\}$. Similarly, orthogonality with the cocircuit $\{b_1, b_2, a_2\}$ implies that $\{a_2, b_2\}$ avoids $\{y_1, y_2, y_3\}$. Moreover, as (ii) of the lemma does not hold, d_1 avoids $\{y_1, y_2, y_3\}$. Finally, if

$u_0 \in \{y_1, y_2, y_3\}$, then orthogonality with the cocircuit $\{b_2, u_0, t_0\}$ implies that $\{y_1, y_2, y_3\} = \{u_0, t_0, v_0\}$; a contradiction. Hence 7.2.20 holds.

Now M has a cocircuit C^* such that $\{y_2, y_3, y_4\} \subsetneq C^* \subseteq \{y_2, y_3, y_4, c_0, c_1, c_2, v_0, v_1, \dots, v_k\}$. As (y_1, y_2, y_3, y_4) is not a 4-fan in $M \setminus \{v_0, v_1, \dots, v_k\}$, it follows that C^* meets $\{c_0, c_1, c_2\}$. Suppose c_0 is in C^* . Then, by orthogonality, $\{y_2, y_3, y_4\}$ meets $\{b_1, b_2\}$ and $\{u_0, b_0\}$, so $\{y_2, y_3\}$ meets $\{b_0, b_1, b_2, u_0\}$; a contradiction to 7.2.20. Likewise, if c_1 is in C^* , then $\{y_2, y_3, y_4\}$ meets $\{a_1, b_1\}$ and $\{d_1, a_2\}$, so $\{y_2, y_3\}$ meets $\{a_1, b_1, a_2, d_1\}$; a contradiction. Thus $c_2 \in C^*$ and $\{y_2, y_3, y_4\}$ meets $\{a_2, b_2\}$ and $\{u_0, b_0, a_1, d_1\}$. This final contradiction to 7.2.20 completes the proof of 7.2.15.

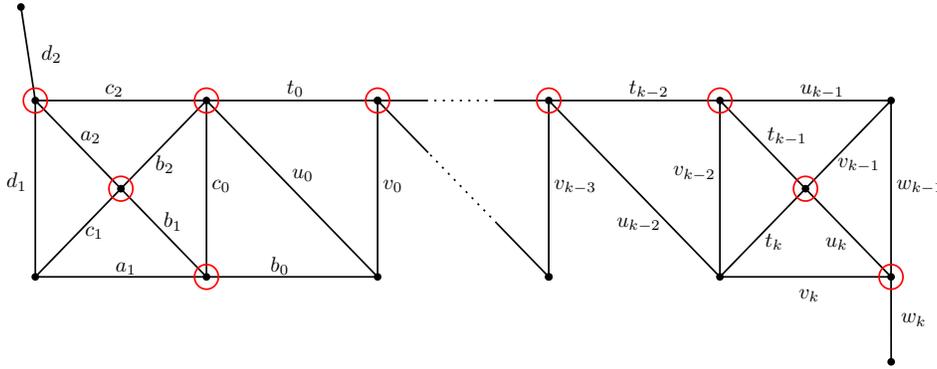


FIGURE 20. $k \geq 2$

Although 7.2.17 showed that $k \geq 2$, that proof was embedded in the proof-by-contradiction of 7.2.15. Temporarily, all we know is that, by 7.2.7, $k \geq 1$. By 7.2.15, we may assume that we have the configuration in Figure 18 and that $\{t_{k-1}, t_k\}$ is contained in a triangle of M . Then $M \setminus \{v_{k-1}\}$ is not $(4, 4, S)$ -connected, so 7.2.13 implies that $k \geq 2$ and, by orthogonality, we have the configuration shown in Figure 20. Now 7.2.10 implies that the elements in Figure 20 are all distinct except that possibly $w_k = d_2$.

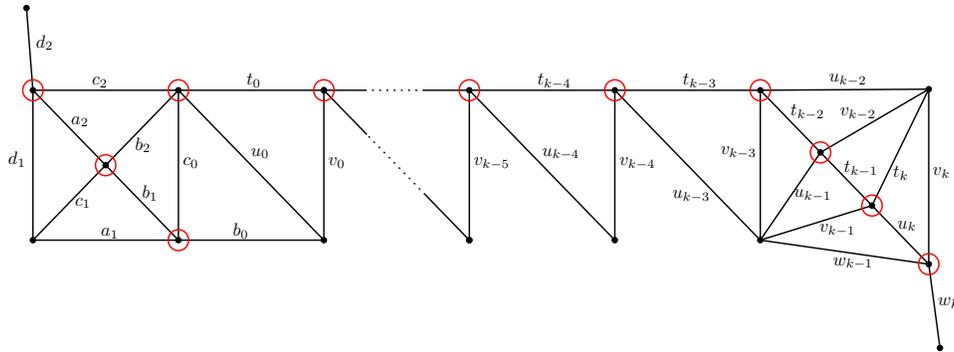


FIGURE 21. The structure that arises if $\{t_{k-2}, u_{k-1}\}$ is contained in a triangle.

Next we show that

7.2.21. $\{t_{k-2}, u_{k-1}\}$ is not contained in a triangle of M .

Suppose that $\{t_{k-2}, u_{k-1}\}$ is contained in a triangle. Then orthogonality implies that $k \geq 3$ and that this triangle contains v_{k-3} , so M contains the configuration shown in Figure 21. This configuration is contained in a right-maximal rotor chain of the form $((v_k, u_k, t_k), (v_{k-1}, t_{k-1}, u_{k-1}), (v_{k-2}, t_{k-2}, u_{k-2}), \dots, (v_\ell, t_\ell, u_\ell))$, where $\ell \leq k-2$. Moreover, $\ell \geq 0$ since orthogonality implies that $\{t_0, u_1\}$ is not in a triangle of M . Lemma 7.1 implies that $M \setminus v_\ell, v_{\ell+1}, \dots, v_k$ is sequentially 4-connected. As $M \setminus v_\ell, v_{\ell+1}, \dots, v_k$ has an N -minor, it is not internally 4-connected otherwise M has an open-rotor-chain win; a contradiction. Let (y_1, y_2, y_3, y_4) be a 4-fan in $M \setminus v_\ell, v_{\ell+1}, \dots, v_k$.

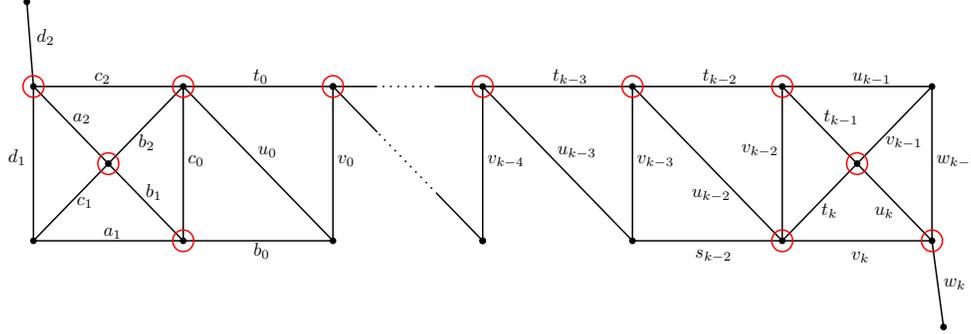
Suppose $\{y_1, y_2, y_3\}$ meets $\{t_\ell, u_{\ell+1}, t_{\ell+1}, \dots, u_k, t_k, w_{k-1}, w_k\}$. Every element in the last set is in a triad of $M \setminus v_\ell, v_{\ell+1}, \dots, v_k$. Orthogonality implies that $\{y_1, y_2, y_3\}$ contains $\{w_{k-1}, w_k\}$ or $\{t_\ell, u_{\ell+1}\}$. The former gives a contradiction to 7.2.4, so we assume the latter. Then $M \setminus v_\ell$ is not $(4, 4, S)$ -connected, so, by 7.2.13, $\ell \geq 1$. Orthogonality implies that $\{y_1, y_2, y_3\}$ is $\{v_{\ell-1}, t_\ell, u_{\ell+1}\}$. By the maximality of the rotor chain, we know that $((v_k, u_k, t_k), (v_{k-1}, t_{k-1}, u_{k-1}), \dots, (v_\ell, t_\ell, u_\ell), (v_{\ell-1}, t_{\ell-1}, u_{\ell-1}))$ is not a rotor chain. Hence these elements are not distinct; a contradiction to 7.2.10. We conclude that $\{y_1, y_2, y_3\}$ avoids $\{t_\ell, u_{\ell+1}, t_{\ell+1}, \dots, u_k, t_k, w_{k-1}, w_k\}$.

Now M has a cocircuit C^* with $\{y_2, y_3, y_4\} \subsetneq C^* \subseteq \{y_2, y_3, y_4, v_\ell, v_{\ell+1}, \dots, v_k\}$. For each i in $\{\ell+1, \ell+2, \dots, k-1\}$, the element v_i is in two circuits whose other elements are contained in $\{u_{\ell+1}, t_{\ell+1}, \dots, u_k, t_k, w_{k-1}\}$. Thus, by orthogonality, $v_i \notin C^*$. Hence $C^* \subseteq \{y_2, y_3, y_4, v_\ell, v_k\}$.

Suppose $v_\ell \in C^*$. Then, by orthogonality, $u_\ell \in \{y_2, y_3\}$ and either $\ell \leq k-3$ and $y_4 \in \{t_{\ell+1}, u_{\ell+2}\}$, or $\ell = k-2$ and $y_4 \in \{t_{k-1}, t_k\}$. But $y_4 \notin \{t_{\ell+1}, u_{\ell+2}\}$ otherwise, by orthogonality, $u_{\ell+1}$ or $t_{\ell+2}$ is in $\{y_2, y_3\}$; a contradiction. Thus $\ell = k-2$. Now $y_4 \neq t_{k-1}$ otherwise, by orthogonality, $t_k \in \{y_2, y_3\}$; a contradiction. We conclude that $y_4 = t_k$, so, as $u_k \notin \{y_2, y_3\}$, we deduce that $v_k \in C^*$. Now, without loss of generality, $u_{k-2} = y_3$. The triangle $\{y_1, y_2, u_{k-2}\}$ meets the cocircuit $\{t_{k-3}, v_{k-3}, t_{k-2}, u_{k-2}\}$ so $\{y_1, y_2\}$ meets $\{t_{k-3}, v_{k-3}, t_{k-2}\}$. But $t_{k-2} \notin \{y_1, y_2\}$ and, using orthogonality, we see that $t_{k-3} \notin \{y_1, y_2\}$. Thus $v_{k-3} \in \{y_1, y_2\}$. By orthogonality between $\{t_{k-3}, u_{k-3}, v_{k-3}\}$ and C^* , we deduce that $v_{k-3} \neq y_2$. Thus $v_{k-3} = y_1$. Now let $Z = \{y_2, v_{k-3}, w_{k-1}\} \cup \{u_i, t_i, v_i : k-2 \leq i \leq k\}$. Then one easily checks that $\lambda(Z) \leq 2$ and $|E - Z| \geq 4$. This contradiction implies that $v_\ell \notin C^*$.

We now conclude that $C^* = \{y_2, y_3, y_4, v_k\}$. Then, by orthogonality, C^* meets $\{t_k, u_k\}$. But $\{y_1, y_2, y_3\}$ avoids the last set, so $y_4 \in \{t_k, u_k\}$. Then orthogonality implies that $\{w_{k-1}, t_{k-1}\}$ meets $\{y_2, y_3\}$; a contradiction. We conclude that 7.2.21 holds.

We now know that M contains the configuration in Figure 20 but not that in Figure 21. We apply Lemma 6.2 to the structure in Figure 20 induced by $\{t_{k-2}, v_{k-2}, t_{k-1}, u_{k-1}, v_{k-1}, t_k, u_k, v_k, w_{k-1}, w_k\}$. Part (i) of that lemma does not hold by assumption, and parts (ii) and (iv) do not hold by 7.2.4 and 7.2.21. If M has a 4-cocircuit containing $\{v_{k-2}, t_k, v_k\}$, then orthogonality implies that the fourth element of this cocircuit is in $\{t_{k-2}, u_{k-2}\}$, so $\lambda(\{t_{k-2}, u_{k-2}, v_{k-2}, t_{k-1}, u_{k-1}, v_{k-1}, t_k, u_k, v_k\}) \leq 2$; a contradiction. We deduce that part (v) of Lemma 6.2 does not hold. Thus part (iv) of that lemma holds,

FIGURE 22. $k \geq 2$

that is, M has a triangle $\{y_1, y_2, y_3\}$ such that $\{v_{k-2}, t_k, v_k, y_2, y_3\}$ is a cocircuit. Orthogonality with the triangle $\{t_{k-2}, u_{k-2}, v_{k-2}\}$ implies that $\{y_2, y_3\}$ meets $\{t_{k-2}, u_{k-2}\}$. But 7.2.21 and orthogonality with the cocircuits containing t_{k-2} imply that $u_{k-2} \in \{y_2, y_3\}$. Then orthogonality implies, without loss of generality, that (y_1, y_2, y_3) is either (c_0, b_0, u_0) when $k = 2$, or is $(v_{k-3}, s_{k-2}, u_{k-2})$ for some element s_{k-2} when $k \geq 3$. We now have the configuration shown in Figure 22.

It may be that M has an element s_{k-3} such that $\{v_{k-4}, u_{k-3}, s_{k-3}\}$ is a triangle and $\{s_{k-3}, u_{k-3}, v_{k-3}, s_{k-2}\}$ is a cocircuit. If there is no such triangle and cocircuit, then let $\ell = k - 2$; otherwise let ℓ be the smallest positive integer such that $(\{u_{k-2}, s_{k-2}, v_{k-3}\}, \{s_{k-2}, v_{k-3}, u_{k-3}, s_{k-3}\}, \{u_{k-3}, s_{k-3}, v_{k-4}\}, \dots, \{u_\ell, s_\ell, v_{\ell-1}\})$ is a string of bowties. Hence M contains the structure shown in Figure 23.

Next we show the following.

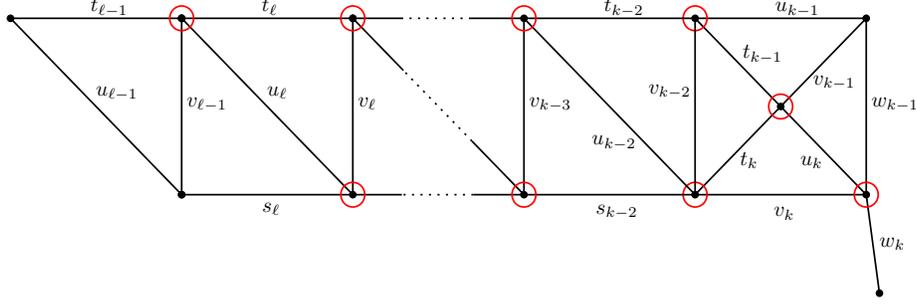
7.2.22. *The elements in Figure 18 and in $\{s_\ell, s_{\ell+1}, \dots, s_{k-2}\}$ are all distinct, with the possible exception that w_k may be the same as d_2 .*

By 7.2.10, the elements in Figure 18 are distinct except that w_k may equal d_2 . By the definition of a string of bowties, the elements in $\{s_\ell, s_{\ell+1}, \dots, s_{k-2}\}$ are distinct. Suppose that one such s_i is an element of Figure 18. As every element of that figure is in a circled cocircuit, s_i is in such a cocircuit C^* . As M has $\{s_i, v_{i-1}, u_i\}$ as a triangle, by orthogonality, $\{v_{i-1}, u_i\}$ meets C^* . Thus $C^* = \{t_{i-1}, v_{i-1}, u_i, t_i\}$, so C^* contains the triangle $\{s_i, v_{i-1}, u_i\}$ contradicting the fact that M is binary. We conclude that 7.2.22 holds.

We show next that

7.2.23. *$M \setminus v_{\ell-1}, v_\ell, \dots, v_k$ is sequentially 4-connected.*

Assume that this is false. By 7.2.3, $M \setminus c_0, c_1, c_2, v_0, v_1, \dots, v_k$ is sequentially 4-connected. Thus $M \setminus v_j, v_{j+1}, \dots, v_k$ is 3-connected for all j in $\{0, 1, \dots, k\}$. Lemma 7.1 implies that $M \setminus v_{k-2}, v_{k-1}, v_k$ is sequentially 4-connected. Let i be the largest integer in $\{0, 1, \dots, k-2\}$ such that $M \setminus v_i, v_{i+1}, \dots, v_k$ is not sequentially 4-connected. Clearly $i \geq \ell - 1$. Let (X, Y) be a non-sequential 3-separation in $M \setminus v_i, v_{i+1}, \dots, v_k$. Suppose $i \geq 1$. Then, without loss of generality, the triangle $\{t_{i-1}, u_{i-1}, v_{i-1}\} \subseteq X$. If t_i or u_i is in X , then we may assume that both are in X , and then $(X \cup v_i, Y)$ is a non-sequential 3-separation of $M \setminus v_{i+1}, v_{i+2}, \dots, v_k$; a contradiction. Thus $\{t_i, u_i\} \subseteq Y$, and $(X, Y \cup v_i)$ is a non-sequential 3-separation in

FIGURE 23. $k \geq 2$ and $1 \leq \ell \leq k - 2$

$M \setminus v_{i+1}, v_{i+2}, \dots, v_k$; a contradiction. We conclude that $i = 0$, so $\ell = 1$. Without loss of generality, $T_2 \subseteq X$. Recall that, because of relabelling, T_0 is now $\{u_0, b_0, c_0\}$. If $T_0 \subseteq X$, then $t_0 \in \text{cl}_{M \setminus v_0, v_1, \dots, v_k}^*(X)$, so $(X \cup t_0 \cup v_0, Y - t_0)$ is a non-sequential 3-separation of $M \setminus v_1, v_2, \dots, v_k$; a contradiction. Thus we may assume that $T_0 \subseteq Y$. If $b_1 \in X$, then we can assume that c_1 and a_1 are in X . Then we can move c_0, s_0 , and u_0 into X ; a contradiction. Thus $b_1 \in Y$, and we can move b_2, a_1, c_1, a_2, c_2 , and t_0 into Y and add v_0 to again get the contradiction that $M \setminus v_1, v_2, \dots, v_k$ has a non-sequential 3-separation. We conclude that 7.2.23 holds.

Now suppose $M \setminus v_{\ell-1}, v_{\ell}, \dots, v_k$ is internally 4-connected. Consider the structure in Figure 23. By removing $u_{\ell-1}$, we see that the structure is, after a rotation, the same as that in Figure 3(a), and deleting $\{v_{\ell-1}, v_{\ell}, \dots, v_k\}$ is an enhanced ladder win. Hence (v) holds; a contradiction.

We now know that $M \setminus v_{\ell-1}, v_{\ell}, \dots, v_k$ is not internally 4-connected, so it has a 4-fan (y_1, y_2, y_3, y_4) . Using orthogonality together with 7.2.4 and 7.2.21, we get that $\{y_1, y_2, y_3\}$ avoids the elements in Figure 23 with the possible exception of $u_{\ell-1}$. The matroid M has a cocircuit C^* such that $\{y_2, y_3, y_4\} \subsetneq C^* \subseteq \{y_2, y_3, y_4, v_{\ell-1}, v_{\ell}, v_{\ell+1}, \dots, v_k\}$. We show next that

7.2.24. $C^* = \{y_2, y_3, y_4, v_{\ell-1}\}$ and $y_4 = s_{\ell}$, while $u_{\ell-1} \in \{y_2, y_3\}$.

For each i in $\{\ell, \ell + 1, \dots, k - 1\}$, the element v_i is in two triangles in Figure 23, neither of which meets $u_{\ell-1}$ or any other v_j . Thus if $v_i \in C^*$, then $\{y_2, y_3, y_4, v_i\}$ meets each of these triangles in two elements. Hence $\{y_2, y_3\}$ contains an element in Figure 23 other than $u_{\ell-1}$; a contradiction. Moreover, $v_k \notin C^*$ otherwise orthogonality implies that $\{y_2, y_3, y_4\}$ meets both $\{t_k, u_k\}$ and $\{u_{k-2}, t_{k-2}, u_{k-1}, w_{k-1}\}$; a contradiction. We conclude that $C^* = \{y_2, y_3, y_4, v_{\ell-1}\}$. Then orthogonality with $\{v_{\ell-1}, s_{\ell}, u_{\ell}\}$ implies that $y_4 \in \{s_{\ell}, u_{\ell}\}$ since $\{y_2, y_3\}$ avoids $\{s_{\ell}, u_{\ell}\}$. As $\{y_2, y_3\}$ also avoids $\{t_{\ell}, u_{\ell}, v_{\ell}\}$, we deduce that $y_4 = s_{\ell}$. Orthogonality between C^* and $\{t_{\ell-1}, u_{\ell-1}, v_{\ell-1}\}$ implies that $u_{\ell-1} \in \{y_2, y_3\}$. Thus 7.2.24 holds.

Without loss of generality, $(y_1, y_2, y_3, y_4) = (y_1, y_2, u_{\ell-1}, s_{\ell})$. If $\ell > 1$, then orthogonality between $\{y_1, y_2, y_3\}$ and the vertex cocircuits in Figure 18 implies that $y_1 = v_{\ell-2}$. This means that we can extend the string of bowties $(\{u_{k-2}, s_{k-2}, v_{k-3}\}, \{s_{k-2}, v_{k-3}, u_{k-3}, s_{k-3}\}, \{u_{k-3}, s_{k-3}, v_{k-4}\}, \dots, \{u_{\ell}, s_{\ell}, v_{\ell-1}\})$, which contradicts our choice of ℓ . Hence $\ell = 1$ and orthogonality implies that $(y_1, y_2, u_0, s_1) = (c_0, y_2, u_0, s_1)$. Thus we have the structure shown in Figure 4, where y_2 is s_0 , which is equal to b_0 relabelled.

We will complete the proof of Lemma 7.2 by showing that $M \setminus \{c_0, c_1, c_2, v_0, \dots, v_k\}$ is internally 4-connected. Since, by 7.2.2, the last matroid has an N -minor, this will establish that part (v) of the lemma holds; a contradiction. By 7.2.22, all of the elements in Figure 4 are distinct with the possible exception that d_2 may be w_k . We know, by 7.2.3, that $M \setminus \{c_0, c_1, c_2, v_0, \dots, v_k\}$ is sequentially 4-connected. Let (y_1, y_2, y_3, y_4) be a 4-fan in this matroid. Suppose $\{y_1, y_2, y_3\}$ meets the elements in Figure 4. The vertex cocircuits imply that $\{y_1, y_2, y_3\}$ contains $\{d_1, d_2\}$ or $\{w_{k-1}, w_k\}$. If the former occurs, then part (ii) of the lemma holds; a contradiction. The latter gives a contradiction to 7.2.4. Thus $\{y_1, y_2, y_3\}$ avoids the elements in Figure 4. Now M has a cocircuit C^* such that $\{y_2, y_3, y_4\} \subsetneq C^* \subseteq \{y_2, y_3, y_4, c_0, c_1, c_2, v_0, \dots, v_k\}$. Each element in $\{c_0, c_1, v_0, v_1, \dots, v_{k-1}\}$ is in two triangles in Figure 4. Thus C^* avoids the last set otherwise orthogonality implies that $\{y_2, y_3\}$ contains an element of Figure 4; a contradiction. Moreover, $v_k \notin C^*$ otherwise orthogonality implies that $\{y_2, y_3, y_4\}$ meets both $\{t_k, u_k\}$ and $\{u_{k-2}, t_{k-2}, u_{k-1}, w_{k-1}\}$; a contradiction. By symmetry, $c_2 \notin C^*$. Hence $C^* = \{y_2, y_3, y_4\}$. This contradiction completes the proof that $M \setminus \{c_0, c_1, c_2, v_0, \dots, v_k\}$ is internally 4-connected thereby finishing the proof of Lemma 7.2. \square

8. MORE ON STRINGS OF BOWTIES

When we deal with a string of bowties that does not wrap around on itself, two situations that arise frequently are shown in Figure 24. The next lemma shows that, when one of these situations arises, the main theorem holds. This is another technical lemma although its proof is not as long as that of the preceding lemma. We continue to follow the practice of using T_i to denote the triangle $\{a_i, b_i, c_i\}$.

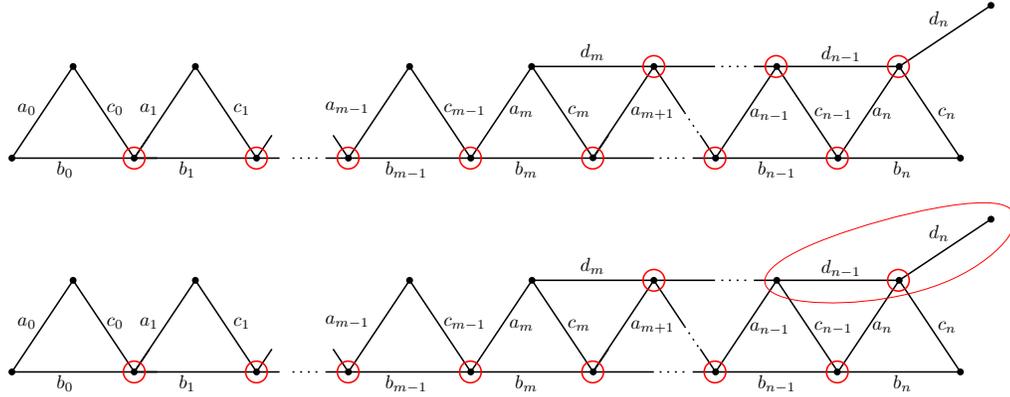


FIGURE 24. Here $0 < m < n$.

Lemma 8.1. *Let M and N be internally 4-connected binary matroids with $|E(M)| \geq 13$ and $|E(N)| \geq 7$. Let M have a right-maximal bowtie string $T_0, D_0, T_1, D_1, \dots, T_n$ and suppose that this string is contained in one of the structures shown in Figure 24, where $0 < m < n$. Suppose that $M \setminus c_0$ is $(4, 4, S)$ -connected, that $M \setminus \{c_0, c_1, \dots, c_n\}$ has an N -minor, and that $M \setminus \{c_0, c_1, b_1\}$ has no N -minor. Suppose that M has no bowtie of the form $(T_n, T_{n+1}, \{x, c_n, a_{n+1}, b_{n+1}\})$, where $x \in \{a_n, b_n\}$, and M has no element d_{m-1} such that $\{c_{m-1}, d_{m-1}, a_m\}$*

is a triangle and either $\{d_{m-1}, a_m, c_m, d_m\}$ is a cocircuit, or $m + 1 = n$ and $\{d_{m-1}, a_m, c_m, a_{m+1}, c_{m+1}\}$ is a cocircuit. Then

- (i) M has a quick win; or
- (ii) M has an open-rotor-chain win or a ladder win; or
- (iii) M has an enhanced-ladder win.

Proof. We assume that neither (i) nor (ii) holds. First we show:

8.1.1. *The elements in each part of Figure 24 are distinct except that (a_0, b_0) may be (c_n, d_n) .*

As $T_0, D_0, T_1, D_1, \dots, T_n$ is a bowtie string, we know that the elements in $T_0 \cup T_1 \cup \dots \cup T_n$ are all distinct except that a_0 may be c_n .

To help prove 8.1.1, next we show the following.

8.1.2. *The elements of $T_m \cup T_{m+1} \cup \dots \cup T_n \cup \{d_m, d_{m+1}, \dots, d_n\}$ are distinct.*

Assume that this fails. Then, by Lemma 6.4, (a_m, b_m) is (c_n, d_n) or (d_{n-1}, d_n) . The first possibility implies that $m = 0$; a contradiction. The second possibility contradicts the hypothesis precluding any bowtie of the form $(T_n, T_{n+1}, \{x, c_n, a_{n+1}, b_{n+1}\})$ for x in $\{a_n, b_n\}$. We conclude that 8.1.2 holds.

Next we establish the following.

8.1.3. *If $d_j \in T_i$ for some i in $\{0, 1, \dots, m-1\}$ and some j in $\{m, m+1, \dots, n-1\}$, then $d_j = a_0$, so $a_0 \neq c_n$ and $\{d_m, d_{m+1}, \dots, d_{n-1}\}$ avoids $\{b_0, c_0\} \cup T_1 \cup T_2 \cup \dots \cup T_n$.*

Suppose d_j meets one of D_0, D_1, \dots, D_{m-1} . Then orthogonality implies that $\{c_j, a_{j+1}\}$ meets the same cocircuit; a contradiction. Thus $d_j = a_0$. Hence, by 8.1.2, $a_0 \neq c_n$, and 8.1.3 follows.

As the next step towards showing that 8.1.1 holds, we show that

8.1.4. *$\{d_m, d_{m+1}, \dots, d_{n-1}\}$ avoids $T_0 \cup T_1 \cup \dots \cup T_n$.*

Assume that this fails. Then, by 8.1.3, $d_j = a_0$ for some j in $\{m, m+1, \dots, n-1\}$, and $a_0 \neq c_n$. Now $\{d_j, a_{j+1}, c_{j+1}, d_{j+1}\}$ or $\{d_j, a_{j+1}, c_{j+1}, a_{j+2}, c_{j+2}\}$ is a cocircuit D^* . By orthogonality between D^* and T_0 , we deduce that $D^* = \{d_j, a_{j+1}, c_{j+1}, d_{j+1}\}$ and $d_{j+1} \in T_0$. If $j < n-1$, then, by 8.1.3, $d_{j+1} = a_0$. This contradicts the fact that $d_j = a_0$. Hence $j = n-1$, so $d_n \in \{b_0, c_0\}$. Therefore $(T_n, T_0, \{a_n, c_n, a_0, d_n\})$ is a bowtie; a contradiction. Thus 8.1.4 holds.

To complete the proof of 8.1.1, we need to consider the possibility that $d_n \in T_i$ for some i in $\{0, 1, \dots, m-1\}$. In that case, by orthogonality between T_i and $\{d_{n-1}, a_n, c_n, d_n\}$ using the fact that d_{n-1} avoids T_i , we deduce that $\{a_n, c_n\}$ meets T_i . Thus $i = 0$ and $a_0 = c_n$. Therefore $d_n \in \{b_0, c_0\}$. If $d_n = c_0$, then $M \setminus c_0, c_1, \dots, c_n$ has $\{d_{n-1}, a_n\}$ as a cocircuit. Hence $M \setminus c_0, c_1, \dots, c_n / a_n$ has an N -minor. But, since $M \setminus c_0, c_1 / b_1$ has no N -minor, this yields a contradiction to Lemma 5.7. We deduce that $d_n = b_0$, so $(a_0, b_0) = (c_n, d_n)$. We conclude that 8.1.1 holds.

Next we show that

8.1.5. *M has no triangle $\{\alpha, \beta, a_m\}$ such that $\{\beta, a_m, c_m, d_m\}$ or $\{\beta, a_m, c_m, a_{m+1}, c_{m+1}\}$ is a cocircuit.*

Assume, instead, that M does have such a triangle and a cocircuit, calling the latter D^* . Suppose $\{\alpha, \beta\}$ meets $\{b_m, c_m\}$. Then $\{\alpha, \beta\} = \{b_m, c_m\}$ and so $\beta = c_m$

otherwise the cocircuit D^* contains a triangle. But then $T_m \cup d_m$ or $T_m \cup T_{m+1}$ is 3-separating in M ; a contradiction. We conclude that $\{\alpha, \beta\}$ avoids $\{b_m, c_m\}$. Orthogonality between $\{\alpha, \beta, a_m\}$ and the cocircuit $\{b_{m-1}, c_{m-1}, a_m, b_m\}$ implies that $\{\alpha, \beta\}$ meets $\{b_{m-1}, c_{m-1}\}$.

Suppose $\beta \in \{b_{m-1}, c_{m-1}\}$. Then, using 8.1.1 and orthogonality between T_{m-1} and D^* , we see that $D^* = \{\beta, a_m, c_m, a_{m+1}, c_{m+1}\}$, that $m+1 = n$, and that $c_{m+1} = a_{m-1}$. Thus $m = 1$ and $n = 2$. Moreover, by 8.1.1, $d_2 = b_0$. Thus $T_1 \cup T_2 \cup \{d_1, d_2, c_0\}$ contains T_0 and is 3-separating; a contradiction. Therefore $\beta \notin \{b_{m-1}, c_{m-1}\}$. We deduce that $\alpha \in \{b_{m-1}, c_{m-1}\}$.

Suppose $\alpha = c_{m-1}$. Then it is not difficult to check, by taking $\beta = d_{m-1}$, that we violate the hypotheses of the lemma unless $m-1 < n$ and $\{\beta, a_m, c_m, a_{m+1}, c_{m+1}\}$ is a cocircuit. In the exceptional case, the triangle $\{c_{m+1}, d_{m+1}, a_{m+2}\}$ implies that $\beta \in \{d_{m+1}, a_{m+2}\}$. But orthogonality using the triangle $\{\alpha, \beta, a_m\}$ and the cocircuits $\{d_m, a_{m+1}, c_{m+1}, d_{m+1}\}$ and D_{m+1} gives a contradiction. Hence $\alpha \neq c_{m-1}$, so $\alpha = b_{m-1}$. Thus $(\beta, b_{m-1}, a_m, b_m, c_m)$ is a 5-fan in $M \setminus c_{m-1}$, so $m-1 > 0$. Then orthogonality between $\{b_{m-1}, \beta, a_m\}$ and the cocircuit $\{b_{m-2}, c_{m-2}, a_{m-1}, b_{m-1}\}$ implies that $\beta \in \{b_{m-2}, c_{m-2}\}$. By orthogonality between T_{m-2} and D^* , we see that T_{m-2} meets $\{a_m, c_m, d_m\}$ or $\{a_m, c_m, a_{m+1}, c_{m+1}\}$. In the first case, by 8.1.1, $m = 2$ and $(a_0, b_0) = (c_2, d_2)$, so $T_0 \subseteq D^*$; a contradiction. We conclude that T_{m-2} meets $\{a_m, c_m, a_{m+1}, c_{m+1}\}$. Then, by 8.1.1 again, we deduce that $m-2 = 0$ and $a_0 = c_3$. Hence $b_0 = d_3$, and $T_2 \cup T_3 \cup \{d_2, d_3, c_0\}$ contains T_0 and D^* . Thus $\lambda(T_2 \cup T_3 \cup \{d_2, d_3, c_0\}) \leq 2$; a contradiction. We conclude that 8.1.5 holds.

Let $S = \{c_m, c_{m+1}, \dots, c_n\}$. Next we observe that

8.1.6. $M \setminus c_n$ does not have a 4-fan, and $\{d_{n-1}, d_n\}$ is not contained in a triangle of M .

To see this, we note that the first assertion is an immediate consequence of the assumption that M has no bowtie of the form $(\{a_n, b_n, c_n\}, \{a_{n+1}, b_{n+1}, c_{n+1}\}, \{x, c_n, a_{n+1}, b_{n+1}\})$, where $x \in \{a_n, b_n\}$. The same assumption also gives the second assertion for it implies that if $\{d_{n-1}, d_n\}$ is contained in a triangle, then that triangle meets T_n , so $\lambda(T_n \cup \{d_{n-1}, d_n\}) \leq 2$; a contradiction.

Next we note that

8.1.7. $n = m + 1$.

Assume that $n > m + 1$. Then we apply Lemma 6.5. Part (i) of that lemma does not hold otherwise M has a ladder win and part (ii) of this lemma holds; a contradiction. By 8.1.6 and the hypothesis forbidding a certain bowtie, neither part (ii) nor part (iv) of Lemma 6.5 holds. Hence part (iii) of Lemma 6.5 holds; that is, $M \setminus S$ is $(4, 4, S)$ -connected and has a 4-fan that is a 4-fan of $M \setminus c_n$ or of $M \setminus c_m$. The first possibility is excluded by 8.1.6. Hence we may assume that $M \setminus S$ has a 4-fan (x_1, x_2, x_3, x_4) that is a 4-fan of $M \setminus c_m$. Then M has $\{x_2, x_3, x_4, c_m\}$ as a cocircuit and, by orthogonality, $\{x_2, x_3, x_4\}$ meets both $\{a_m, b_m\}$ and $\{d_m, a_{m+1}\}$. Thus $\{x_2, x_3\}$ meets $\{a_m, b_m, d_m, a_{m+1}\}$. Suppose $d_m \in \{x_1, x_2, x_3\}$. Lemma 6.3 implies that $m = n - 1$; a contradiction. Thus $d_m \notin \{x_1, x_2, x_3\}$. Furthermore, by Lemma 6.3, we know that $\{a_{m+1}, b_m\}$ avoids $\{x_2, x_3\}$. Thus $a_m \in \{x_2, x_3\}$. Hence $x_4 \in \{d_m, a_{m+1}\}$ and, without loss of generality, $a_m = x_3$. If $x_4 = d_m$, then we obtain a contradiction to 8.1.5. Thus $x_4 = a_{m+1}$. Then orthogonality

between $\{x_2, a_m, a_{m+1}, c_m\}$ and T_{m+1} implies that $x_2 = b_{m+1}$; a contradiction to Lemma 6.3. We conclude that 8.1.7 holds.

We now show that

8.1.8. *M has $\{b_m, b_{m+1}, c_{m-1}\}$ as a triangle.*

To prove this, we apply Lemma 6.1 using a similar argument to that given to prove 8.1.7. The hypotheses of the current lemma, the assumption that (i) of the current lemma does not hold, and 8.1.6 and 8.1.5 imply that either M has a triangle containing $\{b_m, b_{m+1}\}$, or $M \setminus c_m, c_{m+1}$ has a 4-fan of the form (u_1, u_2, u_3, b_{m+1}) and M has a cocircuit C^* with $\{u_2, u_3, c_m, b_{m+1}\} \subseteq C^* \subseteq \{u_2, u_3, c_m, b_{m+1}, c_{m+1}\}$. Suppose the latter. Then orthogonality implies that $\{u_2, u_3\}$ meets $\{a_m, b_m\}$ and $\{d_m, a_{m+1}\}$, so $\lambda(T_m \cup T_{m+1} \cup d_m) \leq 2$; a contradiction. We deduce that $\{b_m, b_{m+1}\}$ is contained in a triangle of M . Let x be the third element of this triangle. Then orthogonality implies that $x \in \{b_{m-1}, c_{m-1}\}$. Suppose $x = b_{m-1}$. Then $(b_{m+1}, b_{m-1}, b_m, a_m, c_m)$ is a 5-fan in $M \setminus c_{m-1}$, so $m - 1 > 0$. Then $|\{b_{m-1}, b_m, b_{m+1}\} \cap \{b_{m-2}, c_{m-2}, a_{m-1}, b_{m-1}\}| = 1$; a contradiction to orthogonality. We conclude that $x = c_{m-1}$, so 8.1.8 holds.

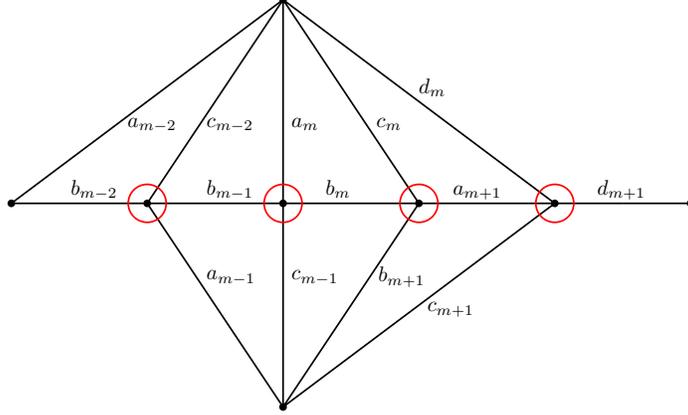


FIGURE 25

Next we show that

8.1.9. *$\{b_{m-1}, a_m\}$ is not contained in a triangle of M .*

Suppose that $\{b_{m-1}, a_m\}$ is contained in a triangle of M . Then $\{b_m, c_m\}$ is in a 5-fan in $M \setminus c_{m-1}$ with the triangle containing $\{b_{m-1}, a_m\}$. As $M \setminus c_0$ is $(4, 4, S)$ -connected, we deduce that c_{m-1} is not c_0 , so $m > 1$. Then orthogonality with the cocircuit $\{b_{m-2}, c_{m-2}, a_{m-1}, b_{m-1}\}$ implies that the triangle containing $\{b_{m-1}, a_m\}$ contains c_{m-2} or b_{m-2} . If $\{b_{m-2}, b_{m-1}, a_m\}$ is a triangle, then $(a_m, b_{m-2}, b_{m-1}, a_{m-1}, c_{m-1})$ is a 5-fan in $M \setminus c_{m-2}$, so $m - 2 > 0$. Thus $|\{b_{m-3}, c_{m-3}, a_{m-2}, b_{m-2}\} \cap \{b_{m-2}, b_{m-1}, a_m\}| = 1$; a contradiction to orthogonality. We deduce that $\{b_{m-2}, b_{m-1}, a_m\}$ is not a triangle, so $\{c_{m-2}, b_{m-1}, a_m\}$ is a triangle. Therefore M contains the configuration shown in Figure 25.

From this configuration, we see that $((c_{m+1}, a_{m+1}, b_{m+1}), (c_m, b_m, a_m), (c_{m-1}, b_{m-1}, a_{m-1}), (c_{m-2}, b_{m-2}, a_{m-2}))$ is a rotor chain in M . Extend this to a

right-maximal rotor chain $((c_{m+1}, a_{m+1}, b_{m+1}), (c_m, b_m, a_m), \dots, (c_k, b_k, a_k))$. Then $k \leq m - 2$. Moreover, since $M \setminus c_0$ is $(4, 4, S)$ -connected, $k \geq 0$. By Lemma 7.1, $M \setminus c_k, c_{k+1}, \dots, c_{m+1}$ is sequentially 4-connected. The last matroid has an N -minor and $n = m + 1$. Since M does not have an open-rotor chain win, we deduce that $M \setminus c_k, c_{k+1}, \dots, c_{m+1}$ is not internally 4-connected.

We now know that $M \setminus c_k, c_{k+1}, \dots, c_{m+1}$ has a 4-fan (y_1, y_2, y_3, y_4) . Suppose first $\{y_1, y_2, y_3\}$ meets $\{b_k, a_{k+1}, b_{k+1}, \dots, a_{m+1}, b_{m+1}, d_m, d_{m+1}\}$. Every element in the last set is in a triad of $M \setminus c_k, c_{k+1}, \dots, c_{m+1}$. Orthogonality implies that $\{y_1, y_2, y_3\}$ contains $\{d_m, d_{m+1}\}$ or $\{b_k, a_{k+1}\}$. The former is a contradiction to 8.1.6, so the latter holds. Then $M \setminus c_k$ is not $(4, 4, S)$ -connected, so $k > 0$. By orthogonality, M has $\{b_k, a_{k+1}, b_{k-1}\}$ or $\{b_k, a_{k+1}, c_{k-1}\}$ as a triangle. The first possibility implies, by orthogonality, that $k = 1$ and hence that $M \setminus c_0$ has a 5-fan; a contradiction. Thus M has $\{b_k, a_{k+1}, c_{k-1}\}$ as a triangle and therefore has $((c_{m+1}, a_{m+1}, b_{m+1}), (c_m, b_m, a_m), \dots, (c_{k-1}, b_{k-1}, a_{k-1}))$ as a rotor chain; a contradiction. We conclude that $\{y_1, y_2, y_3\}$ avoids $\{b_k, a_{k+1}, b_{k+1}, \dots, a_{m+1}, b_{m+1}, d_m, d_{m+1}\}$.

Now M has a cocircuit C^* such that $\{y_2, y_3, y_4\} \subsetneq C^* \subseteq \{y_2, y_3, y_4, c_k, c_{k+1}, \dots, c_{m+1}\}$. Take i in $\{k, k+1, \dots, m+1\}$ such that $c_i \in C^*$. Now c_i is in both T_i and another triangle in the rotor chain unless $i = m+1$. Since $\{y_1, y_2, y_3\}$ avoids $\{b_k, a_{k+1}, b_{k+1}, \dots, a_{m+1}, b_{m+1}, d_m, d_{m+1}\}$, orthogonality implies that either $C^* = \{y_2, y_3, y_4, c_k\}$ where $a_k \in \{y_2, y_3\}$ and $y_4 \in \{b_{k+1}, a_{k+2}\}$; or $C^* = \{y_2, y_3, y_4, c_{m+1}\}$ and $y_4 \in \{a_{m+1}, b_{m+1}\}$. Suppose the former. As y_4 is in $\{b_{k+1}, a_{k+2}\}$, orthogonality implies that a_{k+1} or b_{k+2} is in $\{y_2, y_3\}$; a contradiction. We deduce that $C^* = \{y_2, y_3, y_4, c_{m+1}\}$ and $y_4 \in \{a_{m+1}, b_{m+1}\}$. Then orthogonality implies that d_m or b_m is in $\{y_2, y_3\}$; a contradiction. We conclude that 8.1.9 holds.

At this point, it is worth noting the following.

8.1.10. *All of the elements in $T_{m-1} \cup T_m \cup T_{m+1} \cup \{d_m, d_{m+1}\}$ are distinct.*

To verify this, we simply need to check that $(a_{m-1}, b_{m-1}) \neq (c_{m+1}, d_{m+1})$, since then 8.1.1 will imply that 8.1.10 holds. Suppose instead that $(a_{m-1}, b_{m-1}) = (c_{m+1}, d_{m+1})$. Then T_{m-1} is contained in $T_m \cup T_{m+1} \cup \{d_m, d_{m+1}, c_{m-1}\}$, so the last set is 3-separating; a contradiction. Thus 8.1.10 holds.

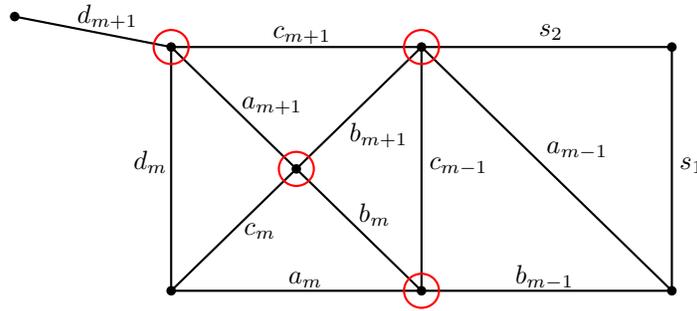


FIGURE 26

We now apply Lemma 6.2 to the configuration induced by $T_m \cup T_{m+1} \cup \{d_m, d_{m+1}, b_{m-1}, c_{m-1}\}$ noting that 8.1.6 and 8.1.9 eliminate the possibility that

(ii) or (iv) holds; and (i) does not hold otherwise (i) of the current lemma holds. Next we observe that (v) of Lemma 6.2 does not hold since if M has a 4-cocircuit containing $\{c_{m-1}, b_{m+1}, c_{m+1}\}$, then, by orthogonality, this cocircuit meets $\{a_{m-1}, b_{m-1}\}$ and so we obtain the contradiction that $\lambda(T_{m-1} \cup T_m \cup T_{m+1}) \leq 2$. We conclude that part (iii) of Lemma 6.2 holds, that is, M has a triangle $\{s_1, s_2, s_3\}$ where $\{c_{m-1}, c_{m+1}, b_{m+1}, s_2, s_3\}$ is a cocircuit, and $\{s_1, s_2, s_3\}$ avoids $\{b_m, c_m, c_{m-1}, a_{m+1}, b_{m+1}, c_{m+1}\}$. Orthogonality with T_{m-1} implies that $\{s_2, s_3\}$ meets $\{a_{m-1}, b_{m-1}\}$. If $b_{m-1} \in \{s_1, s_2, s_3\}$, then, by orthogonality, it follows that $\{b_{m-1}, a_m\} \subseteq \{s_1, s_2, s_3\}$; a contradiction to 8.1.9. Thus, without loss of generality, $a_{m-1} = s_3$, so M contains the configuration shown in Figure 26.

Now $M \setminus c_{m+1}, c_m, c_{m-1}$ has an N -minor and has $(s_1, s_2, a_{m-1}, b_{m+1})$ as a 4-fan. To enable us to apply Lemma 7.2, we show next that

8.1.11. $M \setminus c_{m+1}, c_m, c_{m-1} \setminus s_1$ has an N -minor.

Suppose first that $m = 1$. Then, by hypothesis, $M \setminus c_m, c_{m-1} / b_m$ has no N -minor. Thus $M \setminus c_{m+1}, c_m, c_{m-1} / b_{m+1}$ has no N -minor. It follows by Lemma 4.1 that 8.1.11 holds. We may now assume that $m \geq 2$. Then, by orthogonality between the cocircuit $\{b_{m-1}, a_{m-1}, b_{m-2}, c_{m-2}\}$ and the triangle $\{s_1, a_{m-1}, s_2\}$, we deduce that $\{s_1, s_2\}$ meets $\{b_{m-2}, c_{m-2}\}$. Orthogonality between T_{m-2} and the cocircuit $\{c_{m+1}, b_{m+1}, c_{m-1}, a_{m-1}, s_2\}$ implies that $s_2 \notin T_{m-2}$, so $s_1 \in \{b_{m-2}, c_{m-2}\}$. Suppose $s_1 = b_{m-2}$. Then $M \setminus c_{m-2}$ has a 5-fan, so $m - 2 > 0$. Then, by orthogonality, $s_2 \in \{b_{m-3}, c_{m-3}\}$ so we have a contradiction to orthogonality between T_{m-3} and the cocircuit $\{c_{m+1}, b_{m+1}, c_{m-1}, a_{m-1}, s_2\}$. We conclude that $s_1 = c_{m-2}$. Then, by assumption, $M \setminus c_{m+1}, c_m, c_{m-1}, s_1$ has an N -minor, that is, 8.1.11 holds.

We can now apply Lemma 7.2 noting that 8.1.6 and 8.1.9 eliminate the possibility that part (ii) or (iii) of that lemma holds. Also, by assumption, part (i) and part (iv) of Lemma 7.2 do not hold. Finally, if part (v) of Lemma 7.2 holds, then part (iii) of the current lemma holds where, here, the triple (c_{m-1}, c_m, c_{m+1}) plays the role of the triple (c_0, c_1, c_2) in the configurations in Figure 3 and Figure 4, and $M \setminus c_{m-1}, c_m, c_{m+1}, v_0, v_1, \dots, v_k$ is internally 4-connected with an N -minor. \square

9. WHEN ROTOR CHAINS END

In this section, we specify, in Lemma 9.2, exactly what to expect at the end of a rotor chain. We begin by showing that, when we have a quasi rotor of the type whose existence is guaranteed by Lemma 4.3, then either outcome (i) of the main theorem occurs, or the quasi rotor can be extended to a right-maximal rotor chain with various desirable properties.

Lemma 9.1. *Let $(T_0, T_1, T_2, D_0, D_1, \{c_0, b_1, a_2\})$ be a quasi rotor in an internally 4-connected binary matroid M , where $|E(M)| \geq 13$ and $M \setminus c_0$ is $(4, 4, S)$ -connected. Suppose N is an internally 4-connected proper minor of $M \setminus c_0, c_1$ with $|E(N)| \geq 7$. Then one of the following occurs.*

- (i) M has a quick win; or
- (ii) M has a right-maximal rotor chain $((a_0, b_0, c_0), (a_1, b_1, c_1), \dots, (a_n, b_n, c_n))$, where $M \setminus c_n$ is $(4, 4, S)$ -connected and $M \setminus c_0, c_1, \dots, c_n$ has an N -minor.

Proof. Let $((a_0, b_0, c_0), (a_1, b_1, c_1), \dots, (a_n, b_n, c_n))$ be a right-maximal rotor chain. Then $n \geq 2$. We know that $M \setminus c_0, c_1$ has an N -minor. Suppose that there is an element i in $\{2, 3, \dots, n\}$ such that $M \setminus c_0, c_1, \dots, c_{i-1}$ has an N -minor, but $M \setminus c_0, c_1, \dots, c_i$ does not. As the first matroid has (c_i, b_i, a_i, b_{i-1})

as a 4-fan, we know that M/b_{i-1} has an N -minor. Since M has $(T_{i-2}, T_{i-1}, T_i, D_{i-2}, D_{i-1}, \{c_{i-2}, b_{i-1}, a_i\})$ as a quasi rotor, Lemma 4.5 implies that (i) holds. We may therefore assume that $M \setminus c_0, c_1, \dots, c_n$ has an N -minor.

Suppose $M \setminus c_n$ is not $(4, 4, S)$ -connected. As (T_{n-1}, T_n, D_{n-1}) is a bowtie, Lemma 4.3 implies that T_n is the central triangle of a quasi rotor $(T_{n-1}, T_n, \{d, e, f\}, D_{n-1}, \{y, c_n, d, e\}, \{x, y, d\})$, where $x \in \{b_{n-1}, c_{n-1}\}$ and $y \in \{a_n, b_n\}$. Suppose $y = a_n$. Then orthogonality between $\{y, c_n, d, e\}$ and the triangle $\{c_{n-2}, b_{n-1}, a_n\}$ implies that $\{d, e\}$ meets $\{c_{n-2}, b_{n-1}\}$. As T_{n-1}, T_n , and $\{d, e, f\}$ are disjoint, $b_{n-1} \notin \{d, e\}$, so $c_{n-2} \in \{d, e\}$ and orthogonality implies that $\{d, e\} \subseteq T_{n-2}$. Then $\lambda(T_{n-2} \cup T_{n-1} \cup T_n) \leq 2$, so $|E(M)| \leq 12$; a contradiction. We deduce that $y \neq a_n$, so $y = b_n$. If $x = b_{n-1}$, then orthogonality implies that $\{b_{n-2}, c_{n-2}, a_{n-1}\}$ meets $\{b_n, d\}$, so $d \in \{b_{n-2}, c_{n-2}\}$. Again $\{d, e\} \subseteq T_{n-2}$; a contradiction. Therefore $x \neq b_{n-1}$, so $x = c_{n-1}$.

Now $((a_0, b_0, c_0), (a_1, b_1, c_1), \dots, (a_n, b_n, c_n), (d, e, f))$ is not a rotor chain, so $a_0 = c_n$, or $\{d, e, f\}$ meets $T_0 \cup T_1 \cup \dots \cup T_n$. As $(T_{n-1}, T_n, \{d, e, f\}, D_{n-1}, \{b_n, c_n, d, e\}, \{c_{n-1}, b_n, d\})$ is a quasi rotor, $\{d, e, f\}$ avoids $T_{n-1} \cup T_n$. Suppose $a_0 = c_n$. Then orthogonality implies that $\{b_0, c_0\}$ meets $\{d, e\}$. As $\{b_0, c_0\} \not\subseteq \{d, e, f\}$, orthogonality with D_0 implies that $\{a_1, b_1\}$ meets $\{d, e, f\}$. If $b_0 \in \{d, e, f\}$, then $M \setminus c_0$ has $\{d, e, f\} \cup T_1$ as a 5-fan; a contradiction. Thus $c_0 \in \{d, e\}$ and $M \setminus c_0, c_1, \dots, c_n/b_n$ has an N -minor, so Lemma 5.2 implies that M/b_1 has an N -minor, and Lemma 4.5 implies that (i) holds. We deduce that $a_0 \neq c_n$. Then $\{d, e, f\}$ meets $T_0 \cup T_1 \cup \dots \cup T_{n-2}$.

Suppose $\{d, e, f\}$ meets a cocircuit D_k for some k in $\{0, 1, \dots, n-2\}$. Then orthogonality implies that two elements of $\{d, e, f\}$ are in D_k . Orthogonality with $\{b_n, c_n, d, e\}$ implies that $\{d, e\}$ meets at most one of T_0, T_1, \dots, T_n . Hence $k = 0$ and $\{d, e, f\} = T_0$. As $\{c_0, b_1, a_2\}$ is a triangle, $c_0 \notin \{d, e\}$. Hence $\{a_0, b_0\} = \{d, e\}$, so $c_0 = f$. The triangle $\{c_{n-1}, b_n, d\}$ and the cocircuit $\{b_0, c_0, a_1, b_1\}$ imply that $d \neq b_0$. Hence $d = a_0$. The symmetric difference of all of the triangles in the rotor chain, that is, each T_i and each $\{c_j, b_{j+1}, a_{j+2}\}$, is $\{a_0, b_0, a_1, c_{n-1}, b_n, c_n\}$, and the symmetric difference of the last set with the triangle $\{c_{n-1}, b_n, a_0\}$ is $\{c_n, b_0, a_1\}$, which must also be a triangle. Hence $(c_n, b_0, a_1, b_1, c_1)$ is a 5-fan in $M \setminus c_0$; a contradiction.

We now know that $\{d, e, f\}$ avoids every cocircuit in the rotor chain. Then $\{d, e, f\}$ meets the rotor chain in exactly the element a_0 , so orthogonality implies that $f = a_0$, and we deduce that $((a_0, b_0, c_0), (a_1, b_1, c_1), \dots, (a_n, b_n, c_n), (d, e, f))$ is a rotor chain; a contradiction to our selection of a right-maximal rotor chain. \square

Lemma 9.2. *Let M and N be internally 4-connected binary matroids where $|E(M)| \geq 13$ and $|E(N)| \geq 7$. Suppose that M does not have a proper minor M' such that $|E(M)| - |E(M')| \leq 3$ and M' is internally 4-connected with an N -minor. Let M have a quasi rotor $(T_0, T_1, T_2, D_0, D_1, \{c_0, b_1, a_2\})$ such that $M \setminus c_0$ is $(4, 4, S)$ -connected and $M \setminus c_0, c_1$ has an N -minor. Then M has a right-maximal rotor chain $((a_0, b_0, c_0), (a_1, b_1, c_1), \dots, (a_n, b_n, c_n))$ such that $M \setminus c_0, c_1, \dots, c_n$ is sequentially 4-connected with an N -minor, M/b_i has no N -minor for all i in $\{1, 2, \dots, n-1\}$, and one of the following occurs.*

- (i) $M \setminus c_n$ is $(4, 4, S)$ -connected, M has a triangle $\{a_{n+1}, b_{n+1}, c_{n+1}\}$ and a 4-cocircuit $\{b_n, c_n, a_{n+1}, b_{n+1}\}$ such that $T_0, D_0, T_1, D_1, \dots, T_n, \{b_n, c_n, a_{n+1}, b_{n+1}\}, \{a_{n+1}, b_{n+1}, c_{n+1}\}$ is a bowtie string in M , and $M \setminus c_0, c_1, \dots, c_{n+1}$ has an N -minor; or

- (ii) M has an open-rotor-chain win or a ladder win; or
- (iii) M has an enhanced-ladder win.

Proof. Let $S = \{c_0, c_1, \dots, c_n\}$. By Lemma 9.1, M has a right-maximal rotor chain $((a_0, b_0, c_0), (a_1, b_1, c_1), \dots, (a_n, b_n, c_n))$ such that $M \setminus c_n$ is $(4, 4, S)$ -connected and $M \setminus S$ has an N -minor. Suppose M/b_i has an N -minor, for some i in $\{1, 2, \dots, n-1\}$. Then applying Lemma 4.5 to the quasi rotor $(T_{i-1}, T_i, T_{i+1}, D_{i-1}, D_i, \{c_{i-1}, b_i, a_{i+1}\})$ gives that M has a proper minor M' such that $|E(M)| - |E(M')| \leq 3$ and M' is internally 4-connected with an N -minor; a contradiction. Therefore,

9.2.1. $M \setminus S$ has an N -minor and M/b_i has no N -minor for all i in $\{1, 2, \dots, n-1\}$.

Lemma 7.1 implies that

9.2.2. $M \setminus S$ is sequentially 4-connected.

Next we show the following.

9.2.3. No triangle in $M \setminus S$ meets $\{b_0, a_1, b_1, a_2, b_2, \dots, a_n, b_n\}$.

Suppose $M \setminus S$ has a triangle that meets $\{b_0, a_1, b_1, a_2, b_2, \dots, a_n, b_n\}$. Assume first that T does not contain $\{b_0, a_1\}$. Then orthogonality and the fact that T avoids S implies that $T = \{b_i, b_{i+1}, b_{i+2}\}$ for some i in $\{0, 1, \dots, n-2\}$. Then $\lambda(T_i \cup T_{i+1} \cup T_{i+2}) \leq 2$ so $|E(M)| \leq 12$; a contradiction. We may now assume that T contains $\{b_0, a_1\}$. Then $T \cup \{b_1, c_1\}$ is a 5-fan in $M \setminus c_0$; a contradiction. We conclude that 9.2.3 holds.

If $M \setminus S$ is internally 4-connected, then part (ii) of the lemma holds, so we assume not. Then $M \setminus S$ has a 4-fan (y_1, y_2, y_3, y_4) . Thus M has a cocircuit C^* such that $\{y_2, y_3, y_4\} \subsetneq C^* \subseteq \{y_2, y_3, y_4\} \cup S$. Next we determine the possibilities for C^* .

9.2.4. One of the following occurs.

- (i) $C^* = \{y_2, y_3, y_4, c_n\}$ and $y_4 = b_n$; or
- (ii) $n = 2$ and $a_1 = y_4$ and $C^* = \{y_2, y_3, a_1, c_1\}$; or
- (iii) $a_0 \in \{y_2, y_3\}$ and
 - (a) $n = 2$ and $y_4 = b_1$ and $C^* = \{y_2, y_3, b_1, c_0, c_1\}$; or
 - (b) $n \leq 3$ and $y_4 = a_2$ and $C^* = \{y_2, y_3, a_2, c_0, c_2\}$.

If $c_i \in C^*$ for some i in $\{1, 2, \dots, n\}$, then orthogonality implies that $\{a_i, b_i\}$ meets $\{y_2, y_3, y_4\}$. Hence, by 9.2.3, $y_4 \in \{a_i, b_i\}$. Thus c_i is the only element of $\{c_1, c_2, \dots, c_n\}$ in C^* . Also, orthogonality implies that $\{c_i, b_{i+1}, a_{i+2}\}$ is not a triangle of M . Hence $i \geq n-1$, so $C^* \subseteq \{y_2, y_3, y_4, c_0, c_{n-1}, c_n\}$.

Moving towards obtaining 9.2.4, we now show:

9.2.5. If C^* avoids c_0 but contains c_n , then $y_4 = b_n$.

To see this, note that orthogonality implies that $y_4 \in \{a_n, b_n\}$. By 9.2.3, $\{y_2, y_3\}$ avoids $\{c_{n-2}, b_{n-1}\}$. Hence, by orthogonality, $y_4 \neq a_n$, so $y_4 = b_n$ and 9.2.5 holds.

Next we suppose that $c_{n-1} \in C^* \subseteq \{y_2, y_3, y_4, c_{n-1}, c_n\}$. Then $y_4 \in \{a_{n-1}, b_{n-1}\}$, so 9.2.5 implies that $C^* = \{y_2, y_3, y_4, c_{n-1}\}$. Orthogonality between C^* and the circuit $\{c_0, a_1, c_{n-1}, a_n\}$ implies that either $\{y_2, y_3\}$ meets $\{a_1, a_n\}$; or $a_1 = a_{n-1} = y_4$. The first possibility gives a contradiction to 9.2.3. Hence the second possibility holds and therefore so does 9.2.4(ii).

It remains to consider what happens when $c_0 \in C^*$. In that case, orthogonality with T_0 and $\{c_0, b_1, a_2\}$ implies, using 9.2.3, that $a_0 \in \{y_2, y_3\}$ and $y_4 \in \{b_1, a_2\}$. If

$y_4 = b_1$, then orthogonality implies that $C^* = \{y_2, y_3, b_1, c_0, c_1\}$ and $n = 2$. On the other hand, if $y_4 = a_2$, then $C^* = \{y_2, y_3, a_2, c_0, c_2\}$ and $n \leq 3$. We conclude that 9.2.4 holds.

Next we show the following.

9.2.6. *If 9.2.4(i) holds, then so does the lemma.*

As (y_1, y_2, y_3, b_n) is a 4-fan in $M \setminus S$, and $M \setminus S$ has an N -minor, we deduce that $M \setminus S / b_n$ or $M \setminus S \setminus y_1$ has an N -minor. If $M \setminus S / b_n$ has an N -minor, then so do $M \setminus (S - c_n) / b_n \setminus a_n$ and hence $M \setminus (S - c_n) \setminus a_n / b_{n-1}$; a contradiction to 9.2.1. Thus $M \setminus S \setminus y_1$ has an N -minor. Now 9.2.3 and orthogonality imply that the elements in $T_0 \cup T_1 \cup \dots \cup T_n \cup \{y_1, y_2, y_3\}$ are all distinct except that possibly $y_1 = a_0$. Letting $(y_1, y_2, y_3) = (c_{n+1}, b_{n+1}, a_{n+1})$, we get that part (i) of the lemma holds. Thus 9.2.6 holds.

We may now assume that 9.2.4(i) does not hold. Thus 9.2.4(ii) or 9.2.4(iii) holds, so $n \leq 3$. At this point, we seek to apply Lemma 4.3 to the bowtie $(\{a_{n-1}, b_{n-1}, c_{n-1}\}, \{a_n, b_n, c_n\}, \{b_{n-1}, c_{n-1}, a_n, b_n\})$ to build more structure onto our configuration. We know that $M \setminus c_n$ is $(4, 4, S)$ -connected but not internally 4-connected, and clearly $M \setminus a_{n-1}$ is not internally 4-connected. Thus neither part (i) nor part (iv) of Lemma 4.3 holds. Next we eliminate the possibility that part (ii) of that lemma holds.

9.2.7. *If M has a triangle $\{a_{n+1}, b_{n+1}, c_{n+1}\}$ that is disjoint from $T_{n-1} \cup T_n$ such that $(T_n, \{a_{n+1}, b_{n+1}, c_{n+1}\}, \{x, c_n, a_{n+1}, b_{n+1}\})$ is a bowtie for some x in $\{a_n, b_n\}$, then part (i) of the lemma holds.*

Suppose that M has such a triangle and denote it by T_{n+1} . Then $(c_{n+1}, b_{n+1}, a_{n+1}, x)$ is a 4-fan in $M \setminus c_n$. Suppose $x = a_n$. Then orthogonality implies that $\{c_{n-2}, b_{n-1}\}$ meets $\{a_{n+1}, b_{n+1}\}$. If $c_{n-2} \in \{a_{n+1}, b_{n+1}\}$, then $M \setminus S$ has a_n in a 2-element cocircuit, so $M \setminus S / a_n$ has an N -minor. Hence $M \setminus (S - c_n) \setminus b_n / b_{n-1}$ has an N -minor, so M / b_{n-1} has an N -minor; a contradiction. Thus $b_{n-1} \in \{a_{n+1}, b_{n+1}\}$, and $\{a_n, c_n, a_{n+1}, b_{n+1}\}$ meets both T_{n-1} and T_n ; a contradiction to Lemma 4.2. We conclude that $x \neq a_n$. Thus $x = b_n$ and $\{b_n, c_n, a_{n+1}, b_{n+1}\}$ is a cocircuit.

Suppose that $T_0, D_0, T_1, D_1, \dots, T_n, D_{n+1}, T_{n+1}$ is a bowtie string. As $M \setminus S$ has $(c_{n+1}, b_{n+1}, a_{n+1}, b_n)$ as a 4-fan, either $M \setminus S \setminus c_{n+1}$ has an N -minor and (i) holds, as desired, or $M \setminus S / b_n$ has an N -minor, and Lemma 5.2 implies that $M \setminus c_0, c_1 / b_1$ has an N -minor; a contradiction. It follows that $T_0, D_0, T_1, D_1, \dots, T_n, D_{n+1}, T_{n+1}$ is not a bowtie string. Therefore T_{n+1} meets $\{b_0, c_0\} \cup T_1 \cup T_2 \cup \dots \cup T_{n-2}$ since we know that T_{n+1} avoids $T_{n-1} \cup T_n$. If $n = 2$, then T_{n+1} , which is T_3 , meets $\{b_0, c_0, a_1, b_1\}$, so $T_3 = T_0$ and $\lambda(T_0 \cup T_1 \cup T_2) \leq 2$; a contradiction. Hence we may assume that $n = 3$. Therefore 9.2.4(iii)(b) holds, so $a_0 \in \{y_2, y_3\}$ and $C^* = \{y_2, y_3, a_2, c_0, c_2\}$.

Suppose T_4 meets $\{b_1, c_1\}$. Since T_4 avoids $T_2 \cup T_3$, it follows, by orthogonality with the cocircuit $\{b_1, c_1, a_2, b_2\}$, that $\{b_1, c_1\} \subseteq T_4$, so $T_1 = T_4$. Then $\lambda(T_1 \cup T_2 \cup T_3) \leq 2$; a contradiction. Hence T_4 avoids $\{b_1, c_1\}$. Thus T_4 meets $\{b_0, c_0, a_1, b_1\}$. As T_4 avoids b_1 , it follows that either $T_4 = T_0$, or T_4 meets $\{b_0, c_0, a_1, b_1\}$ in $\{b_0, a_1\}$ or $\{c_0, a_1\}$. Next we shall eliminate each of these possibilities.

Suppose $T_4 = T_0$. We know that $a_0 \in \{y_2, y_3\}$. If $a_0 \in \{a_4, b_4\}$, then, since $\{y_1, y_2, y_3\}$ is a circuit of $M \setminus S$, it follows by orthogonality between $\{y_1, y_2, y_3\}$ and the cocircuit $\{b_3, c_3, a_4, b_4\}$ that $b_3 \in \{y_1, y_2, y_3\}$ otherwise $T_4 = \{y_1, y_2, y_3\}$; a contradiction. Thus $\{y_1, y_2, y_3\}$ contains $\{a_0, b_3\}$ and we get a contradiction

9.2.9. $\{w, b_n\}$ avoids $\{y_2, y_3\}$.

By 9.2.3, $b_n \notin \{y_1, y_2, y_3\}$. Suppose $w \in \{y_2, y_3\}$. Then, by orthogonality between the circuit $\{y_1, y_2, y_3\}$ and the cocircuit $\{w, x, b_n, c_n\}$, we deduce that $\{w, x\} \subseteq \{y_1, y_2, y_3\}$. Thus $M \setminus c_n$ has a 5-fan. This contradiction establishes that 9.2.9 holds.

By 9.2.6, we may assume that 9.2.4(i) does not hold. Now suppose that 9.2.4(iii) holds. Then $n \leq 3$ and, without loss of generality, $a_0 = y_3$. If 9.2.4(iii)(a) holds, then orthogonality between the cocircuit $\{y_2, y_3, b_1, c_0, c_1\}$ and the circuits $\{c_1, b_2, w\}$ and T_2 implies that $w \in \{y_2, b_1, c_0\}$. Since $n = 2$, we see that $w \notin \{b_1, c_0\}$, so $w = y_2$; a contradiction to 9.2.9. We may now assume that 9.2.4(iii)(b) holds. If $n = 2$, then we have a configuration of the form shown in Figure 16. Now $M \setminus c_0, c_1, c_2$ has an N -minor and has (y_1, y_2, a_0, a_2) as a 4-fan. Thus $M \setminus c_0, c_1, c_2/a_2$ or $M \setminus c_0, c_1, c_2 \setminus y_1$ has an N -minor. The first case does not arise because M/b_1 has no N -minor, yet

$$M \setminus c_0, c_1, c_2/a_2 \cong M/a_2 \setminus b_1, b_2, c_1 \cong M \setminus a_2, b_1, b_2, c_1 \cong M/b_1 \setminus a_2, b_2, c_1.$$

Hence $M \setminus c_0, c_1, c_2, y_1$ has an N -minor, so we can apply Lemma 7.2 to obtain that the lemma holds unless $\{w, x\}$ or $\{b_0, a_1\}$ is contained in a triangle, T . In the exceptional cases, either $T \cup \{c_1, b_2\}$ contains a 5-fan in $M \setminus c_2$, or $T \cup \{b_1, c_1\}$ contains a 5-fan in $M \setminus c_0$, so we get a contradiction. Thus $n \neq 2$ so $n = 3$. By orthogonality between $\{w, b_3, c_2\}$ and $\{b_0, c_0, a_1, b_1\}$, we see that $w \neq c_0$. Then orthogonality between $\{w, b_3, c_2\}$ and $\{y_2, y_3, a_2, c_0, c_2\}$ implies that $\{w, b_3\}$ meets $\{y_2, y_3\}$; a contradiction to 9.2.9. We conclude that 9.2.4(iii) does not hold.

We may now assume that 9.2.4(ii) holds, that is, $n = 2$ and $C^* = \{y_2, y_3, a_1, c_1\}$. Since $\{y_1, y_2, y_3\}$ is a triangle and $\{y_2, y_3, a_1, c_1\}$ is a cocircuit, we see that $\{y_2, y_3\}$ avoids T_1 . Then orthogonality between $\{y_2, y_3, a_1, c_1\}$ and the triangle $\{w, c_{n-1}, b_n\}$ implies that $\{w, b_n\}$ meets $\{y_2, y_3\}$. This contradiction to 9.2.9 completes the proof. \square

10. PROOF OF THE MAIN THEOREM

In this section, we complete the proof of Theorem 1.3. We begin by proving a lemma that treats the case in which a right-maximal bowtie string does not wrap around into a bowtie ring. One fact that will be used repeatedly in the next proof is that if $T_0, D_0, T_1, D_1, \dots, T_n$ is a string of bowties, then, relative to this string, there is symmetry between the elements a_n and b_n .

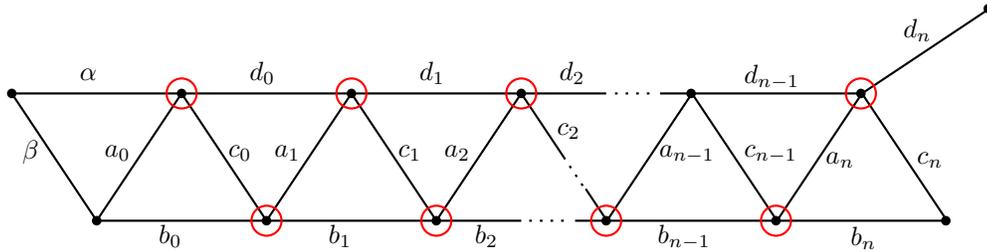


FIGURE 28. $\{d_{n-2}, a_{n-1}, c_{n-1}, d_{n-1}\}$ or $\{d_{n-2}, a_{n-1}, c_{n-1}, a_n, c_n\}$ is a cocircuit where $d_{-1} = \alpha$ when $n = 1$.

Lemma 10.1. *Let M and N be internally 4-connected binary matroids such that $|E(M)| \geq 13$ and $|E(N)| \geq 7$. Suppose that M has a bowtie (T_0, T_1, D_0) , such that $M \setminus c_0$ is $(4, 4, S)$ -connected, and $M \setminus c_0, c_1$ has an N -minor. Let $T_0, D_0, T_1, D_1, \dots, T_n$ be a right-maximal string of bowties. If $(T_n, T_0, \{x, c_n, a_0, b_0\})$ is not a bowtie for all x in $\{a_n, b_n\}$, then one of the following holds.*

- (i) M has a quick win; or
- (ii) M has an open-rotor-chain win or a ladder win; or
- (iii) M contains the configuration in Figure 28, up to switching the labels of a_n and b_n , and $M \setminus c_0, c_1, \dots, c_n$ has an N -minor; or
- (iv) $M \setminus c_0, c_1/b_1$ has an N -minor; or $n = 1$ and $M \setminus c_0, c_1/b_1$ or $M \setminus c_0, c_1/a_1$ has an N -minor; or
- (v) M has an enhanced-ladder win.

Proof. Suppose that none of (i), (iii), or (iv) holds. Then, by Lemma 5.7, $M \setminus c_0, c_1, \dots, c_n$ has an N -minor, but $M \setminus c_0, c_1, \dots, c_i/b_i$ has no N -minor for all i in $\{1, 2, \dots, n\}$, and $M \setminus c_0, c_1, \dots, c_n/a_n$ has no N -minor.

First we show that

10.1.1. *M has no bowtie of the form $(T_n, \{a_{n+1}, b_{n+1}, c_{n+1}\}, \{x, c_n, a_{n+1}, b_{n+1}\})$, where $x \in \{a_n, b_n\}$.*

Since $T_0, D_0, T_1, D_1, \dots, T_n$ is a right-maximal string of bowties, we may assume that $n \geq 2$ otherwise 10.1.1 certainly holds. By the symmetry between a_n and b_n , it suffices to prove that (T_n, T_{n+1}, D_n) is not a bowtie. Assume the contrary. Observe that $\{a_{n+1}, b_{n+1}\}$ avoids $\{c_0, c_1, \dots, c_n\}$ otherwise $M \setminus c_0, c_1, \dots, c_n$ has b_n in a 1- or 2-element cocircuit, so $M \setminus c_0, c_1, \dots, c_n/b_n$ has an N -minor; a contradiction.

To enable us to apply Lemma 5.4, we now show that $a_0 \neq c_n$. Assume otherwise. Then, by orthogonality and symmetry, we may assume that $a_{n+1} \in \{b_0, c_0\}$. From the last paragraph, we know that $a_{n+1} \neq c_0$, so $a_{n+1} = b_0$. Now $c_0 \notin T_{n+1}$ otherwise $T_{n+1} = T_0$, so $c_n \in T_{n+1}$; a contradiction. By orthogonality between D_0 and T_{n+1} , we see that $\{a_1, b_1\}$ meets $\{b_{n+1}, c_{n+1}\}$. If $b_{n+1} \in \{a_1, b_1\}$, then orthogonality between T_1 and the cocircuit $\{b_n, c_n, a_{n+1}, b_{n+1}\}$ gives the contradiction that T_1 meets $\{b_n, c_n\}$. Thus $c_{n+1} \in \{a_1, b_1\}$. Then $M \setminus c_0$ has $T_{n+1} \cup T_1$ as a 5-fan; a contradiction. We conclude that $a_0 \neq c_n$.

By Lemma 5.4, $T_{n+1} = T_j$ for some j in $\{0, 1, \dots, n-2\}$ otherwise we contradict the fact that $T_0, D_0, T_1, D_1, \dots, T_n$ is a right-maximal string of bowties. If $j = 0$, then the hypothesis forbidding $(T_n, T_0, \{x, c_n, a_0, b_0\})$ from being a bowtie implies that $\{a_0, b_0\} \neq \{a_{n+1}, b_{n+1}\}$, so $c_0 \in \{a_{n+1}, b_{n+1}\}$; a contradiction. Thus j is in $\{1, 2, \dots, n-2\}$, and $n \geq 3$. Since $\{a_{n+1}, b_{n+1}\}$ avoids c_j , we see that $\{a_{n+1}, b_{n+1}\} = \{a_j, b_j\}$, and $\{b_n, c_n, a_{n+1}, b_{n+1}\} \Delta \{b_{j-1}, c_{j-1}, a_j, b_j\}$, which is $\{b_n, c_n, b_{j-1}, c_{j-1}\}$, is a cocircuit. Thus $M \setminus c_0, c_1, \dots, c_n/b_n$ has an N -minor; a contradiction to Lemma 5.7. We conclude that 10.1.1 holds.

We now show the following.

10.1.2. *Up to relabelling a_n and b_n , the matroid M has elements d_{n-1} and d_n such that $\{c_{n-1}, d_{n-1}, a_n\}$ is a triangle and $\{d_{n-1}, a_n, c_n, d_n\}$ is a cocircuit.*

We shall apply Lemma 4.3 to the bowtie (T_{n-1}, T_n, D_{n-1}) . By 10.1.1, $\{a_n, b_n, c_n\}$ is not the central triangle of a quasi rotor of the form described in Lemma 4.3, and part (ii) of that lemma does not hold. Moreover, as $M \setminus c_n$ has an N -minor, it is not internally 4-connected otherwise part (i) of the current lemma holds. We deduce that (iii) or (iv) of Lemma 4.3 holds.

Suppose that part (iv) of Lemma 4.3 holds. Then, as $M \setminus a_{n-1}$ is internally 4-connected, we must have that $n = 1$ and that M has a triangle $\{a_0, 7, 8\}$ and a cocircuit $\{x, c_1, 7, 8\}$ for some $x \in \{a_1, b_1\}$. Then, as part (i) of the current lemma does not hold, we deduce that $M \setminus a_0$ has no N -minor. Now $M \setminus c_0, c_1$ has $(a_0, 7, 8, x)$ as a 4-fan, so $M \setminus c_0, c_1, a_0$ or $M \setminus c_0, c_1/x$ has an N -minor. The former gives the contradiction that $M \setminus a_0$ has an N -minor. Thus the latter holds. Let y be the element in $\{a_1, b_1\} - x$. As $M \setminus c_0, c_1/x \cong M \setminus c_0, y/x \cong M \setminus c_0, y/b_0 \cong M/b_0 \setminus a_0, y$, we deduce that $M \setminus a_0$ has an N -minor; a contradiction. We conclude that part (iv) of Lemma 4.3 does not hold.

Finally, suppose that part (iii) of Lemma 4.3 holds. Then, up to relabelling a_n and b_n , the matroid M has elements d_{n-1} and d_n that are not in $T_{n-1} \cup T_n$ such that $\{u, d_{n-1}, a_n\}$ is a triangle and $\{d_{n-1}, a_n, c_n, d_n\}$ is a cocircuit for some u in $\{b_{n-1}, c_{n-1}\}$. If $u = c_{n-1}$, then 10.1.2 holds. Thus we may suppose that $u = b_{n-1}$. Then $n > 1$ otherwise we obtain the contradiction that $M \setminus c_0$ has a 5-fan. By orthogonality between the triangle $\{b_{n-1}, d_{n-1}, a_n\}$ and the cocircuit D_{n-2} , we deduce that $d_{n-1} \in \{b_{n-2}, c_{n-2}\}$. Then, by orthogonality between T_{n-2} and the cocircuit $\{d_{n-1}, a_n, c_n, d_n\}$, we see that either $d_n \in T_{n-2}$, or T_{n-2} meets $\{a_n, c_n\}$. The latter implies that $n = 2$ and $a_0 = c_2$. Thus both cases give the contradiction that $\lambda(T_{n-2} \cup T_{n-1} \cup T_n) \leq 2$. We conclude that 10.1.2 holds.

Next we show that

10.1.3. $n \geq 2$.

Suppose that $n = 1$. By 10.1.2 M contains the configuration shown in Figure 12. We now apply Lemma 6.1. Since part (i) of the current lemma does not hold but 10.1.1 does, we deduce that neither part (i) nor part (ii) of Lemma 6.1 holds. Moreover, part (iii) of Lemma 6.1 does not hold since if $\{b_0, b_1\}$ is contained in a triangle, then we obtain the contradiction that $M \setminus c_0$ has a 5-fan containing this triangle and $\{a_1, c_1\}$. If part (iv) of Lemma 6.1 holds, then part (iii) of the current lemma holds. It remains to consider the case when part (v) of Lemma 6.1 holds, that is, when $M \setminus c_0, c_1$ has a 4-fan of the form (y_1, y_2, y_3, b_1) . Then M has a cocircuit C^* such that $\{y_2, y_3, b_1\} \subsetneq C^* \subseteq \{y_2, y_3, b_1, c_0, c_1\}$. Then 10.1.1 implies that $c_0 \in C^*$, so orthogonality implies that $\{y_2, y_3\}$ meets $\{a_0, b_0\}$ and $\{d_0, a_1\}$. If $\{y_2, y_3\} \neq \{b_0, a_1\}$, then $\lambda(T_0 \cup T_1 \cup d_0) \leq 2$; a contradiction. Thus $\{y_2, y_3\} = \{b_0, a_1\}$. Hence, by orthogonality, $y_1 = d_1$, so $\lambda(T_0 \cup T_1 \cup d_0 \cup d_1) \leq 2$; a contradiction. We conclude that 10.1.3 holds.

We show next that

10.1.4. M does not contain the ladder segment shown in Figure 14, nor does M contain the ladder segment in Figure 14 after switching the labels on a_n and b_n .

By the symmetry between a_n and b_n in the lemma statement, it suffices to show that M does not contain the first of these ladder segments. Assume the contrary. We show first that the elements in this ladder segment are distinct. If not, then Lemma 6.4 implies that either $\{b_n, c_n, c_0, d_0\}$ is a cocircuit of M , or $(a_0, b_0) = (d_{n-1}, d_n)$. The former implies that $M \setminus c_0, c_1, \dots, c_n$ has $\{b_n, d_0\}$ as a cocircuit, so $M \setminus c_0, c_1, \dots, c_n/b_n$ has an N -minor; a contradiction. The latter implies that $(T_n, T_0, \{a_0, b_0, a_n, c_n\})$ is a bowtie; a contradiction to 10.1.1. We conclude that the elements in the ladder segment are distinct.

We now apply Lemma 6.5. Since $M \setminus c_0, c_1, \dots, c_n$ has an N -minor, if it is internally 4-connected, then part (ii) of the current lemma holds. Thus we may

assume that part (i) of Lemma 6.5 does not hold. Moreover, by 10.1.1, part (ii) of Lemma 6.5 does not hold. Also, part (iv) of Lemma 6.5 does not hold otherwise M is the cycle matroid of a quartic Möbius ladder as shown in Figure 15 and we get a contradiction to 10.1.1. We deduce that part (iii) of Lemma 6.5 holds; that is, either $M \setminus c_0$ has a 4-fan of the form $(\alpha, \beta, a_0, d_0)$, or $M \setminus c_n$ has a 4-fan of the form (y_1, y_2, y_3, b_n) . The latter implies that $(\{a_n, b_n, c_n\}, \{y_1, y_2, y_3\}, \{y_2, y_3, b_n, c_n\})$ is a bowtie. This gives a contradiction to 10.1.1, so the former holds. Hence so does part (iii) of the current lemma; a contradiction. Thus 10.1.4 holds.

By 10.1.2, M contains one of the configurations in Figure 24 with $m = n - 1$. By 10.1.4, if we take m to be as small as possible such that M contains one of the configurations in Figure 24, then $m > 0$. Moreover, the hypotheses of Lemma 8.1 hold. It follows, by that lemma, that (ii) or (v) of the current lemma holds. \square

Next we note a helpful property of bowtie rings.

Lemma 10.2. *Let $(T_0, D_0, T_1, D_1, \dots, T_n, \{b_n, c_n, a_0, b_0\})$ be a ring of bowties in a matroid M . Then*

$$M \setminus c_0, c_1, \dots, c_n / a_1 \cong M \setminus a_0, b_1, a_2, a_3, \dots, a_n / b_2.$$

Proof. We have

$$\begin{aligned} M \setminus c_0, c_1, \dots, c_n / a_1 &\cong M \setminus c_0, b_1, c_2, c_3, \dots, c_n / a_1 \\ &\cong M \setminus c_0, b_1, c_2, c_3, \dots, c_n / b_0 \\ &\cong M \setminus a_0, b_1, c_2, c_3, \dots, c_n / b_0 \\ &\cong M \setminus a_0, b_1, c_2, c_3, \dots, c_n / b_n \\ &\cong M \setminus a_0, b_1, c_2, c_3, \dots, c_{n-1}, a_n / b_{n-1} \\ &\vdots \\ &\cong M \setminus a_0, b_1, a_2, a_3, \dots, a_n / b_2. \end{aligned}$$

\square

We showed in Lemma 5.5 that, when we have a bowtie ring, we may obtain one of the structures shown in Figure 11. The next two lemmas deal with these two structures.

Lemma 10.3. *Let M and N be internally 4-connected binary matroids such that $|E(M)| \geq 13$ and $|E(N)| \geq 7$. Suppose that M has a bowtie ring $(T_0, D_0, T_1, D_1, \dots, T_n, \{b_n, c_n, a_0, b_0\})$, that $M \setminus c_0$ is $(4, 4, S)$ -connected, that $M \setminus c_0, c_1$ has an N -minor, and that $M \setminus c_0, c_1 / b_1$ does not have an N -minor. Let M have a bowtie (T_j, S_1, C_0) for some j in $\{1, 2, \dots, n\}$ where C_0 is $\{z, c_j, e_1, f_1\}$ for some z in $\{a_j, b_j\}$ and S_1 is a triangle $\{e_1, f_1, g_1\}$ that avoids $T_0 \cup T_1 \cup \dots \cup T_n$. Let $T_0, D_0, T_1, D_1, \dots, T_j, C_0, S_1, C_1, \dots, S_\ell$ be a right-maximal bowtie string where, for all i in $\{2, 3, \dots, \ell\}$, the set S_i is a triangle, $\{e_i, f_i, g_i\}$, and C_{i-1} is a cocircuit, $\{f_{i-1}, g_{i-1}, e_i, f_i\}$. If M has $\{y, g_\ell, a_0, b_0\}$ as a cocircuit for some y in $\{e_\ell, f_\ell\}$, then $j = 1$ and $z = a_j$, and both of the matroids $M \setminus c_0, c_1, \dots, c_n / a_1$ and $M \setminus a_0, b_1, a_2, a_3, \dots, a_n / b_2$ have N -minors.*

Proof. Assume that the lemma fails. By symmetry, we may suppose that M has $\{f_\ell, g_\ell, a_0, b_0\}$ as a cocircuit. Now either this cocircuit equals $\{b_n, c_n, a_0, b_0\}$, or the symmetric difference of these two cocircuits is $\{f_\ell, g_\ell, b_n, c_n\}$ and the last

set is a cocircuit of M . By Lemma 5.7, $M \setminus c_0, c_1, \dots, c_n$ has an N -minor but $M \setminus c_0, c_1, \dots, c_i/b_i$ has no N -minor for all i in $\{1, 2, \dots, n\}$, and $M \setminus c_0, c_1, \dots, c_h/b_h$ has no N -minor for all h in $\{2, 3, \dots, n\}$. We show first that

10.3.1. $j \leq n - 1$.

Suppose $j = n$. By Lemma 5.7, $M \setminus c_0, c_1, \dots, c_n, g_1, \dots, g_\ell$ has an N -minor. Now S_ℓ avoids T_n , so $\{f_\ell, g_\ell, b_n, c_n\}$ is a cocircuit of M . Thus $M \setminus c_0, c_1, \dots, c_n, g_1, \dots, g_\ell$ has b_n in a cocircuit of size at most two, so $N \not\preceq M \setminus c_0, c_1, \dots, c_n/b_n$; a contradiction. Hence 10.3.1 holds.

Next we show that

10.3.2. S_ℓ meets T_n .

Assume that 10.3.2 fails. Then $\{f_\ell, g_\ell, b_n, c_n\}$ is a cocircuit of M , and we can adjoin this cocircuit and T_n to the end of the bowtie string $T_0, D_0, T_1, D_1, \dots, T_j, C_0, S_1, C_1, \dots, S_\ell$ to give a contradiction. Thus 10.3.2 holds.

Now choose m to be the least integer such that S_m meets $T_0 \cup T_1 \cup \dots \cup T_n$. Since $S_1 \cup S_2 \cup \dots \cup S_\ell$ avoids $T_0 \cup T_1 \cup \dots \cup T_j$, we see that S_m meets T_p for some p in $\{j+1, j+2, \dots, n\}$. We show next that

10.3.3. $S_m = T_p$.

By hypothesis, $m > 1$. First observe that if $\{e_m, f_m\}$ meets T_p , then 10.3.3 follows by orthogonality between T_p and C_{m-1} . We may now assume that $g_m \in T_p$. Then orthogonality between S_m and one of the cocircuits D_{p-1} and D_p implies that $\{e_m, f_m\}$ meets $T_{p-1} \cup T_p \cup T_{p+1}$ where we interpret the subscripts on D_i and T_i modulo $n+1$. Then $\{e_m, f_m\}$ meets T_{p-1}, T_p , or T_{p+1} . Thus, by the first part of the argument, we see that $S_m \in \{T_{p-1}, T_p, T_{p+1}\}$. As $g_m \in T_p$, we deduce that $S_m = T_p$, so 10.3.3 holds.

Define $X = \{c_0, c_1, \dots, c_n\} - \{c_{j-1}, c_j\}$. Clearly $T_{j-1}, D_{j-1}, T_j, C_0, S_1, \dots, C_{m-1}, S_{m-1}$ is a bowtie string in $M \setminus X$. Moreover, $N \preceq M \setminus X \setminus c_{j-1}, c_j$. Applying Lemma 5.7 gives that either

- (a) $N \preceq M \setminus X \setminus c_{j-1}, c_j/z$; or
- (b) $N \preceq M \setminus X \setminus c_{j-1}, c_j, g_1, \dots, g_{m-1}$ and

$$N \not\preceq M \setminus X \setminus c_{j-1}, c_j, g_1, \dots, g_{m-1}/f_{m-1}.$$

Consider the second case. Then $M \setminus c_0, c_1, \dots, c_n, g_1, g_2, \dots, g_{m-1}$ has an N -minor but does not have a cocircuit containing f_{m-1} and having at most two elements. As $S_m = T_p$, either $c_p \in \{e_m, f_m\}$, or $\{e_m, f_m\} = \{a_p, b_p\}$. Thus either C_{m-1} or $C_{m-1} \triangle D_{p-1}$ contains a 1- or 2-element cocircuit of $M \setminus c_0, c_1, \dots, c_n, g_1, g_2, \dots, g_{m-1}$ containing f_{m-1} . Hence (b) does not hold.

We now know that (a) holds. Then we get a contradiction unless $j = 1$ and $z = a_j$. In the exceptional case, the lemma follows using Lemma 10.2. \square

Lemma 10.4. *Let M and N be internally 4-connected binary matroids such that $|E(M)| \geq 13$ and $|E(N)| \geq 7$. Suppose that M has a bowtie ring $(T_0, D_0, T_1, D_1, \dots, T_n, \{b_n, c_n, a_0, b_0\})$, that $M \setminus c_0$ is $(4, 4, S)$ -connected, and that $M \setminus c_0, c_1$ has an N -minor. Suppose M has a bowtie $(\{a_j, b_j, c_j\}, \{e_1, f_1, g_1\}, \{z, c_j, e_1, f_1\})$ for some j in $\{1, 2, \dots, n\}$ and some z in $\{a_j, b_j\}$, where $\{e_1, f_1, g_1\}$ avoids $T_0 \cup T_1 \cup \dots \cup T_n$. Then*

- (i) M has a quick win; or

- (ii) M has an open-rotor-chain win, a bowtie-ring win, or a ladder win; or
- (iii) M has an enhanced-ladder win; or
- (iv) $M \setminus c_0, c_1/b_1$ has an N -minor; or
- (v) $j = 1$ and $z = a_j$, and both of the matroids $M \setminus c_0, c_1, \dots, c_n/a_1$ and $M \setminus a_0, b_1, a_2, a_3, \dots, a_n/b_2$ have N -minors.

Proof. Assume that the lemma fails. First we show the following.

10.4.1. *When $z = a_j$ and $j = 1$, the matroid $M \setminus c_0, c_1/a_1$ has no N -minor.*

Assume that $M \setminus c_0, c_1/a_1$ has an N -minor. As $M \setminus c_0, c_1/a_1$ has (c_2, b_2, a_2, b_1) as a 4-fan, by Lemma 4.1, $M \setminus c_0, c_1/a_1/b_1$ or $M \setminus c_0, c_1/a_1 \setminus c_2$ has an N -minor. The first option gives the contradiction that $M \setminus c_0, c_1/b_1$ has an N -minor, so we assume the latter. Then $M \setminus c_0, c_1/a_1 \setminus c_2$ has an N -minor and has (c_3, b_3, a_3, b_2) as a 4-fan. By repeatedly applying this argument, we deduce that $M \setminus c_0, c_1/a_1 \setminus c_2, c_3, \dots, c_n$ has an N -minor. By Lemma 10.2, $M \setminus c_0, c_1, \dots, c_n/a_1 \cong M \setminus a_0, b_1, a_2, a_3, \dots, a_n/b_2$. Thus we obtain the contradiction that (v) holds, so 10.4.1 is proved.

Take a right-maximal bowtie string $T_0, D_0, T_1, D_1, \dots, T_j, C_0, S_1, C_1, \dots, S_\ell$, where, for all i in $\{2, 3, \dots, \ell\}$, the set S_i is $\{e_i, f_i, g_i\}$, a triangle, and C_{i-1} is $\{f_{i-1}, g_{i-1}, e_i, f_i\}$, a cocircuit. By Lemma 10.3, M has no cocircuit $\{y, g_\ell, a_0, c_0\}$ with y in $\{e_\ell, f_\ell\}$. Then, by Lemma 10.1 and 10.4.1, we deduce that M contains the configuration in Figure 28 with suitable adjustments to the labelling. Let z' be the element of $\{a_j, b_j\} - z$. Now M has $\{c_j, e_1, d_j\}$ as a triangle, and S_1 avoids $T_0 \cup T_1 \cup \dots \cup T_n$. Thus orthogonality between $\{c_j, e_1, d_j\}$ and D_j implies that $d_j \in \{a_{j+1}, b_{j+1}\}$. Furthermore, M has $\{d_j, d_{j+1}, e_1, g_1\}$ or $\{d_{j-1}, d_j, z', c_j\}$ as a cocircuit. If $\{d_j, d_{j+1}, e_1, g_1\}$ is a cocircuit, then orthogonality between this cocircuit and T_{j+1} implies that $d_{j+1} \in T_{j+1}$. Thus $\lambda(T_j \cup T_{j+1} \cup S_1) \leq 2$; a contradiction. We deduce that M has $\{d_{j-1}, d_j, z', c_j\}$ as a cocircuit. As this cocircuit meets both T_j and T_{j+1} , Lemma 4.2 implies that it equals D_j . Thus $z' = b_j$ and $\{d_{j-1}, d_j\} = \{a_{j+1}, b_{j+1}\}$. Then M has $\{c_{j-1}, d_{j-1}, b_j\}$ as a triangle, and orthogonality between it and D_{j+1} implies that $d_{j-1} = a_{j+1}$, so $d_j = b_{j+1}$. Thus the triangle $\{c_j, e_1, d_j\}$ meets D_{j+1} in a single element; a contradiction. \square

We now assemble the pieces already proved to finish the proof of the main result.

Proof of Theorem 1.3. For notational convenience, we suppose that our bowtie is (T_0, T_1, D_0) , where $M \setminus c_0, c_1$ has an N -minor, $M \setminus c_0$ is $(4, 4, S)$ -connected, and $M \setminus c_1$ is not $(4, 4, S)$ -connected. Assume that the theorem fails.

Lemma 4.3 implies that T_1 is the central triangle of a quasi rotor $(T_0, T_1, T_2, D_0, \{y, c_1, a_2, b_2\}, \{x, y, a_2\})$, for some x in $\{b_0, c_0\}$ and some y in $\{a_1, b_1\}$. By possibly switching the labels of a_1 and b_1 , we may assume that $y = b_1$. If $x = b_0$, then $(a_2, b_0, b_1, a_1, c_1)$ is a 5-fan in $M \setminus c_0$; a contradiction. Thus $x = c_0$.

By Lemma 9.2, M has a right-maximal rotor chain $((a_0, b_0, c_0), (a_1, b_1, c_1), \dots, (a_n, b_n, c_n))$ for some $n \geq 2$, and M has a triangle T_{n+1} and a 4-cocircuit D_n where $T_{n+1} = \{a_{n+1}, b_{n+1}, c_{n+1}\}$ and $D_n = \{b_n, c_n, a_{n+1}, b_{n+1}\}$ such that $T_0, D_0, T_1, D_1, \dots, T_n, D_n, T_{n+1}$ is a bowtie string, $M \setminus c_0, c_1, \dots, c_{n+1}$ has an N -minor, $M \setminus c_n$ is $(4, 4, S)$ -connected, and M/b_i has no N -minor for all i in $\{1, 2, \dots, n-1\}$. Thus M contains one of the structures shown in Figure 29.

Take a right-maximal bowtie string $T_0, D_0, T_1, \dots, T_n, D_n, T_{n+1}, \dots, T_k$. Then $k \geq n+1 \geq 3$. By Lemma 4.5, M/a_1 has no N -minor. Thus, by Lemma 5.7 and the final part of the previous paragraph,

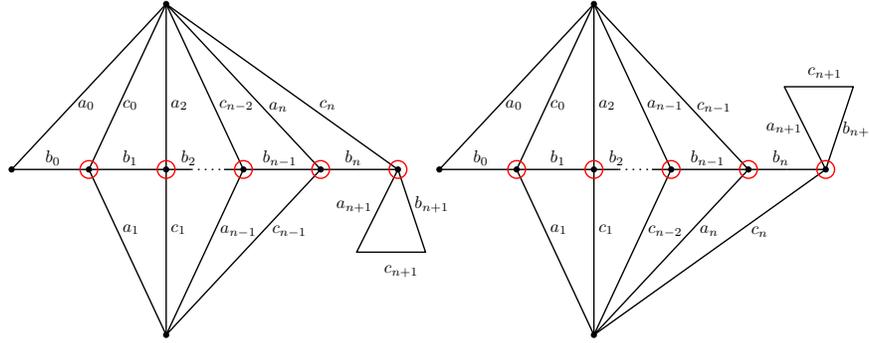


FIGURE 29. n is even on the left and odd on the right.

10.5.1. $M \setminus \{c_0, c_1, \dots, c_k\}$ has an N -minor, and M/b_i has no N -minor for all i in $\{1, 2, \dots, n-1\}$. Moreover, for all j in $\{1, 2, \dots, k\}$, neither $M \setminus \{c_0, c_1, \dots, c_j\}/b_j$ nor $M \setminus \{c_0, c_1, \dots, c_j\}/a_j$ has an N -minor.

Next we show that

10.5.2. $(T_k, T_0, \{x, c_k, a_0, b_0\})$ is a bowtie for some x in $\{a_k, b_k\}$.

Assume that this fails. Then, by Lemma 10.1, the bowtie string $T_0, D_0, T_1, \dots, T_n, D_n, T_{n+1}, \dots, T_k$ is contained in a ladder segment as shown in Figure 28, where k takes the place of n in the figure. Thus $\{c_0, d_0, a_1\}$ and $\{c_1, d_1, a_2\}$ are triangles. As $\{c_0, a_2, c_1, a_1\}$ is a circuit, using symmetric difference, we deduce that so too is $\{c_1, a_2, d_0\}$. Hence $d_0 = d_1$. But $\{d_0, a_1, c_1, d_1\}$ is a cocircuit as $k \geq 3$, so $\{d_0, a_1, c_1\}$ is a triad that meets a triangle of M ; a contradiction. Thus 10.5.2 holds.

By symmetry between a_k and b_k , we may assume that M has $(T_k, T_0, \{b_k, c_k, a_0, b_0\})$ as a bowtie. By 10.5.1 and Lemma 5.5, $M \setminus \{c_0, c_1, \dots, c_k\}$ is $(4, 4, S)$ -connected and M has a triangle $\{e_1, f_1, g_1\}$ that is disjoint from $T_0 \cup T_1 \cup \dots \cup T_k$ such that, for some j in $\{0, 1, \dots, k\}$, there is a cocircuit $\{e_1, f_1, h_1, c_j\}$ in M for some h_1 in $\{b_j, a_j\}$. Suppose $j = 0$. Then orthogonality implies that $\{e_1, f_1\}$ meets $\{b_1, a_2\}$, a contradiction. Thus $j \geq 1$.

By Lemma 10.4 and 10.5.1, part (v) of that lemma holds. Thus $h = a_1$ and $j = 1$, and $M \setminus \{a_0, b_1, a_2, a_3, \dots, a_k\}/b_2$, and hence M/b_2 , has an N -minor. Then, by 10.5.1, $n = 2$ and M contains the structure in Figure 30.

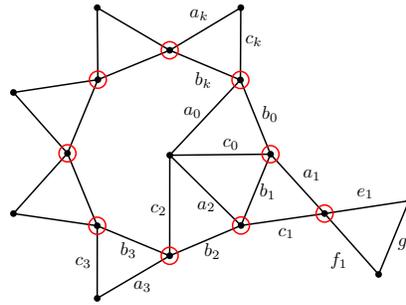


FIGURE 30

To complete the proof of the theorem, we show that

10.5.3. $M/b_2 \setminus c_2$ is internally 4-connected with an N -minor.

Since M/b_2 has an N -minor, $M/b_2 \setminus c_2$ also has an N -minor. By Lemma 9.1, $M \setminus c_2$ is $(4, 4, S)$ -connected and, as it has b_2 as the coguts element of a 4-fan, we deduce that $M/b_2 \setminus c_2$ is 3-connected. Suppose that $M/b_2 \setminus c_2$ has (U, V) as a non-sequential 3-separation. Then, by [6, Lemma 3.3], we may assume that $T_0 \cup T_1 \cup a_2 \subseteq U$. Thus $(U \cup \{b_2, c_2\}, V)$ is a non-sequential 3-separation of M ; a contradiction. Therefore $M/b_2 \setminus c_2$ is sequentially 4-connected.

Let $(\alpha, \beta, \gamma, \delta)$ be a 4-fan in $M/b_2 \setminus c_2$. Suppose $\{\beta, \gamma, \delta\}$ is a triad in M . Then none of β, γ , or δ is in a triangle of M . Hence $\{b_2, \alpha, \beta, \gamma\}$ is a circuit in M , and orthogonality implies that this circuit meets both $\{b_1, c_1, a_2\}$ and $\{a_3, b_3\}$. Thus $\{\beta, \gamma\}$ meets a triangle; a contradiction. We deduce that $\{\beta, \gamma, \delta\}$ is not a triad in M . Thus $\{c_2, \beta, \gamma, \delta\}$ is a cocircuit of M . Then, by orthogonality, $a_2 \in \{\beta, \gamma, \delta\}$, so $\{c_0, b_1\}$ also meets $\{\beta, \gamma, \delta\}$, and either T_0 or T_1 contains two elements of $\{\beta, \gamma, \delta\}$. Thus $\lambda(T_0 \cup T_1 \cup T_2) \leq 2$; a contradiction. We conclude that $M/b_2 \setminus c_2$ is internally 4-connected, so 10.5.3 holds and, hence, so does the theorem. \square

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