

Circuit-difference matroids

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Abstract

One characterization of binary matroids is that the symmetric difference of every pair of intersecting circuits is a disjoint union of circuits. This paper considers circuit-difference matroids, that is, those matroids in which the symmetric difference of every pair of intersecting circuits is a single circuit. Our main result shows that a connected regular matroid is circuit-difference if and only if it contains no pair of skew circuits. Using a result of Pfeil, this enables us to explicitly determine all regular circuit-difference matroids. The class of circuit-difference matroids is not closed under minors, but it is closed under series minors. We characterize the infinitely many excluded series minors for the class.

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1 Introduction

A matroid M is *circuit-difference* if $C_1 \Delta C_2$ is a circuit whenever C_1 and C_2 are distinct intersecting circuits of M . Evidently, all such matroids are binary. An example of such a matroid is the tipless binary r -spike, that is, the matroid whose binary representation is $[I_r \mid J_r - I_r]$, where J_r is the $r \times r$ matrix of all ones. Subsets X and Y of $E(M)$ are *skew* if $r(X \cup Y) = r(X) + r(Y)$. It is easy to check that no two circuits of the tipless binary r -spike are skew. The following is the main result of the paper.

Theorem 1. *Let M be a connected regular matroid. Then M is a circuit-difference matroid if and only if it has no pair of skew circuits.*

To see that this theorem does not extend to all binary matroids, consider the matroid S_8 for which a binary representation is shown in Figure 1. In this matroid, $\{1, 4, 7, 8\}$ and $\{2, 3, 5, 6, 8\}$ are circuits whose symmetric difference is the disjoint union of the circuits $\{1, 2, 6\}$ and $\{3, 4, 5, 7\}$. Thus S_8 is not a circuit-difference matroid. However, since $r(S_8) = 4$ and the only 3-circuits of S_8 contain 6, the matroid S_8 has no two skew circuits. Thus one implication of the last theorem fails for arbitrary connected binary matroids. However, as the next result shows, the other implication does hold in the more general context. The proof of this lemma will be given in the next section.

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \left[\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right] \end{array}.$$

Figure 1: A binary representation for S_8 .

Lemma 2. *Let M be a connected binary matroid. If M has a pair of skew circuits, then M is not circuit-difference.*

The terminology used here will follow [1] with one exception. In [1], the term “series extension” for matroids is defined as the addition of an element e to a matroid M to create a matroid M' in which $\{e, f\}$ is a cocircuit where f is an element of M , and $M'/e = M$. It will be expedient here to use the term “series extension” more broadly. We shall call a matroid M' a *series extension* of M if it is obtained from M by a sequence of one-element series-extension moves.

Pfeil [2] defined a connected matroid M to be *unbreakable* if M/F is connected for every flat F of M . He proved that a matroid is unbreakable if and only if its dual has no

two skew circuits, and he determined all unbreakable regular matroids. Combining Pfeil's two results gives the following.

Theorem 3. *A non-empty connected regular matroid M has no two skew circuits if and only if M is a series extension of one of the following matroids: $U_{0,1}$, $U_{1,m}$ for some $m \geq 1$; $M^*(K_n)$ for some $n \geq 1$; $M(K_{3,3})$; or R_{10} .*

By combining this theorem with our main result, we have the following explicit description of all regular circuit-difference matroids.

Corollary 4. *A regular matroid M is circuit-difference if and only if every component of M is a series extension of one of the following matroids: $U_{0,1}$, $U_{1,m}$ for some $m \geq 1$; $M^*(K_n)$ for some $n \geq 1$; $M(K_{3,3})$; or R_{10} .*

One can check explicitly, or deduce from the last corollary, that $M(K_4)$ is a circuit-difference matroid, but that $M(K_4)/e$ is not circuit-difference for each element e . Thus the class of circuit-difference matroids is not minor-closed. We shall show that this class is closed under series minors and will characterize the excluded series minors in Section 4. In the next section, we prove some auxiliary results that will be used in the proofs of the main results. The proof of Theorem 1 will be given in the third section.

2 Preliminary Results

We begin this section with the proof deferred from the last section.

Proof of Lemma 2. Let C_1 and C_2 be skew circuits of M , and let D be a circuit that meets C_1 and C_2 . Since C_1 and C_2 are skew, $|D - (C_1 \cup C_2)| > 0$. As is easily checked,

$$D - (C_1 \cup C_2) = (C_1 \Delta D) \cap (C_2 \Delta D).$$

Thus $C_1 \Delta D$ meets $C_2 \Delta D$. As their symmetric difference is the disjoint union of the circuits C_1 and C_2 , we deduce that M is not circuit-difference for either $C_1 \Delta D$ or $C_2 \Delta D$ is not a circuit, or both are circuits but their symmetric difference is not. \square

The straightforward proof of the next lemma is omitted.

Lemma 5. *Let M be a matroid. If M has a pair of skew circuits, then so does every series extension of M .*

The next result makes repeated use of the fact that if a circuit in a matroid meets a 2-cocircuit, then it contains that 2-cocircuit.

Lemma 6. Let M be a circuit-difference binary matroid and suppose that M' is obtained from M by adding an element e in series to an element f of M . Then M' is circuit-difference.

Proof. Let D_1 and D_2 be an intersecting pair of circuits of M' . Suppose first that $e \in D_1 \cap D_2$. Then $f \in D_1 \cap D_2$ and $\{D_1 - e, D_2 - e\}$ is an intersecting pair of circuits of M'/e . Thus $(D_1 - e) \Delta (D_2 - e)$, which equals $D_1 \Delta D_2$, is a circuit of M'/e . Hence $D_1 \Delta D_2$ or $(D_1 \Delta D_2) \cup e$ is a circuit of M' . Because $f \notin D_1 \Delta D_2$, the latter cannot occur. Hence $D_1 \Delta D_2$ is a circuit of M' .

Assume next that $e \in D_1 - D_2$. Then $f \in D_1 - D_2$. Now, $D_1 - e$ is a circuit of M'/e . Moreover, D_2 is a circuit of M'/e as otherwise M' would have a circuit that contains e and is contained in $D_2 \cup e$. As such a circuit would avoid f , we have a contradiction. We now know that $(D_1 - e) \Delta D_2$ is a circuit of M'/e containing f , so $D_1 \Delta D_2$ is a circuit of M' .

Finally, assume that $e \notin D_1 \cup D_2$. Then $f \notin D_1 \cup D_2$ and so D_1 and D_2 are circuits of M'/e . Hence so is $D_1 \Delta D_2$. As this set avoids f , it must also be a circuit of M' and the lemma is proved. \square

In the proof of Theorem 1, we will encounter a matroid with the property that the complement of every circuit is a circuit. We call such matroids *circuit-complementary*. Such matroids that are binary form an interesting subclass of the class of circuit-difference matroids; they are crucial in Section 4 when considering the excluded series minors for the latter class.

Lemma 7. Let M be a connected binary matroid that is circuit-complementary. Then M is a circuit-difference matroid.

Proof. Let C_1 and C_2 be an intersecting pair of circuits of M . Then $C_1 \Delta C_2$ is a disjoint union of circuits. If there are at least two circuits in this union, then, since this union avoids $C_1 \cap C_2$, we violate the property that the complement of every circuit is a circuit. \square

Again, the proof of the next result is elementary and is omitted.

Lemma 8. Let M be a connected binary matroid that is circuit-complementary.

- (i) If $\{e, f\}$ is a cocircuit of M , then M/e is circuit-complementary.
- (ii) If M' is a series extension of M , then M' is circuit-complementary.

Lemma 9. Let M be a cosimple connected graphic matroid that is circuit-complementary. Then $M \cong U_{1,4}$.

Proof. Let M be $M(G)$, the cycle matroid of the graph G . By Lemmas 2 and 7, M has no two skew circuits. Let C be a cycle of G . Then $E(G) - C$ is a cycle C' of G . Now, C and C' must have exactly two common vertices, otherwise G is not 2-connected or M has two skew circuits. It follows that G has two vertices u and v that are joined by four internally disjoint paths where these paths use every edge of G . As $M(G)$ is cosimple, we deduce that $M \cong U_{1,4}$. \square

The following lemma makes repeated use of the fact that in a loopless 2-connected graph, the set of edges meeting a vertex is a bond.

Lemma 10. *Let M be a cosimple connected cographic matroid that is circuit-complementary. Then $M \cong U_{1,4}$.*

Proof. Let M be $M^*(G)$, the bond matroid of the graph G . Then G is 2-connected and simple. Take a vertex v of G and let C_1 be the set of edges meeting v . Then C_1 is a bond in G and so is a circuit of M . Thus $E(G) - C$ is also a bond of G . Hence G has a vertex w that is not adjacent to v . Let C_2 be the set of edges meeting w . Then $E(G) = C_1 \cup C_2$ and G is isomorphic to $K_{2,n}$ for some $n \geq 2$. Let u be a vertex of G other than v or w . The complement of the set of edges meeting u is a bond of G . Thus $n = 2$ and G is a 4-cycle. Hence $M \cong U_{1,4}$. \square

Lemma 11. *Let M be a connected cosimple regular matroid that is circuit-complementary. Then M is isomorphic to $U_{1,4}$ or R_{10} .*

Proof. If M is graphic or cographic, then, by Lemmas 9 and 10, $M \cong U_{1,4}$. Now assume that M is neither graphic nor cographic and is not isomorphic to R_{10} . Then, by Seymour's Regular Matroids Decomposition Theorem [3], as M is connected, it can be obtained from graphic matroids, cographic matroids and copies of R_{10} by a sequence of 2-sums and 3-sums. Moreover, each matroid that is used to build M occurs as a minor of M .

11.1. M is 3-connected.

If M is not 3-connected, then M has a 2-separation, (X, Y) . Then M is the 2-sum, with basepoint p say, of matroids M_X and M_Y with ground sets $X \cup p$ and $Y \cup p$, respectively. As M is connected, so are M_X and M_Y . Suppose X is independent in M . Then M_X must be a circuit with at least three elements. Thus M is not cosimple, a contradiction. We may now assume that both X and Y contain circuits of M . Hence, by the circuit-complementary property, both X and Y are circuits of M . As $r(X) + r(Y) = r(M) + 1$, we see that $(|X| - 1) + (|Y| - 1) = r(M) + 1$ and, consequently, $r^*(M) = |X| + |Y| - r(M) = 3$. Then M^* is a rank-3 simple binary connected matroid having X and Y as disjoint cocircuits. It follows that M^* is graphic, a contradiction. We conclude that 11.1 holds.

We may now assume that there are matroids M_1 and M_2 each with at least seven elements such that $E(M_1) \cap E(M_2)$ is a triangle, T , in both matroids, and M is the 3-sum

of M_1 and M_2 across this triangle. Moreover, M_1 and M_2 are both minors of M , and $E(M_i) - T$ spans T in M_i for each i . Let $X_i = E(M_i) - T$. Then (X_1, X_2) is a 3-separation of M . Suppose X_1 is independent in M . As X_1 spans T , it follows that M_1 has rank $|X_1|$, so M_1^* has rank three and has T as a triad. Since M is cosimple, no element of X_1 is in a 2-circuit of M_1^* . As M_1 is binary, it follows that $|X_1| \leq 3$ so $|E(M_1)| \leq 6$, a contradiction. We conclude by the circuit-complementary property that both X_1 and X_2 must be circuits of M . Then, as $r(X_1) + r(X_2) = r(M) + 2$, we have $r(M) = |X_1| + |X_2| - 4$ so M^* has rank four, has (X_1, X_2) as a 3-separation and has each of X_1 and X_2 as a cocircuit. Since M^* is a disjoint union of cocircuits, it is affine. As M^* is simple, $|E(M^*)| \leq 8$. But $|X_i| \geq 4$ for each i , so $|E(M^*)| = 8$ and $M^* \cong AG(3, 2)$. This contradicts the fact that M is regular and thereby completes the proof of the lemma. \square

Lemma 12. *Let M be a connected regular matroid and let X be a series class in M . Then $M \setminus X$ and M/X cannot both be connected and circuit-complementary.*

Proof. Assume that the lemma fails. By Lemma 11, each of $M \setminus X$ and M/X is a series extension of $U_{1,4}$ or R_{10} . Note that X is independent in M . For every series extension M' of $U_{1,4}$ or R_{10} , we have that

$$(r(M'), r^*(M')) \in \{(k+1, 3), (k+5, 5) : k \geq 0\}.$$

Thus, for some non-negative integer m ,

$$(r(M \setminus X), r^*(M \setminus X)) \in \{(m+1, 3), (m+5, 5)\}.$$

Now let $|X| = t$. Then $(r(M), r^*(M)) \in \{(m+t, 4), (m+4+t, 6)\}$. Thus $(r(M/X), r^*(M/X)) \in \{(m, 4), (m+4, 6)\}$, so M/X cannot be a series extension of $U_{1,4}$ or R_{10} , a contradiction. \square

Although the following lemma is well known, we include a proof for completeness.

Lemma 13. *Let Y be a set in a connected matroid M such that $|Y| \geq 2$ and $M|Y$ is connected. Let W be a minimal non-empty subset of $E(M) - Y$ such that M has a circuit C such that $C \cap Y \neq \emptyset$ and $C - Y = W$. Then W is a series class of $M|(Y \cup W)$.*

Proof. This is certainly true if $|W| = 1$. Now, suppose that w_1 and w_2 are distinct elements of W that are not in series in $M|(Y \cup W)$. Then $M|(Y \cup W)$ has a circuit K containing w_1 and not w_2 . As W is independent, K meets Y . But $K \cap W \subseteq W - w_2$. Thus we have a contradiction to the choice of W . We deduce that every two elements of W are in series in $M|(Y \cup W)$. Since $M|Y$ is connected, no element of Y is in series with an element of W . Thus W is indeed a series class of $M|(Y \cup W)$. \square

3 The Proof of Main Theorem

In this section, we prove the main result of the paper.

Proof of Theorem 1. Let M be a regular connected matroid. By Lemma 2, if M has a pair of skew circuits, then M is not circuit-difference. To prove the converse, consider all connected regular matroids with no two skew circuits that are not circuit-difference, and choose M to be such a matroid with the minimum number of elements. Then, by Lemma 6, M is cosimple. Let C_1 and C_2 be a pair of intersecting circuits of M such that $C_1 \Delta C_2$ is not a circuit and $|C_1 \cup C_2|$ is a minimum among such pairs. As $M|(C_1 \cup C_2)$ is connected, we must have that $E(M) = C_1 \cup C_2$ by our choice of M . Now, $C_1 \Delta C_2$ is a disjoint union of at least two circuits.

13.1. *If D is a circuit of M contained in $C_1 \Delta C_2$, then $(C_1 \Delta C_2) - D$ is a circuit of M .*

Clearly, D meets both $C_1 - C_2$ and $C_2 - C_1$ but contains neither of these sets. The choice of $\{C_1, C_2\}$ implies that $C_1 \Delta D$ is a circuit and hence that $(C_1 \Delta D) \Delta C_2$ is a circuit. The last set is $(C_1 \Delta C_2) - D$, so 13.1 holds.

Let $Z = C_1 \Delta C_2$. As M has no two skew circuits, $M|Z$ is connected and, by 13.1, it is circuit-complementary. Thus, by Lemma 11, $M|Z$ is a series extension of $U_{1,4}$ or of R_{10} . Let X be a minimal non-empty subset of $C_1 \cap C_2$ such that M has a circuit whose intersection with $C_1 \cap C_2$ is X . Then, by Lemma 13, X is a series class of $M|(Z \cup X)$. Thus every circuit of $M|(Z \cup X)$ that meets X must contain X .

13.2. *Every circuit of $M|Z$ is a circuit of $(M|(Z \cup X))/X$.*

Let D be a circuit of M that is contained in Z . Then D meets both $C_1 - C_2$ and $C_2 - C_1$ and, by 13.1, $Z - D$ is a circuit of M that also meets both $C_1 - C_2$ and $C_2 - C_1$. Assume that D is not a circuit of $(M|(Z \cup X))/X$. Then $M|(Z \cup X)$ has a circuit K such that $K \subseteq D \cup X$ and $K \cap D \neq D$. Thus K meets and so contains X . Hence $K - D = X$. As $|K \cup D| = |X \cup D| < |C_1 \cup C_2|$, it follows that $K \Delta D$ is a circuit of M and hence that $K \Delta D$ meets $C_1 - C_2$ and $C_2 - C_1$.

As $|C_1 \cup K| < |C_1 \cup D| < |C_1 \cup C_2|$, the choice of $\{C_1, C_2\}$ implies that $C_1 \Delta K$ is a circuit C of M and that $C_1 \Delta (Z - D)$ is a circuit C' of M . As C and C' both contain the non-empty set $(D - K) \cap C_1$ and both avoid the non-empty set $(D - K) \cap C_2$, we see that $|C \cup C'| < |C_1 \cup C_2|$ and $C \Delta C'$ is a circuit of M . This circuit is $[C_1 \Delta K] \Delta [C_1 \Delta (Z - D)]$, which equals $K \Delta (Z - D)$. But the last set is a disjoint union of two circuits, a contradiction. Thus 13.2 holds.

We know that $M|Z$ is connected and circuit-complementary. Moreover, the choice of X implies that $M|(Z \cup X)$ has a circuit that meets $C_1 \cap C_2$ in X . Therefore $M|(Z \cup X)$ is connected. Moreover, by 13.2, $(M|(Z \cup X))/X$ is connected. It follows by Lemma 12

that $(M|(Z \cup X))/X$ is not circuit-complementary. Thus $(M|(Z \cup X))/X$ has a circuit J such that $Z - J$ is not a circuit of $(M|(Z \cup X))/X$. If J is a circuit of $M|Z$, then, as $M|Z$ is circuit-complementary, $Z - J$ is a circuit of $M|Z$. Thus, by 13.2, we obtain the contradiction that $Z - J$ is a circuit of $(M|(Z \cup X))/X$. We deduce that J is not a circuit of $M|Z$. Then $J \cup X'$ is a circuit, K , of $M|Z$ for some non-empty subset X' of X . By the choice of X , it follows that $X' = X$. Now, $Z = D \cup D'$ for some disjoint circuits D and D' . We deduce using 13.2 that K meets both D and D' but contains neither. Hence $|K \cup D| < |C_1 \cup C_2|$, so $K \triangle D$ is a circuit of M . As $|D' \cup (K \triangle D)| < |C_1 \cup C_2|$, we see that $D' \triangle (K \triangle D)$ is a circuit of M , that is, $(Z - K) \cup X$ is a circuit of M . Thus $Z - J$ is a circuit of $(M|(Z \cup X))/X$, a contradiction. \square

4 Excluded Series Minors of Circuit-Difference Matroids

In this section, we show that the class of circuit-difference matroids is closed under series minors, and we characterize the infinitely many excluded series minors for this class.

Lemma 14. *The class of circuit-difference matroids is closed under series minors.*

Proof. Let M be a circuit-difference matroid. Evidently, $M \setminus e$ is circuit-difference for all $e \in E(M)$. Now let $\{e, f\}$ be a cocircuit of M and consider M/e . A circuit C of M/e contains f if and only if $C \cup e$ is a circuit of M . Thus the collection $\mathcal{C}(M/e)$ of circuits of M/e is $\mathcal{C}(M \setminus e) \cup \{C - e : f \in C \in \mathcal{C}(M)\}$. It is now routine to check that M/e is a circuit-difference matroid. \square

Let N_5 be the 5-element matroid that is obtained from a triangle by adding single elements in parallel to exactly two of its elements. This is easily seen to be an excluded series minor for the class of circuit-difference matroids. Although the next proposition is not needed for the proof of the main result of this section, it seems to be of independent interest.

Proposition 15. *A connected binary matroid M has a pair of skew circuits if and only if M has a series minor isomorphic to N_5 .*

Proof. If M has a series minor isomorphic to N_5 , then, by Lemma 5, as N_5 has a pair of skew circuits, so does M . For the converse, let C_1 and C_2 be a pair of skew circuits of M , and let D be a circuit meeting both such that $|D - (C_1 \cup C_2)|$ is a minimum. Let $M' = M|(C_1 \cup C_2 \cup D)$. Next we show the following.

15.1. *If $C_1 - D$ or $C_1 \cap D$ contains $\{x, y\}$, then $\{x, y\}$ is a cocircuit of M' .*

Suppose that this fails. Then M' has a circuit K that contains x but not y . Assume first that K meets C_2 . Then, by the choice of D , we must have that $K - (C_1 \cup C_2) =$

$D - (C_1 \cup C_2)$. Then $K \Delta D$ is a disjoint union of circuits that is contained in $(C_1 \cup C_2) - y$ or $(C_1 \cup C_2) - x$. But, for each z in C_1 , the matroid $(M|(C_1 \cup C_2)) \setminus z$ has C_2 as its only circuit. As $K \Delta D \neq C_2$, we have a contradiction. We deduce that K avoids C_2 . As $y \notin K$, we must have that $K \cap (D - (C_1 \cup C_2))$ is non-empty. Then $K \Delta D$ is a disjoint union of circuits that does not contain $D - (C_1 \cup C_2)$. One such circuit must meet $C_2 \cap D$ and C_1 . But this violates the choice of D . Thus 15.1 holds.

By 15.1 and symmetry, we can perform a sequence of series contractions in M' , reducing each of the sets $C_1 - D$, $C_1 \cap D$, $C_2 \cap D$, and $C_2 - D$ to a single element. The resulting matroid is a series minor of M that has two disjoint 2-circuits such that deleting one element from each leaves a circuit with at least three elements. It follows that M has N_5 as a series minor. \square

We call a matroid *hyperplane-complementary* if the complement of every hyperplane is a hyperplane. One such matroid is the binary affine geometry $AG(r-1, 2)$ of rank at least two. The next result determines all simple binary hyperplane-complementary matroids. For all k , every rank- k flat of $AG(r-1, 2)$ is isomorphic to $AG(k-1, 2)$.

Lemma 16. *A simple rank- r binary matroid M is hyperplane-complementary if and only if $r \geq 2$ and $M \cong AG(r-1, 2) \setminus X$ for some set X such that $AG(r-1, 2)|X$ does not contain a copy of $AG(r-3, 2)$.*

Proof. Suppose that M is hyperplane-complementary. Then $r \geq 2$. Moreover, $E(M)$ is a disjoint union of cocircuits, so every circuit of M has even cardinality. Hence we can view M as $AG(r-1, 2) \setminus X$ for some set X . Let $E = E(AG(r-1, 2))$. Then $E(M) = E - X$. Assume that $AG(r-1, 2)|X$ contains a copy, Z , of $AG(r-3, 2)$. For $y \in E - X$, consider the closure, $\text{cl}_A(Z \cup y)$, of $Z \cup y$ in $AG(r-1, 2)$. This closure is a rank- $(r-1)$ flat of $AG(r-1, 2)$ and is thus isomorphic to $AG(r-2, 2)$. Let $Y = \text{cl}_A(Z \cup y) \cap (E - X)$ and $W = (E - X) - Y$. Then Y is contained in some copy of $AG(r-3, 2)$, and W is contained in some copy of $AG(r-2, 2)$. Thus $r(Y) \leq r-2$ and $r(W) \leq r-1$. Hence W is contained in a hyperplane W' of M whose complement in $E(M)$ is not a hyperplane. Thus M is not hyperplane-complementary, a contradiction.

Now let $M = AG(r-1, 2) \setminus X$ where $r \geq 2$ and $AG(r-1, 2)|X$ does not contain a copy of $AG(r-3, 2)$. Let H be a hyperplane of $AG(r-1, 2)$. Then $AG(r-1, 2)|H = AG(r-2, 2)$. If $r(H - X) \leq r-2$, then $H - X$ is contained in some copy of $AG(r-3, 2)$ that is contained in H and so, as $AG(r-2, 2)$ is hyperplane-complementary, X contains a copy of $AG(r-3, 2)$. This contradiction implies that the hyperplanes of M are all of the sets of the form $H - X$ where H is a hyperplane of $AG(r-1, 2)$. As $AG(r-1, 2)$ is hyperplane-complementary, so is M . \square

Recall that $AG(r-1, 2)$ is obtained from the projective geometry $PG(r-1, 2)$ by deleting a hyperplane, that is, by deleting a copy of $PG(r-2, 2)$. It is a well-known

consequence of the unique representability of binary matroids that if $PG(r-1, 2)|E_1 \cong PG(r-1, 2)|E_2$, then $PG(r-1, 2)\setminus E_1 \cong PG(r-1, 2)\setminus E_2$. Thus, as all single-element deletions of $PG(r-2, 2)$ are isomorphic, there is, up to isomorphism, a unique simple binary rank- r single-element extension of $AG(r-1, 2)$. We shall denote this extension by $AG(r-1, 2) + e$.

Let \mathbb{M} be the set of all matroids of rank at least three of the form $[AG(r-1, 2) + e]\setminus X$ such that $AG(r-1, 2)\setminus X$ is hyperplane-complementary of rank r . Thus N_5^* is the unique rank-3 member of \mathbb{M} while its rank-4 members are the tipped binary 4-spike and a non-tip deletion thereof, that is, S_8 . We now show that the duals of the matroids in \mathbb{M} are precisely the excluded series minors for the class of circuit-difference matroids.

Theorem 17. *A binary matroid M is an excluded series minor for the class of circuit-difference matroids if and only if $M^* \in \mathbb{M}$.*

Proof. Let M be an excluded series minor for the class of circuit-difference matroids. By Lemma 6, M is cosimple. Let C_1 and C_2 be intersecting circuits of M such that $C_1 \Delta C_2$ is not a circuit and $|C_1 \cup C_2|$ is minimal.

17.1. $M^* \in \mathbb{M}$.

Evidently, $E(M) = C_1 \cup C_2$. Then C_1 and C_2 are the only circuits of M containing $C_1 - C_2$ and $C_2 - C_1$, respectively. Now, letting $x \in C_1 \cap C_2$, suppose that $(C_1 \cap C_2) - x$ contains an element y . Then, as x and y are not in series, M has a circuit D containing x but not y . As D meets both C_1 and C_2 , we have, by the choice of $\{C_1, C_2\}$, that $C_1 \Delta D$, $C_2 \Delta D$, and hence $(C_1 \Delta D) \Delta (C_2 \Delta D)$ are circuits of M . This last circuit is $C_1 \Delta C_2$, so we have a contradiction. Thus $C_1 \cap C_2 = \{x\}$.

To see that M/x is circuit-complementary, suppose that $C_1 \Delta C_2$ has a partition (D_1, D_2, \dots, D_k) into circuits of M/x such that $k \geq 3$. Then each D_i meets $C_1 - C_2$ and $C_2 - C_1$. For each i , we see that D_i or $D_i \cup x$ is a circuit, D' , of M . The choice of $\{C_1, C_2\}$ implies that $C_1 \Delta D'$, and hence $C_1 \Delta D' \Delta C_2$, is a circuit of M . As $(C_1 \Delta C_2) - D_i$ is not a circuit of M/x , we deduce that $D' = D_i$ for all i . Hence the circuit $C_1 \Delta D_i \Delta C_2$ of M is the union of $k-1$ circuits of M , a contradiction. Thus M/x is circuit-complementary. Therefore, $M^* \setminus x$ is hyperplane-complementary. As M is cosimple, M^* is simple. Moreover, M^* has $C_1 - x$ and $C_2 - x$ as hyperplanes, so M^* has the form $[AG(r-1, 2) + e]\setminus X$ where $AG(r-1, 2)\setminus X$ is hyperplane-complementary of rank r . Since M^* is connected, $r(M^*) \geq 2$. But if $r(M^*) = 2$, then $M^* \cong U_{2,3}$, so $M \cong U_{1,3}$ and M is circuit-difference, a contradiction. Thus $M^* \in \mathbb{M}$, so 17.1 holds.

To prove the converse, let $M^* = [AG(r-1, 2) + e]\setminus X$ where $AG(r-1, 2)\setminus X$ is hyperplane-complementary of rank r and $r \geq 3$. By Lemma 16, $AG(r-1, 2)|X$ does not contain a copy of $AG(r-3, 2)$. Consider $AG(r-1, 2) + e$ and let H_0 be the hyperplane of $PG(r-1, 2)$ whose deletion gives $AG(r-1, 2)$. Take a rank- $(r-2)$ flat F of $PG(r-1, 2)$ that is contained in H_0 and avoids e . Apart from H_0 , there are exactly two hyperplanes,

H_1 and H_2 , of $PG(r-1, 2)$ that contain F . Then $H_1 - H_0$ and $H_2 - H_0$ are hyperplanes of $AG(r-1, 2) + e$, and $H_1 - (H_0 \cup X)$ and $H_2 - (H_0 \cup X)$ are hyperplanes of $[AG(r-1, 2) + e] \setminus X$. The complements of these two hyperplanes are circuits C_1, C_2 of M that meet in the element e . We now note that $C_1 \Delta C_2$ is not a circuit of M otherwise $\{e\}$ is a hyperplane of M^* , and we obtain the contradiction that $r(M^*) \leq 2$. Hence M is not circuit-difference.

17.2. *If D is a circuit of $M \setminus e$, then $e \notin \text{cl}(D)$.*

Suppose that $e \in \text{cl}(D)$ for some circuit D of $M \setminus e$. Then there is a partition $\{X_1, X_2\}$ of D such that X_i is a circuit of M/e for both i . As M/e is circuit-complementary having X_1 and X_2 as disjoint circuits, $X_1 \cup X_2 = E(M/e) = E(M) - e$. This is a contradiction as $X_1 \cup X_2 = D \subsetneq E(M) - e$. Hence 17.2 holds.

17.3. *$M \setminus f$ is circuit-difference for all f in $E(M)$.*

Suppose some $M \setminus f$ is not circuit-difference. Then it has a pair of intersecting circuits D_1, D_2 such that $D_1 \Delta D_2$ contains a pair of disjoint circuits K_1, K_2 . Suppose first that both D_1 and D_2 avoid e . Then so do K_1 and K_2 . Thus, by 17.2, none of D_1, D_2, K_1 , or K_2 has e in its closure. Hence all of D_1, D_2, K_1 , and K_2 are circuits of M/e , so M/e is not circuit-difference, a contradiction. Hence at least one of D_1 and D_2 must contain e , so $f \neq e$.

Now suppose $e \in D_1 - D_2$ and $e \in K_1$. Then $D_1 - e$ and D_2 are intersecting circuits of M/e with circuits $K_1 - e$ and K_2 in their symmetric difference. This again contradicts the fact that M/e is circuit-difference. Hence, by symmetry, we must have that $e \in D_1 \cap D_2$. Consequently, $K_1, K_2, D_1 - e$, and $D_2 - e$ are circuits of M/e . If $D_1 \cap D_2 = \{e\}$, then $D_1 - e$ and $D_2 - e$ are disjoint. Thus their union is $E(M/e)$. But this union avoids f , a contradiction. Hence $(D_1 \cap D_2) - e$ must be non-empty. But then $D_1 - e$ and $D_2 - e$ are intersecting circuits of M/e so their symmetric difference, which equals $D_1 \Delta D_2$, is a circuit of M/e . However, this symmetric difference contains K_1 and K_2 , which are circuits of M/e . This contradiction completes the proof of 17.3.

As M^* is simple, M is cosimple and hence no series contractions can be performed. Thus, by 17.3, every series minor of M is circuit-difference and the theorem holds. \square

The next result follows immediately by combining the last theorem with Tutte's excluded-minor characterization of binary matroids [4].

Corollary 18. *A matroid M is an excluded series minor for the class of circuit-difference matroids if and only if $M \cong U_{n,n+2}$ for some $n \geq 2$, or M^* can be obtained from $AG(r-1, 2) + e$ for some $r \geq 3$ by deleting some set X such that $e \notin X$ and $AG(r-1, 2)|X$ does not contain a copy of $AG(r-3, 2)$.*

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