

# THE BINARY MATROIDS WITH NO ODD CIRCUITS OF SIZE EXCEEDING FIVE

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ABSTRACT. Generalizing a graph-theoretical result of Maffray to binary matroids, Oxley and Wetzler proved that a connected simple binary matroid  $M$  has no odd circuits other than triangles if and only if  $M$  is affine,  $M$  is isomorphic to  $M(K_4)$  or  $F_7$ , or  $M$  is the cycle matroid of a graph consisting of a collection of triangles all sharing a common edge. In this paper, we show that if  $M$  is a 3-connected binary matroid having a 5-element circuit but no larger odd circuit, then  $M$  has rank less than six; or  $M$  has rank six and is one of nine sporadic matroids; or  $M$  can be obtained by attaching together, via generalized parallel connection across a common triangle, a collection of copies of  $F_7$  and  $M(K_4)$  and then possibly deleting up to two elements of the common triangle. From this, we deduce that a 3-connected simple graph with a 5-cycle but no larger odd cycle is obtained from  $K_{3,n}$  for some  $n \geq 3$  by adding one, two, or three edges between the vertices in the 3-vertex class.

## 1. INTRODUCTION

The terminology used here will follow Oxley [8]. Generalizing the well-known result that a graph is bipartite if and only if it has no odd cycles, Maffray [7, Theorem 2] proved the following theorem. For each  $n \geq 1$ , the graph  $K'_{2,n}$  is obtained from  $K_{2,n}$  by adding an edge joining the vertices in the two-vertex class (see Figure 1).

**Theorem 1.1.** *A 2-connected simple graph  $G$  has no odd cycles of length exceeding three if and only if*

- (i)  $G$  is bipartite; or
- (ii)  $G \cong K_4$ ; or
- (iii)  $G \cong K'_{2,n}$  for some  $n \geq 1$ .

It is well-known [2, 4] that the cycle matroid of a graph  $G$  is bipartite if and only if  $M(G)$  is an affine binary matroid. The following generalization of the last theorem to binary matroids was proved in [10]. A circuit in a matroid is *odd* if it has odd cardinality; otherwise, it is *even*.

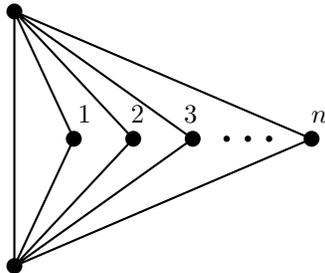
**Theorem 1.2.** *A 2-connected simple binary matroid  $M$  has no odd circuits other than triangles if and only if*

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FIGURE 1.  $K'_{2,n}$ 

- (i)  $M$  is affine; or
- (ii)  $M \cong M(K_4)$  or  $F_7$ ; or
- (iii)  $M \cong M(K'_{2,n})$  for some  $n \geq 1$ .

The goal of this paper is to prove the next theorem. First we introduce a class of matroids that play a key role in the theorem. Let  $M_1, M_2, \dots, M_k$  be binary matroids, each of which contains a triangle  $T$  and is isomorphic to  $M(K_4)$  or  $F_7$ . Suppose that  $E(M_i) \cap E(M_j) = T$  for all distinct  $i$  and  $j$  in  $\{1, 2, \dots, k\}$ . Let  $P_2$  be the binary matroid  $P_T(M_1, M_2)$ , the generalized parallel connection of  $M_1$  and  $M_2$  across the triangle  $T$ . For all  $i$  in  $\{3, 4, \dots, k\}$ , let  $P_i = P_T(P_{i-1}, M_i)$ . We call  $P_k$  a *book*  $B_{k,T}$  with  $k$  pages,  $M_1, M_2, \dots, M_k$ , and we refer to  $T$  as the *spine of the book*. Note that if  $k \geq 3$ , then  $r(B_{k,T}) \geq 5$ .

**Theorem 1.3.** *A 3-connected binary matroid  $M$  has no odd circuits of size exceeding five if and only if*

- (i)  $M$  is affine; or
- (ii)  $r(M) \leq 5$ ; or
- (iii)  $M$  is obtained from an  $n$ -page book for some  $n \geq 4$  by deleting up to two elements of its spine; or
- (iv)  $M$  has rank six and is one of nine non-regular matroids.

Binary representations of the nine matroids in (iv) are given in Figure 2. It is routine to check that each has an  $F_7$ - or  $F_7^*$ -minor, so none is regular. No special insight into these nine matroids can be offered except for the two with ten elements and rank six. Their duals are the complements in  $PG(3, 2)$  of  $U_{3,4} \oplus U_{1,1}$  and of the parallel connection of two copies of  $U_{2,3}$ . Apart from the two duals just noted, there are exactly two other simple binary rank-4 matroids with ten elements. They are  $M(K_5)$  and  $\widetilde{K}_5$  where the latter [14] is the unique rank-4 simple single-element extension of  $M^*(K_{3,3})$ .

An immediate consequence of the last theorem is the following graph result. The graph  $K'''_{3,t}$  is the simple graph that is obtained from  $K_{3,t}$  by adding three *special edges* each of which joins two vertices in the three-vertex class of the vertex bipartition.

$$\begin{array}{ccc}
\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} \\
\text{(A) } A_{21} & \text{(B) } A_{22} & \text{(C) } A_{23} \\
\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \\
\text{(D) } A_{24} & \text{(E) } A_{25} & \text{(F) } A_{72} \\
\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix} \\
\text{(G) } A_{76} & \text{(H) } A_{77} & \text{(I) } A_{78}
\end{array}$$

FIGURE 2. The nine matroids in Theorem 1.3(iv) are represented by  $[I_6|A]$  where  $A$  is one of the matrices above.

**Corollary 1.4.** *A 3-connected simple graph  $G$  has no odd cycles of size exceeding five if and only if*

- (i)  $G$  is bipartite; or
- (ii)  $|V(G)| \leq 6$ ; or
- (iii) for some  $t \geq 4$ , the graph  $G$  is obtained from  $K_{3,t}'''$  by deleting up to two special edges.

By applying Theorem 1.3 to the bond matroid of a 3-connected simple graph, we obtain the following result.

**Corollary 1.5.** *A 3-connected simple graph  $G$  has no odd bonds of size exceeding five if and only if*

- (i)  $G$  is Eulerian; or
- (ii)  $|E(G)| \leq |V(G)| + 4$ .

We observe that (ii) implies that  $|V(G)| \leq 8$ . Corollary 1.4 and Theorem 1.1 are the first steps in proving the following result [13].

**Theorem 1.6.** *Suppose  $n \geq 2$ . Let  $G$  be an  $n$ -connected simple graph having a cycle of length  $2n - 1$ . Then  $G$  has no odd cycle of length exceeding  $2n - 1$  if and only if*

- (i)  $|V(G)| \leq 2n$ ; or
- (ii) for some  $t \geq n + 1$ , the graph  $G$  is isomorphic to a graph that is obtained from  $K_{n,t}$  by adding at least one and at most  $\binom{n}{2}$  edges each having both ends in the  $n$ -vertex class of the vertex bipartition.

## 2. PRELIMINARIES

In the proofs of the main results, much of the analysis will focus on the cycle matroids of graphs. We shall also use single-element extensions of graphic matroids. A *graft* is a pair  $(G, \gamma)$  consisting of a graph  $G$  and a subset  $\gamma$  of  $V(G)$ . The matroid  $M(G, \gamma)$  of  $(G, \gamma)$  is the matroid that has as a binary representation the matrix that is obtained from the mod-2 vertex-edge incidence matrix of  $G$  by adjoining a new column  $e_\gamma$  that is the incidence vector of the set  $\gamma$ . We indicate that we are considering a graft by enclosing each vertex of  $\gamma$  in a box and labelling one of these boxed vertices by  $e_\gamma$ . As an example, let  $G = K_4$  and  $\gamma = V(G)$ . We draw  $(G, \gamma)$  as  $K_4$  with each of the four vertices in its own box. Clearly,  $M(K_4, \gamma) \cong F_7$ . Similarly,  $F_7^* \cong M(K_{2,3}, \gamma')$  where  $\gamma'$  is obtained from  $V(K_{2,3})$  by omitting one degree-3 vertex. The reader unfamiliar with grafts will find them discussed in [8] beginning in Section 10.3. It will be convenient here to abbreviate the graft matroid  $M(G, \gamma)$  as  $M(G)$  and to call it the *cycle matroid* of  $(G, \gamma)$ .

For a matroid  $M$ , we shall use three functions relating to connectivity. The *connectivity function*,  $\lambda_M$ , of  $M$  is defined by  $\lambda_M(X) = r(X) + r(E(M) - X) - r(M)$  for all subsets  $X$  of  $E(M)$ . Equivalently,  $\lambda_M(X) = r(X) + r^*(X) - |X|$ . For subsets  $X$  and  $Y$  of  $E(M)$ , the *local connectivity*,  $\Pi(X, Y)$ , between  $X$  and  $Y$  is defined by  $\Pi(X, Y) = r(X) + r(Y) - r(X \cup Y)$ . When  $X$  and  $Y$  are disjoint, the connectivity,  $\kappa_M(X, Y)$ , between  $X$  and  $Y$  equals  $\min\{\lambda_M(Z) : X \subseteq Z \subseteq E(M) - Y\}$ .

We shall make frequent implicit use of the following elementary result.

**Lemma 2.1.** *If  $X$  is a disjoint union of circuits in a matroid and  $r(X) = |X| - 1$ , then  $X$  is a circuit.*

We will also use the following result of Lemos [6].

**Theorem 2.2.** *Suppose  $M$  is a 3-connected matroid with at least four elements and let  $C$  be a circuit of  $M$ . If  $M \setminus e$  is not 3-connected for every  $e$  in  $C$ , then  $C$  meets two distinct triads of  $M$ .*

Leo [5] proved the following variant of the last theorem.

**Theorem 2.3.** *Suppose  $M$  is a 3-connected matroid with at least four elements,  $C$  is a circuit of  $M$ , and  $f$  is an element of  $C$ . If  $M \setminus e$  is not 3-connected for every  $e$  in  $C - f$ , then*

- (i)  $M$  has at least two triads containing  $f$ ; or

(ii)  $M$  has a triad meeting  $C$  but avoiding  $f$ .

Next we prove two further variants on Lemos's theorem.

**Lemma 2.4.** *Let  $Z$  be a subset of a 3-connected matroid  $M$  where  $M|Z$  is connected and  $|Z| \geq 2$ . Suppose that  $M \setminus e$  is not 3-connected for all  $e$  in  $E(M) - \text{cl}(Z)$ . If there is a circuit  $D$  of  $M$  that avoids  $\text{cl}(Z)$ , then  $M$  has a triad that avoids  $\text{cl}(Z)$ .*

*Proof.* The hypotheses imply that  $|E(M)| \geq 4$ . By Theorem 2.2,  $D$  meets a triad of  $M$ . By orthogonality, this triad must contain at least two elements of  $D$ . By orthogonality again, it cannot meet  $\text{cl}(Z)$ .  $\square$

**Lemma 2.5.** *Let  $Z$  be a subset of a 3-connected matroid  $M$  where  $M|Z$  is connected and  $r(Z) \geq 3$ . Suppose that  $M \setminus e$  is not 3-connected for all  $e$  in  $E(M) - Z$ . If  $e \in E(M) - \text{cl}(Z)$ , then*

- (i)  $e$  is in a triad that avoids  $Z$ ; or
- (ii)  $e$  is in a triad that contains exactly two elements of  $Z$  and these elements are in series in  $M|Z$ ; or
- (iii)  $e$  is in neither a triad nor a triangle of  $M$ ; moreover,  $M/e$  is 3-connected and  $M/e \setminus f$  is not 3-connected for all  $f$  in  $E(M) - (Z \cup e)$ .

*Proof.* As  $M|Z$  is connected of rank at least three,  $Z$  is a union of circuits of  $M$ . By orthogonality, a triad of  $M$  that meets  $E(M) - \text{cl}(Z)$  must either be contained in  $E(M) - \text{cl}(Z)$  or must contain exactly two elements of  $Z$ . If the second possibility occurs, then, as  $M|Z$  is connected, the two elements of  $Z$  are in series in  $M|Z$ .

Suppose  $e \in E(M) - \text{cl}(Z)$ . We may now assume that  $e$  is not in a triad, otherwise (i) or (ii) holds. Suppose  $e$  is in a triangle  $T$  of  $M$ . As  $e \notin \text{cl}(Z)$ , at least one element of  $T - e$  is in  $E(M) - \text{cl}(Z)$ . Then Tutte's Triangle Lemma [12] (see also [8, Lemma 8.7.7]) gives us the contradiction that  $e$  is in a triad. Hence  $e$  is not in a triangle. As  $M \setminus e$  is not 3-connected and  $e$  is not in a triad,  $\text{co}(M \setminus e)$  is not 3-connected. By Bixby's Lemma [1] (see also [8, Lemma 8.7.3]),  $\text{si}(M/e)$  is 3-connected. Since  $e$  is in no triangle,  $\text{si}(M/e) = M/e$ . Hence  $M/e$  is 3-connected.

Take  $f \in E(M) - (Z \cup e)$ . Then  $M \setminus f$  is not 3-connected. If  $M \setminus f/e$  is 3-connected, then  $\{e, f\}$  is in a triad of  $M$ , a contradiction. We conclude that (iii) holds.  $\square$

**Lemma 2.6.** *Suppose that  $M$  is a 3-connected binary matroid and that  $M|Z$  is connected having rank at least two for some  $Z \subseteq E(M)$ . Suppose  $T$  is a triangle of  $M$  such that  $M \setminus e$  is not 3-connected for all  $e$  in  $T$ . If  $T \cap Z = \{t\}$  and  $C$  is a circuit of  $M|Z$  containing  $t$ , then either  $M|Z$  has a series class of size at least three containing  $t$ , or  $M$  has a circuit of size  $|C| + 1$ .*

*Proof.* Let  $T = \{t, f, g\}$ . By Tutte's Triangle Lemma,  $M$  has two distinct triads that meet  $T$  in exactly two elements. Suppose these triads meet  $T$  in  $\{t, f\}$  and  $\{t, g\}$ . Then, by orthogonality, the third elements of these triads

must be in  $Z$ . Then  $M \setminus f, g$ , and hence  $M|Z$ , has a series class of size at least three containing  $t$ . Thus we may assume that  $\{f, g\}$  is in a triad,  $T^*$ , of  $M$ . By hypothesis,  $\{f, g\}$  avoids  $Z$ . As  $M|Z$  is connected, by orthogonality,  $T^*$  avoids  $Z$ . Thus  $C \Delta T$  contains  $\{f, g\}$  and so has rank  $|C|$  and size  $|C| + 1$ . Hence  $C \Delta T$  is a circuit of  $M$  of size  $|C| + 1$ .  $\square$

We conclude this section with a result that we will use to construct minimally 3-connected matroids.

**Lemma 2.7.** *Let  $M$  be a 3-connected matroid having a 5-circuit. Then  $M$  has a 3-connected restriction  $M'$  having a 5-circuit such that  $r(M) = r(M')$  and*

- (i)  $M'$  is minimally 3-connected; or
- (ii)  $E(M')$  has a non-empty subset  $A$  that is contained in every 5-circuit of  $M'$ , and  $M' \setminus A'$  is minimally 3-connected of rank  $r(M)$  for some  $A' \subseteq A$ .

*Proof.* Let  $M'$  be a 3-connected matroid having a 5-circuit that is obtained from  $M$  by deleting elements one by one so that each intermediate matroid is 3-connected and, for all  $e$  in  $E(M')$ , either  $M' \setminus e$  has no 5-circuit, or  $M' \setminus e$  is not 3-connected. Suppose  $M'$  is not minimally 3-connected. Let  $X$  be the set of elements  $x$  of  $M'$  for which  $M' \setminus x$  is 3-connected. Let  $A$  be the set of elements of  $M$  that are in all of its 5-circuits. Then  $X$  is non-empty and  $X \subseteq A$ . Let  $A'$  be a maximal subset of  $X$  for which  $M' \setminus A'$  is 3-connected of rank  $r(M)$ . Then  $M' \setminus A'$  is minimally 3-connected.  $\square$

### 3. PROOF STRATEGY

As the proof of the main theorem occupies the rest of the paper, this section will briefly discuss the proof strategy. Let  $M$  be a 3-connected binary matroid of rank at least six that has a 5-circuit  $C$  but no odd circuit of size exceeding five. Our goal is to identify the possibilities for  $M$ . Initially, we determine all of these possibilities under the added assumption that  $M \setminus e$  is not 3-connected for all  $e \in E(M) - C$ . Adding this assumption enables us to use the tools identified in the previous section. Moreover, it turns out to be relatively straightforward to solve the problem in general once we have done it in the special case.

Using the minimality assumption on  $M$ , we show in Lemma 4.1 that  $M$  has a triad  $T_1^*$  avoiding  $C$ . When  $M = M(G)$  and  $T_1^*$  corresponds to a degree-3-vertex  $v$  of  $G$ , Menger's Theorem implies that there are three paths from  $v$  to  $V(C)$  that are disjoint except that each uses the vertex  $v$ . Tutte's Linking Lemma generalizes Menger's Theorem to matroids and we use it in Theorem 4.2 to identify seven matroids one of which must occur as a restriction  $M|Z_1$  of  $M$  where  $C \cup T_1^* \subseteq Z_1$ . Following that, for the case when  $Z_1$  does not span  $M$ , the rest of Section 4 is devoted to finding another triad  $T_2^*$  of  $M$  that is disjoint from the set  $Z_1$ . By applying Theorem 4.2 again, this time to the circuit  $C$  and the triad  $T_2^*$ , we get a subset  $Z_2$  of

$$\begin{array}{c}
5 \quad e_2 \quad e_4 \quad e_5 \quad e_6 \\
\begin{array}{l}
1 \\
2 \\
3 \\
4 \\
e_1 \\
e_3 \\
e' \\
e''
\end{array}
\left[ \begin{array}{ccccc}
1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 0
\end{array} \right]
\end{array}$$

FIGURE 3.  $A$ .

$E(M)$  that contains  $C \cup T_2^*$  such that  $M|Z_2$  is one of the seven previously identified matroids.

In Section 5, we begin by treating the case when  $r(Z_1) = r(M)$ . In that case, with the aid of the matroid functionality of the Sage mathematics package, we confirm that  $M$  is one of the nine exceptional matroids identified in Theorem 1.3(iv). We then assume that  $r(Z_1) < r(M)$ , so  $T_2^*$  and  $Z_2$  exist and we combine what we know about  $M|Z_1$  and  $M|Z_2$  to determine the structure of  $M$ . Because there are seven choices for each of  $M|Z_1$  and  $M|Z_2$ , this involves quite a bit of case analysis. Finally, in Section 7, we complete the proof of the main theorem. We know the structure of  $M$  in the case that we have imposed our minimality assumption, so we need only consider what elements can be added to these choices for  $M$  to maintain a simple matroid of rank  $r(M)$  having no odd circuit of size exceeding five. It turns out that the number of such choices is very limited so the completion of the proof of the main theorem is relatively straightforward.

#### 4. FINDING TRIADS

In this section, we begin the implementation of the proof strategy outlined in the last section. The matrix  $A$  in Figure 3 is important in the next lemma where, to prevent  $M(\mathcal{W}_5)$  from being a counterexample, we require  $r(M) \geq 6$ .

**Lemma 4.1.** *Let  $M$  be a 3-connected binary matroid of rank at least six. Assume that  $\{1, 2, 3, 4, 5\}$  is a circuit  $C$  of  $M$  and that  $M \setminus e$  is not 3-connected for all  $e$  in  $E(M) - C$ . If  $M$  has no triad avoiding  $C$ , then  $M$  has a 7-circuit and can be obtained from  $M[I_8|A]$  by contracting one of the four subsets of  $\{e', e''\}$ .*

*Proof.* First we show the following.

**4.1.1.** *At most six elements of  $E(M) - C$  are in triads of  $M$ . Moreover, if  $M$  contains exactly six triads, after a possible permutation of  $\{1, 2, 3, 4, 5\}$ , these triads are  $\{e_1, 1, 4\}$ ,  $\{e_2, 1, 5\}$ ,  $\{e_3, 2, 4\}$ ,  $\{e_4, 2, 5\}$ ,  $\{e_5, 3, 4\}$ ,  $\{e_6, 3, 5\}$ .*

Let  $T$  be a triad of  $M$  meeting  $E(M) - C$ . By orthogonality, since  $T$  is not contained in  $E(M) - C$ , exactly one element of  $T$  is in  $E(M) - C$ . If  $M$  has distinct triads  $T$  and  $T'$  meeting  $C$  in the same pair of elements, then their symmetric difference contains a cocircuit of size at most two, a contradiction. Hence, as  $M$  is binary,  $T \cap C$  is not contained in any triad of  $M$  other than  $T$ .

Now form an auxiliary graph  $G$  having vertices  $\{1, 2, 3, 4, 5\}$  with two such vertices being joined if the corresponding elements lie in some triad. The graph  $G$  has no triangles, otherwise the symmetric difference of the corresponding three triads is a triad that avoids  $C$ . Thus  $G$  is a subgraph of  $K_5$  having no triangles. Therefore either  $G$  is bipartite and hence is a subgraph of  $K_{2,3}$  or  $K_{1,4}$ , or  $G$  is a 5-cycle. Thus  $G$  contains at most six edges. Hence  $M$  contains at most six triads and, when equality holds,  $G$  is isomorphic to  $K_{2,3}$ . Thus 4.1.1 follows.

We shall complete the proof of the lemma by arguing by induction on  $r(M)$  supposing first that  $r(M) = 6$ . Then  $E(M) - \text{cl}(C)$  is the union of two cocircuits, each having at least four elements. Their symmetric difference is also a disjoint union of cocircuits and so contains at least four elements. Thus  $|E(M) - \text{cl}(C)| \geq 6$ . By Lemma 2.4,  $E(M) - \text{cl}(C)$  is independent. Hence  $E(M) - \text{cl}(C)$  is a basis of  $M$  and  $|E(M) - \text{cl}(C)| = 6$ . Moreover,  $E(M) - \text{cl}(C)$  is the union of two 4-cocircuits,  $C_1^*$  and  $C_2^*$ , whose symmetric difference is also a 4-cocircuit of  $M$ .

Next we show the following.

**4.1.2.** *Each element of  $E(M) - \text{cl}(C)$  is in a triad of  $M$ .*

Suppose that some element  $e$  of  $E(M) - \text{cl}(C)$  is not in a triad of  $M$ . Then  $M \setminus e$  has a 2-separation  $(A, B)$  where  $r(A) \leq r(B)$ , say. Then  $r(A) + r(B) = r(M \setminus e) + 1 = 7$ . As  $M \setminus e$  has no 2-element cocircuits, it follows that each of  $r(A)$  and  $r(B)$  is at least three. Thus  $r(A) = 3$  and  $r(B) = 4$ . Then  $M \setminus e$  is the 2-sum, with basepoint  $p$ , of matroids with ground sets  $A \cup p$  and  $B \cup p$ , respectively. The first of these matroids,  $M_1$ , can be obtained from a copy of the Fano matroid with  $p$  added in parallel to one of the points by deleting some set of elements other than  $p$ . There are three lines in the Fano matroid through the point  $p$ . Let  $L_1, L_2$  and  $L_3$  be the sets of elements of  $A$  that lie on these lines and are not spanned by  $p$  in  $M_1$ . In  $M \setminus e$ , we see that  $L_i \cup L_j$  is a cocircuit for all distinct  $i$  and  $j$  in  $\{1, 2, 3\}$ . Hence  $|L_i| + |L_j| \geq 3$ . It follows that  $2|L_1| + 2|L_2| + 2|L_3| \geq 9$ , so  $|L_1| + |L_2| + |L_3| \geq 5$ . Thus  $|A| \geq 5$ . If  $A$  contains an element  $q$  that is parallel to  $p$  in  $M_1$ , then, since  $(A - q, B \cup q)$  is a 2-separation of  $M \setminus e$ , we can replace  $(A, B)$  by  $(A - q, B \cup q)$ . We deduce that we may assume  $M|A$  is isomorphic to  $M(K_4)$  or to a single-element deletion of  $M(K_4)$ .

Since  $r(B) = 4$  but  $r(E(M) - (\text{cl}(C) \cup e)) = 5$ , it follows that  $A$  meets  $E(M) - (\text{cl}(C) \cup e)$ . In  $M \setminus e$ , the set  $E(M) - (\text{cl}(C) \cup e)$  is the union of two triads that meet in a single element. As  $A$  is a union of circuits of  $M$ , it follows by orthogonality that  $|A \cap (E(M) - (\text{cl}(C) \cup e))| \geq 2$ . Since

$E(M) - \text{cl}(C)$  is independent and  $r(A) = 3$ , we deduce that  $A$  contains at least two and at most three elements of  $E(M) - \text{cl}(C)$ . From the cocircuits  $C_1^*, C_2^*$ , and  $C_1^* \Delta C_2^*$ , whose union is  $E(M) - \text{cl}(C)$ , we deduce that, in  $M|A$ , each of  $C_1^* \cap A$ ,  $C_2^* \cap A$ , and  $(C_1^* \Delta C_2^*) \cap A$  is either empty or contains a cocircuit of  $M|A$ . But  $M(K_4)$  has no 2-element cocircuits, so  $M|A$  must be a single-element deletion of  $M(K_4)$  and  $A$  contains exactly two elements of  $E(M) - \text{cl}(C)$ . Now, the remaining three elements of  $A$  form a triangle  $T$  that, in  $M_1$ , avoids the basepoint  $p$ . Since these elements must be in  $\text{cl}(C)$ , at least two of them are in  $C$ . But then  $C$  is not a circuit of  $M \setminus e$  as  $T$  is skew to  $p$ . This contradiction completes the proof of 4.1.2.

Label  $E(M) - \text{cl}(C)$  as  $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ . In 4.1.1, we identified six triads in  $M$ . Using symmetric difference, we deduce that  $M$  also has the following sets as cocircuits:  $\{e_1, e_2, 4, 5\}$ ,  $\{e_3, e_4, 4, 5\}$ ,  $\{e_5, e_6, 4, 5\}$ ,  $\{e_1, e_3, 1, 2\}$ ,  $\{e_1, e_5, 1, 3\}$ ,  $\{e_3, e_5, 2, 3\}$ ,  $\{e_2, e_4, 1, 2\}$ ,  $\{e_2, e_6, 1, 3\}$ ,  $\{e_4, e_6, 2, 3\}$ .

We now construct a basis  $B$  for  $M$  such that  $B$  contains  $\{1, 2, 3, 4\}$ . A straightforward check involving the triads in 4.1.1 confirms that none of  $e_1, e_2$ , and  $e_3$  is in  $\text{cl}(\{1, 2, 3, 4\})$ . Hence  $\{1, 2, 3, 4, e_1\}$ ,  $\{1, 2, 3, 4, e_2\}$  and  $\{1, 2, 3, 4, e_3\}$  are independent. The proofs of 4.1.3–4.1.5 will make repeated use of orthogonality.

**4.1.3.**  $\{1, 2, 3, 4, e_1, e_2, e_3\}$  contains no circuit containing  $e_3$ .

Assume  $\{1, 2, 3, 4, e_1, e_2, e_3\}$  contains such a circuit  $D_3$ . Then the elements 3 and 4 are in  $D_3$  by orthogonality between  $D_3$  and the cocircuits  $\{e_3, e_4, 4, 5\}$  and  $\{e_5, 3, 4\}$ , respectively. Hence  $|D_3 \cap \{e_6, 3, 5\}| = 1$ , a contradiction to orthogonality. Thus 4.1.3 holds.

It follows from 4.1.3 and the fact that  $r(M) = 6$  that  $\{1, 2, 3, 4, e_1, e_2\}$  is not a basis of  $M$  but  $\{1, 2, 3, 4, e_1, e_3\}$  is.

**4.1.4.**  $\{1, e_1, e_2\}$  is a circuit of  $M$ .

As  $\{1, 2, 3, 4, e_1, e_2\}$  is not a basis of  $M$ , it contains a circuit  $D$ . Moreover,  $D$  must contain  $e_1$  and  $e_2$ . We deduce that  $D$  contains 1 by orthogonality between  $D$  and  $\{e_2, 1, 5\}$ . As  $M$  is binary, the triad  $\{e_1, 1, 4\}$  implies that  $4 \notin D$ . By orthogonality between  $D$  and the cocircuits  $\{e_3, 2, 4\}$  and  $\{e_5, 3, 4\}$ , neither 2 nor 3 is in  $D$ . Hence  $D = \{1, e_1, e_2\}$  and 4.1.4 holds.

By symmetry, it follows that

**4.1.5.**  $\{2, e_3, e_4\}$  and  $\{3, e_5, e_6\}$  are circuits of  $M$ .

Let  $D_5$  be the fundamental circuit of  $e_5$  with respect to the basis  $\{1, 2, 3, 4, e_1, e_3\}$ . Then  $4 \in D_5$  and  $3 \in D_5$  by orthogonality between  $D_5$  and the cocircuits  $\{e_5, e_6, 4, 5\}$  and  $\{e_3, e_4, 4, 5\}$ , respectively. Since  $M$  is binary and  $\{e_3, 2, 4\}$  and  $\{e_5, 3, 4\}$  are triads,  $D_5$  avoids  $\{2, 3\}$ . Then  $1 \notin D_5$  and  $e_1 \in D_5$  by orthogonality between  $D_5$  and the cocircuits  $\{e_2, e_4, 1, 2\}$  and  $\{e_1, 1, 4\}$ , respectively. We deduce that  $D_5 = \{4, e_1, e_3, e_5\}$ . Taking the symmetric difference of  $D$  with the circuit  $\{3, e_5, e_6\}$  gives that  $\{3, 4, e_1, e_3, e_6\}$  is a circuit of  $M$ . It is the fundamental circuit of  $e_6$  with respect to the



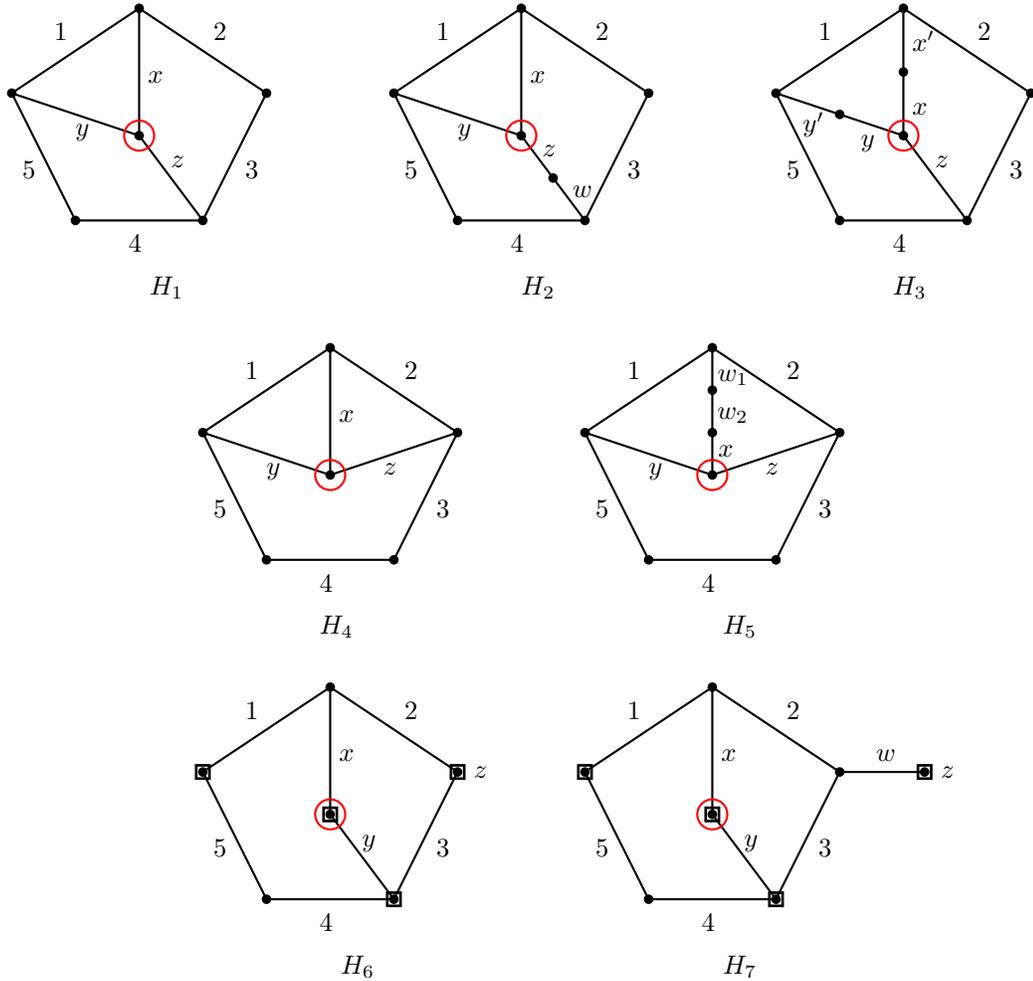
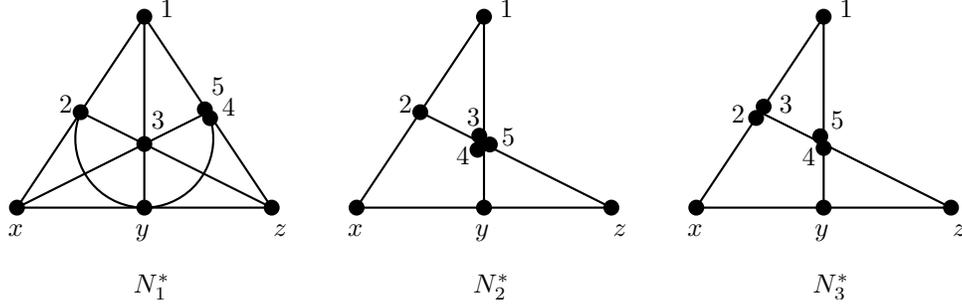


FIGURE 5. Each of  $H_1$  through  $H_7$  is a graph or graft whose cycle matroid is a restriction of  $M$ , where  $\{x, y, z\}$  is a triad of  $M$ .

**Theorem 4.2.** *Let  $M$  be a 3-connected binary matroid. Suppose that  $M$  has a 5-circuit,  $C$ , and that, for all  $e \in E(M) - C$ , the matroid  $M \setminus e$  is not 3-connected. If  $M$  has no odd circuit of size exceeding five, then*

- (i)  $r(M) \leq 5$ ; or
- (ii)  $M$  has a triad avoiding  $C$  and, for each such triad  $T$ , the matroid  $M$  contains a restriction isomorphic to the cycle matroid of a graph or graft in Figure 5, where  $C$  is represented by the cycle  $\{1, 2, \dots, 5\}$  and  $T$  is represented by  $\{x, y, z\}$ .

*Proof.* We let  $C = \{1, 2, 3, 4, 5\}$  and assume that neither (i) nor (ii) holds. Then  $r(M) \geq 6$ . By Lemma 4.1,  $M$  has a triad  $T^*$  avoiding

FIGURE 6. The three possibilities for  $N^*$ .

$C$ . By Geelen, Gerards, and Whittle's extension of Tutte's Linking Theorem [3, Theorem 4.2] (see also [8, Theorem 8.5.7]),  $M$  has a minor  $N$  with ground set  $T^* \cup C$  such that  $N|T^* = M|T^*$  and  $N|C = M|C$  while  $\kappa_M(T^*, C) = \kappa_N(T^*, C)$ . As  $M$  is 3-connected,  $\kappa_M(T^*, C) \geq 2$ . Since  $T^*$  is a triad of  $M$ , it follows that  $\lambda_M(T^*) = 2$ , so  $\kappa_M(T^*, C) = 2$ . Thus  $2 = r_N(T^*) + r_N(C) - r(N) = 3 + 4 - r(N)$ , so  $r(N) = 5$ . Moreover,  $T^*$  is a disjoint union of cocircuits of  $N$ . By orthogonality, it follows that  $\text{cl}_N(C) = C$ , so  $T^*$  is a triad,  $\{x, y, z\}$  say, of  $N$ . As  $|E(N)| = 8$ , we deduce that  $N^*$  is an 8-element rank-3 matroid having  $T^*$  as a triangle and  $C$  as a disjoint 5-cocircuit. Moreover,  $\lambda_N(T^*) = 2$ .

Since  $N^*$  is binary and loopless, it follows that, after possibly permuting the labels within  $C$  and  $T^*$ , the matroid  $N^*$  is one of the matroids shown in Figure 6.

The complement of each hyperplane of  $N^*$  is a circuit in  $N$ . Furthermore, by finding the circuits in  $N^*$ , we find the cocircuits of  $N$ . Now  $N = M \setminus X / Y$  for some coindependent set  $X$  and independent set  $Y$  of  $M$ .

Assume  $N^*$  is the matroid  $N_1^*$  in Figure 6. The hyperplanes of  $N_1^*$  are the rank-two flats, namely  $\{1, 2, x\}$ ,  $\{1, 3, y\}$ ,  $\{1, 4, 5, z\}$ ,  $\{2, 4, 5, y\}$ ,  $\{3, 4, 5, x\}$ ,  $\{2, 3, z\}$ , and  $\{x, y, z\}$ . Thus the circuits of  $N_1$  are  $\{3, 4, 5, y, z\}$ ,  $\{2, 4, 5, x, z\}$ ,  $\{2, 3, x, y\}$ ,  $\{1, 3, x, z\}$ ,  $\{1, 2, y, z\}$ ,  $\{1, 4, 5, x, y\}$ , and  $\{1, 2, 3, 4, 5\}$ . Some circuits of  $N_1^*$ , and hence cocircuits of  $N_1$ , are  $\{4, 5\}$ ,  $\{2, 4, y\}$ ,  $\{2, 3, z\}$ , and  $\{x, y, z\}$ .

Recall that  $N = M \setminus X / Y$  for some independent set  $Y$  and coindependent set  $X$  of  $M$ . Let us examine  $M \setminus X$ . Each cocircuit of  $N_1$  is a cocircuit of  $M \setminus X$ . For each circuit  $D$  of  $N$ , there is a circuit  $D \cup Y_D$  of  $M \setminus X$  where  $Y_D \subseteq Y$ . Thus, for some subsets  $Y_1, Y_2$ , and  $Y_3$  of  $Y$ , each of  $\{1, 2, y, z\} \cup Y_1$ ,  $\{1, 3, x, z\} \cup Y_2$ , and  $\{2, 3, x, y\} \cup Y_3$  is a circuit of  $M$ . Next we show the following.

**4.2.1.** *When  $N = N_1$ , each of  $\{3, 4, 5, y, z\} \cup Y_1$ ,  $\{2, 4, 5, x, z\} \cup Y_2$ , and  $\{1, 4, 5, x, y\} \cup Y_3$  is a circuit of  $M$ . Moreover, either*

- (i)  $Y_1 = Y_2 = Y_3 = \emptyset$ ; or

- (ii) for some  $\{i, j, k\} = \{1, 2, 3\}$ , the set  $Y_i$  is empty and there is an element  $w$  of  $Y$  such that  $Y_j = Y_k = \{w\}$ .

As  $\{1, 2, 3, 4, 5\}$  is a circuit of both  $N$  and  $M$ , it follows by taking the symmetric difference of this circuit and  $\{1, 2, y, z\} \cup Y_1$  that  $\{3, 4, 5, y, z\} \cup Y_1$  is a disjoint union of circuits of  $M$ . Let  $W$  be the circuit in this disjoint union that contains  $z$ . Since  $\{2, 3, z\}$  and  $\{x, y, z\}$  are cocircuits of  $M \setminus X$ , by orthogonality  $3 \in W$  and  $y \in W$ . By orthogonality with the cocircuits  $\{2, 4, y\}$  and  $\{4, 5\}$  of  $M \setminus X$ , we see that both 4 and 5 are contained in  $W$ , so  $W = \{3, 4, 5, y, z\} \cup Y_1$ . It follows, using the symmetry of  $N_1$ , that  $\{2, 4, 5, x, z\} \cup Y_2$ , and  $\{1, 4, 5, x, y\} \cup Y_3$  are also circuits of  $M$ .

Since  $M$  has no odd circuits of size exceeding 5 and, for each  $i$  in  $\{1, 2, 3\}$ , the matroid  $M$  has circuits of size  $4 + |Y_i|$  and  $5 + |Y_i|$ , we deduce that each  $|Y_i| \leq 1$ . Next we note that  $(\{1, 2, y, z\} \cup Y_1) \Delta (\{1, 3, x, z\} \cup Y_2)$ , which equals  $\{2, 3, x, y\} \cup (Y_1 \Delta Y_2)$ , is a disjoint union of circuits of  $M$ . But  $\{2, 3, x, y\} \cup Y_3$  is also a circuit of  $M$ . Hence  $Y_1 \Delta Y_2 \Delta Y_3$  is a disjoint union of circuits of  $M$ . But  $Y_1 \Delta Y_2 \Delta Y_3$  is contained in the independent set  $Y$  of  $M$ . Hence  $Y_3 = Y_1 \Delta Y_2$ . Now  $|Y_3| \leq 1$ . If  $|Y_3| = 0$ , then  $Y_1 = Y_2$ , so (i) or (ii) holds. If  $|Y_3| = 1$ , then either  $Y_3 = Y_1$  and  $Y_2 = \emptyset$ , or  $Y_3 = Y_2$  and  $Y_1 = \emptyset$ . We conclude that (i) or (ii) holds. Hence 4.2.1 is proved.

Next assume that  $N^* = N_2^*$ . The circuits of  $N_2$  include  $\{3, 4, 5, y, z\}$ ,  $\{2, x, z\}$ ,  $\{1, x, y\}$ , and  $\{1, 2, 3, 4, 5\}$ . The cocircuits of  $N_2$  include  $\{1, 2, x\}$ ,  $\{1, 3, y\}$ ,  $\{3, 4\}$ ,  $\{3, 5\}$ ,  $\{4, 5\}$ , and  $\{x, y, z\}$ .

Again we examine  $M \setminus X$  where  $N_2 = M \setminus X / Y$  for some independent set  $Y$  and coindependent set  $X$ . For some subsets  $Y_1, Y_2$ , and  $Y_3$  of  $Y$  each of  $\{1, x, y\} \cup Y_1$ ,  $\{2, x, z\} \cup Y_2$ , and  $\{3, 4, 5, y, z\} \cup Y_3$  is a circuit of  $M$ . Next we show the following.

**4.2.2.** *When  $N = N_2$ , each of  $\{2, 3, 4, 5, x, y\} \cup Y_1$ ,  $\{1, 3, 4, 5, x, z\} \cup Y_2$ , and  $\{1, 2, y, z\} \cup Y_3$  is a circuit of  $M$ . Moreover, either*

- (i)  $Y_1 = Y_2 = Y_3 = \emptyset$ ; or
- (ii)  $Y_3 = \emptyset$  and  $Y$  has distinct elements,  $w_1$  and  $w_2$ , such that  $Y_1 = Y_2 = \{w_1, w_2\}$ .

Since  $\{1, x, y\} \cup Y_1$  and  $\{1, 2, 3, 4, 5\}$  are circuits of  $M$ , their symmetric difference,  $\{2, 3, 4, 5, x, y\} \cup Y_1$ , is a disjoint union of circuits in  $M \setminus X$ . Let  $W$  be the circuit in this disjoint union that contains  $x$ . Since  $\{x, y, z\}$  and  $\{1, 2, x\}$  are cocircuits in  $M \setminus X$ , by orthogonality,  $y \in W$  and  $2 \in W$ . By orthogonality with the cocircuits  $\{1, 3, y\}$ ,  $\{3, 4\}$ , and  $\{3, 5\}$  of  $M \setminus X$ , the circuit  $W$  contains  $\{3, 4, 5\}$ . We deduce that  $\{2, 3, 4, 5, x, y\} \cup Y_1$  is a circuit of  $M$ . By symmetry,  $\{1, 3, 4, 5, x, z\} \cup Y_2$  is a circuit of  $M$ .

Since  $\{3, 4, 5, y, z\} \cup Y_3$  and  $\{1, 2, 3, 4, 5\}$  are circuits of  $M$ , their symmetric difference,  $\{1, 2, y, z\} \cup Y_3$  is a disjoint union of circuits of  $M$ . Let  $U$  be the circuit in this disjoint union that contains  $y$ . Since  $\{x, y, z\}$ ,  $\{1, 3, y\}$ , and  $\{1, 2, x\}$  are cocircuits of  $M \setminus X$ , by orthogonality, each of  $z, 1$ , and  $2$  is in  $U$ . Thus  $\{1, 2, y, z\} \cup Y_3$  is a circuit of  $M$ .

Now  $M$  has  $\{2, 3, 4, 5, x, y\} \cup Y_1$  and  $\{1, x, y\} \cup Y_1$  as circuits of sizes  $6 + |Y_1|$  and  $3 + |Y_1|$ , respectively. Thus  $|Y_1| \in \{0, 2\}$ . By symmetry,  $|Y_2| \in \{0, 2\}$ . Finally,  $\{1, 2, y, z\} \cup Y_3$  and  $\{3, 4, 5, y, z\} \cup Y_3$  are circuits of  $M$  of sizes  $4 + |Y_3|$  and  $5 + |Y_3|$ , respectively. Thus  $Y_3 \in \{0, 1\}$ .

Next note that  $(\{1, x, y\} \cup Y_1) \Delta (\{2, x, z\} \cup Y_2)$ , which equals  $\{1, 2, y, z\} \cup (Y_1 \Delta Y_2)$ , is a disjoint union of circuits of  $M$ . As  $\{1, 2, y, z\} \cup Y_3$  is a circuit of  $M$  and  $Y$  is independent, it follows that  $Y_1 \Delta Y_2 = Y_3$ . As  $|Y_1|, |Y_2| \in \{0, 2\}$ , it follows that  $|Y_3|$  is even. Thus  $Y_3 = \emptyset$  so  $Y_1 = Y_2$ . As  $|Y_1| \in \{0, 2\}$ , we deduce that (i) or (ii) holds so the proof of 4.2.2 is complete.

Assume  $N = N_3$ . The circuits in  $N_3$  include  $\{1, x, y\}$ ,  $\{2, 3, x, z\}$ ,  $\{4, 5, y, z\}$ , and  $\{1, 2, 3, 4, 5\}$ . The cocircuits of  $N_3$  include  $\{1, 2, x\}$ ,  $\{1, 4, y\}$ ,  $\{2, 4, z\}$ ,  $\{2, 3\}$ ,  $\{4, 5\}$ , and  $\{x, y, z\}$ .

As before, we examine  $M \setminus X$  where  $N_3 = M \setminus X / Y$  for some independent set  $Y$  and coindependent set  $X$ . There are subsets  $Y_1, Y_2$ , and  $Y_3$  of  $Y$  such that  $\{1, x, y\} \cup Y_1$ ,  $\{2, 3, x, z\} \cup Y_2$ , and  $\{4, 5, y, z\} \cup Y_3$  are circuits of  $M$ .

**4.2.3.** *When  $N = N_3$ , each of  $\{2, 3, 4, 5, x, y\} \cup Y_1$ ,  $\{1, 4, 5, x, z\} \cup Y_2$ , and  $\{1, 2, 3, y, z\} \cup Y_3$  is a circuit of  $M$ . Moreover, either*

- (i)  $Y_1 = Y_2 = Y_3 = \emptyset$ ; or
- (ii)  $Y_1 = \emptyset$  and  $Y$  has an element  $w$  such that  $Y_2 = \{w\} = Y_3$ ; or
- (iii)  $Y$  has distinct elements,  $x'$  and  $y'$ , such that  $Y_2 = \{x'\}$  and  $Y_3 = \{y'\}$  while  $Y_1 = \{x', y'\}$ .

The symmetric difference of the circuits  $\{1, x, y\} \cup Y_1$  and  $\{1, 2, 3, 4, 5\}$ , which equals  $\{2, 3, 4, 5, x, y\} \cup Y_1$ , is a disjoint union of circuits in  $M \setminus X$ . Let  $W$  be the circuit in this disjoint union that contains  $x$ . Orthogonality between  $W$  and the cocircuits  $\{x, y, z\}$ ,  $\{1, 2, x\}$ ,  $\{2, 3\}$ ,  $\{1, 4, y\}$ , and  $\{4, 5\}$  of  $M \setminus X$  implies that  $W$  contains each of  $y, 2, 3, 4$ , and  $5$ . Hence  $\{2, 3, 4, 5, x, y\} \cup Y_1$  is a circuit of  $M$ .

As the disjoint union of  $\{2, 3, x, z\} \cup Y_2$  and  $\{1, 2, 3, 4, 5\}$ , the set  $\{1, 4, 5, x, z\} \cup Y_2$ , is a disjoint union of circuits of  $M$ . Let  $V$  be the circuit in this disjoint union containing  $x$ . Orthogonality between  $V$  and the cocircuits  $\{x, y, z\}$ ,  $\{1, 2, x\}$ ,  $\{2, 4, z\}$ , and  $\{4, 5\}$  of  $M \setminus X$  implies that  $V$  contains each of  $z, 1, 4$ , and  $5$ . Hence  $\{1, 4, 5, x, z\} \cup Y_2$  is a circuit of  $M$ . By symmetry,  $\{1, 2, 3, y, z\} \cup Y_3$  is a circuit of  $M$ .

Now  $\{2, 3, 4, 5, x, y\} \cup Y_1$  and  $\{1, x, y\} \cup Y_1$  are circuits of  $M$  of size  $6 + |Y_1|$  and  $3 + |Y_1|$ , respectively. Thus  $|Y_1| \in \{0, 2\}$ . The circuits  $\{1, 4, 5, x, z\} \cup Y_2$  and  $\{2, 3, x, z\} \cup Y_2$  have sizes  $5 + |Y_2|$  and  $4 + |Y_2|$ , respectively. Hence  $|Y_2| \in \{0, 1\}$ . By symmetry,  $|Y_3| \in \{0, 1\}$ .

Note that  $(\{4, 5, y, z\} \cup Y_3) \Delta (\{1, 4, 5, x, z\} \cup Y_2) = (\{1, x, y\} \cup (Y_2 \Delta Y_3))$ . Thus  $Y_2 \Delta Y_3 = Y_1$ . Suppose  $|Y_1| = 0$ . Then  $Y_2 = Y_3$  and either each of these sets is empty, or  $Y$  contains an element  $w$  such that  $Y_2 = \{w\} = Y_3$ . On the other hand, if  $|Y_1| = 2$ , then  $Y$  has distinct elements,  $x'$  and  $y'$ , such that  $Y_2 = \{x'\}$  and  $Y_3 = \{y'\}$ . Moreover,  $Y_1 = \{x', y'\}$ . We conclude that 4.2.3 holds.

Now, for each of the three choices for  $N$ , we have defined subsets  $Y_1, Y_2$ , and  $Y_3$  of  $Y$ . Let  $Z = \{1, 2, 3, 4, 5, x, y, z\} \cup (Y_1 \cup Y_2 \cup Y_3)$ . Finally, we show that each possible case gives the cycle matroid of a graph or graft in Figure 5. If  $N = N_1$  and 4.2.1(i) holds, then  $M|Z = M(H_6)$ . If  $N = N_1$  and 4.2.1(ii) holds, then, by symmetry, we may assume that  $Y_3 = \emptyset$ , and  $M|Z = M(H_7)$ . If  $N = N_2$  and 4.2.2(i) holds, then  $M|Z = M(H_4)$ . If  $N = N_2$  and 4.2.2(ii) holds, then  $M|Z = M(H_5)$ . If  $N = N_3$  and 4.2.3(i) holds, then  $M|Z = M(H_1)$ . If  $N = N_3$  and 4.2.3(ii) holds, then  $M|Z = M(H_2)$ . If  $N = N_3$  and 4.2.3(iii) holds, then  $M|Z = M(H_3)$ .  $\square$

The rest of this section is dedicated to finding a triad disjoint from the restriction of  $M$  that is the cycle matroid of a graph or graft in Figure 5. We will then apply Theorem 4.2 to this second triad in Section 6.

**Lemma 4.3.** *Let  $Z$  be a subset of a 3-connected binary matroid  $M$ . Assume that  $M|Z$  is connected and that  $r(M) - 1 = r(Z) \geq 3$ . Suppose  $e \in E(M) - \text{cl}(Z)$  and  $e$  is not in a triad of  $M$ . Let  $(A_e, B_e)$  be a 2-separation of  $M \setminus e$ . Then*

- (i)  $r(A_e) \geq 3$  and  $r(B_e) \geq 3$ ; and
- (ii)  $(A_e \cap Z, B_e \cap Z)$  is a 2-separation of  $M|Z$ ; and
- (iii) either  $M \setminus f$  is 3-connected for some  $f$  in  $E(M) - \text{cl}(Z) - e$ , or neither  $A_e \cap Z$  nor  $B_e \cap Z$  is a series pair in  $M|Z$ .

*Proof.* We have  $r(A_e) + r(B_e) = r(M) + 1 = r(Z) + 2$ . Since  $e$  is not in a triad of  $M$ , and  $M$  is simple, both  $|A_e|$  and  $|B_e|$  exceed two. Thus  $r(A_e) \geq 3$  unless  $A_e$  is a triangle of  $M$ . In the exceptional case, since  $A_e$  is a 2-separating triangle in the binary matroid  $M \setminus e$ , it must contain a series pair of  $M \setminus e$ . Thus  $e$  is in a triad of  $M$ , a contradiction. We deduce using symmetry that (i) holds.

Now

$$\cap(A_e \cap Z, B_e \cap Z) \leq \cap(A_e, B_e) = 1.$$

Suppose  $A_e$  contains at least  $|Z| - 1$  elements of  $Z$ . Then, as  $M|Z$  is connected,  $r(A_e) \geq r(Z)$  and  $r(B_e) \leq 2$ , a contradiction. We deduce that  $|A_e \cap Z| \leq |Z| - 2$ , so  $|B_e \cap Z| \geq 2$ . By symmetry,  $|A_e \cap Z| \geq 2$ , so (ii) holds.

Assume that  $A_e \cap Z$  is a series pair,  $\{\alpha, \beta\}$ , of  $M|Z$  and that, for all  $f$  in  $E(M) - (\text{cl}(Z) \cup e)$ , the matroid  $M \setminus f$  is not 3-connected. Then  $r(B_e \cap Z) = r(Z) - 1$ . As  $r(A_e) \geq 3$ , we deduce that  $r(B_e) = r(Z) - 1$  and  $r(A_e) = 3$ . Moreover, no element of  $E(M) - \text{cl}(Z)$  is in  $B_e$ . As  $e$  is not in a triad,  $|E(M) - \text{cl}(Z)| \geq 4$ , and, by Lemma 2.4, the set  $E(M) - \text{cl}(Z)$  is independent. Since  $E(M) - (\text{cl}(Z) \cup e) \subseteq A_e$  and  $r(A_e) = 3$ , it follows that  $|E(M) - (\text{cl}(Z) \cup e)| \leq 3$ . Hence  $|E(M) - \text{cl}(Z)| = 4$ .

Let  $E(M) - \text{cl}(Z) = \{a_1, a_2, a_3, e\}$ . Then, by assumption,  $M \setminus a_i$  is not 3-connected for all  $i$  in  $\{1, 2, 3\}$ . As  $r(A_e) = 3$  and  $\{\alpha, \beta\} \subseteq Z$ , it follows that  $\{\alpha, \beta, a_i\}$  spans  $A_e$  for all  $i$ . Thus, for each distinct  $j$  and  $k$  in  $\{1, 2, 3\}$ , there is a non-empty subset  $D_{jk}$  of  $\{\alpha, \beta\}$  such that  $\{a_j, a_k\} \cup D_{jk}$  is a circuit. As  $\{a_1, a_2, a_3\}$  is independent,  $D_{jk} \neq D_{j'k'}$  unless  $\{j, k\} = \{j', k'\}$ . Thus we

may assume that  $M$  has  $\{a_1, a_2, \alpha\}$ ,  $\{a_1, a_3, \beta\}$ , and  $\{a_2, a_3, \alpha, \beta\}$  as circuits. By Tutte's Triangle Lemma,  $M$  has a triad  $T_1^*$  containing  $a_1$  and exactly one of  $a_2$  and  $\alpha$ . But, as  $M|_{\text{cl}(Z)}$  is connected,  $|T_1^* \cap \text{cl}(Z)| \geq 2$ , so  $a_2 \notin T_1^*$ . Hence  $\alpha \in T_1^*$ . Moreover, by orthogonality with the triangle  $\{a_1, a_3, \beta\}$ , it follows that  $\beta \in T_1^*$ , so  $T_1^* = \{a_1, \alpha, \beta\}$ . Similarly,  $M$  has a triad  $T_2^*$  containing  $\{a_2, \alpha\}$  whose third element  $\gamma_2$  is in  $\text{cl}(Z)$ . As  $M$  is binary,  $T_2^*$  is not contained in the circuit  $\{a_2, a_3, \alpha, \beta\}$ , so  $\gamma_2 \neq \beta$ . By symmetry,  $M$  has a triad  $T_3^*$  containing  $\{a_3, \beta\}$  whose third element  $\gamma_3$  is in  $\text{cl}(Z)$ . Moreover,  $\gamma_3 \neq \alpha$ . Then  $T_1^* \Delta T_2^* \Delta T_3^* \Delta \{a_1, a_2, a_3, e\}$ , which contains  $e$  and has at most three elements, is a disjoint union of cocircuits, a contradiction.  $\square$

**Lemma 4.4.** *Let  $Z$  be a subset of a 3-connected binary matroid  $M$  such that  $M|Z$  is connected having no series class of size exceeding three. Assume that  $|E(M) - \text{cl}(Z)| \geq 4$ . Let  $r(M) = r(Z) + 1$  and suppose that  $M$  has triads  $T_f^*$  and  $T_g^*$  that contain distinct elements  $f$  and  $g$ , respectively, of  $E(M) - \text{cl}(Z)$ . Then  $T_f^* \cap Z$  and  $T_g^* \cap Z$  are 2-cocircuits of  $M|Z$ . Moreover,*

- (i)  $T_f^* \cap T_g^* = \emptyset$ ; or
- (ii)  $|T_f^* \cap T_g^*| = 1$  and  $(T_f^* \cup T_g^*) \cap Z$  is a 3-element series class of  $M|Z$ ; also there is no element  $h$  of  $E(M) - \text{cl}(Z) - \{f, g\}$  and triad  $T_h^*$  for  $M$  containing  $h$  such that  $T_h^* \cap Z \subseteq (T_f^* \cup T_g^*) \cap Z$ .

*Proof.* Since  $|E(M) - \text{cl}(Z)| \geq 4$  and  $r(M) = r(Z) + 1$ , it follows by orthogonality that  $T_f^* \cap Z$  and  $T_g^* \cap Z$  are 2-cocircuits of  $M|Z$ . These cocircuits are distinct, otherwise  $T_f^* \Delta T_g^*$  equals  $\{f, g\}$  and this set is a cocircuit of  $M$ , a contradiction. Assume that (i) does not hold. Then  $|T_f^* \cap T_g^*| = 1$  and  $T_f^* \Delta T_g^*$  is a 4-cocircuit of  $M$ . Thus  $(T_f^* \Delta T_g^*) - \{f, g\}$  is a 2-cocircuit of  $M|Z$ . Hence  $(T_f^* \cup T_g^*) \cap Z$  is a 3-element series class of  $M|Z$ . Suppose  $E(M) - \text{cl}(Z) - \{f, g\}$  contains an element  $h$  such that  $h$  is in a triad  $T_h^*$  where  $T_h^* \cap Z \subseteq (T_f^* \cup T_g^*) \cap Z$ . As  $T_h^* \cap Z$  is distinct from each of  $T_f^* \cap Z$  and  $T_g^* \cap Z$ , we must have  $T_h^* \cap Z = (T_f^* \cap Z) \Delta (T_g^* \cap Z)$ . Thus  $T_f^* \Delta T_g^* \Delta T_h^* = \{f, g, h\}$  and this set is a cocircuit of  $M$  that is properly contained in the cocircuit  $E(M) - \text{cl}(Z)$ , a contradiction.  $\square$

**Lemma 4.5.** *Let  $Z$  be a subset of a 3-connected binary matroid  $M$  such that  $r(Z) = r(M) - 1$  and  $M|Z$  is the cycle matroid of one of the graphs  $H_4$  and  $H_5$  in Figure 5. Assume that  $M \setminus f$  is not 3-connected for all  $f$  in  $E(M) - Z$  and that  $\{x, y, z\}$  is a triad of  $M$ . Let  $e$  be an element of  $E(M) - \text{cl}(Z)$  that is not in a triad. Let  $(A_e, B_e)$  be a 2-separation of  $M \setminus e$ . Then, up to relabeling  $A_e$  and  $B_e$ ,*

- (i)  $A_e \cap Z = \{3, 4, 5\}$ ; and
- (ii)  $|E(M) - \text{cl}(Z)| = 4$  and, for  $E(M) - \text{cl}(Z) = \{e, a_1, a_2, a_3\}$ , there are circuits  $\{a_1, a_2\} \cup A_{12}$ ,  $\{a_1, a_3\} \cup A_{13}$ , and  $\{a_2, a_3\} \cup A_{23}$  where  $\{A_{12}, A_{13}, A_{23}\} = \{\{3, 4\}, \{3, 5\}, \{4, 5\}\}$ . Moreover, at most two members of  $\{a_1, a_2, a_3\}$  are in triads of  $M$ .

*Proof.* By Lemma 4.3(ii),  $(A_e \cap Z, B_e \cap Z)$  is a 2-separation of  $M|Z$ . Note that each 2-separation of  $M|Z$  has one side contained in  $\{3, 4, 5\}$  or  $\{w_1, w_2, x\}$ . By Lemma 4.3(iii), we may assume that  $A_e \cap Z$  consists of a 3-element series class of  $M|Z$ . Assume  $A_e \cap Z = \{w_1, w_2, x\}$ . Then  $\{y, z\} \subseteq B_e$ . As  $\{x, y, z\}$  is a triad of  $M \setminus e$ , it follows that  $(A_e - x, B_e \cup x)$  is a 2-separation of  $M \setminus e$ . Then  $(A_e - x) \cap Z$  is a series pair of  $M|Z$ , a contradiction to Lemma 4.3(iii). Hence  $A_e \cap Z = \{3, 4, 5\}$ , so  $r(A_e \cap Z) = 3$  and  $r(B_e \cap Z) = r(Z) - 2 = r(M) - 3$ . Thus  $(r(A_e), r(B_e)) \in \{(3, r(M) - 2), (4, r(M) - 3)\}$ . Let  $E(M) - \text{cl}(Z) = \{e, a_1, a_2, \dots, a_m\}$ . Note that  $E(M) - \text{cl}(Z)$  is a cocircuit of  $M$ . As  $e$  is not in a triad,  $m \geq 3$ . By Lemma 2.4,  $E(M) - \text{cl}(Z)$  is independent. Next we show the following.

**4.5.1.** *If  $r(A_e) = 4$ , then (ii) holds.*

As  $r(B_e) = r(B_e \cap Z)$ , it follows that  $E(M) - \text{cl}(Z) - e$  is contained in  $A_e$ . Hence  $A_e = \{3, 4, 5, a_1, a_2, \dots, a_m\}$ . Moreover, since  $r(A_e) = 4$  and  $E(M) - \text{cl}(Z) = \{e, a_1, a_2, \dots, a_m\}$ , we see that  $m \in \{3, 4\}$ . Then, for all  $i$  in  $\{1, 2, \dots, m\}$ , there is a circuit of  $M$  containing  $\{a_i, e\}$  and contained in  $\{a_i, e, 3, 4, 5\}$ . If  $a_i$  is in a triad of  $M$ , then, by orthogonality, this triad contains two members of  $\{3, 4, 5\}$ . If  $\{a_i, 3, 4\}$ ,  $\{a_j, 3, 5\}$ , and  $\{a_k, 4, 5\}$  are triads for distinct  $i, j$ , and  $k$ , then their symmetric difference,  $\{a_i, a_j, a_k\}$  is a triad, a contradiction. Thus there are at most two elements of  $\{a_1, a_2, \dots, a_m\}$  that are in triads, so we may assume that  $a_j$  is not in a triad for  $j \geq 3$ .

Since  $\{3, 4, 5, a_i\}$  spans  $A_e$  for all  $i$  in  $\{1, 2, \dots, m\}$ , orthogonality implies that, for all 2-element subsets  $\{j, k\}$  of  $\{1, 2, \dots, m\}$ , there is a non-empty subset  $A_{jk}$  of  $\{3, 4, 5\}$  such that  $\{a_j, a_k\} \cup A_{jk}$  is a circuit of  $M$ . As  $\{a_1, a_2, \dots, a_m\}$  is independent,  $A_{jk} \neq A_{j'k'}$  unless  $\{j, k\} = \{j', k'\}$ . If  $|A_{jk}| = 1$ , then, by Lemma 2.5,  $a_j$  and  $a_k$  are both in triads. Therefore  $\{j, k\} = \{1, 2\}$ . If  $m = 4$ , then we need six distinct non-empty sets  $A_{jk}$ . As at most one has size one, there are only five possible such sets, a contradiction. Hence  $m = 3$ .

Since  $\{3, 4, 5\}$  is independent, we may assume that  $A_{12} \triangle A_{13} = A_{23}$ . Thus (ii) holds unless  $|A_{12}| = 1$ . Consider the exceptional case. Assume, without loss of generality, that  $A_{12} = \{5\}$  and  $A_{13} = \{3, 4, 5\}$ . Applying Tutte's Triangle Lemma to  $\{a_1, a_2, 5\}$  and using orthogonality, we deduce that  $\{a_1, 5\}$  is contained in a triad  $T^*$  of  $M$ . As  $M$  is binary,  $T^*$  meets the circuit  $\{a_1, a_3, 3, 4, 5\}$  in  $\{a_1, 5\}$ . Thus  $M|Z$  has a 2-cocircuit that contains 5 but avoids  $\{3, 4\}$ , a contradiction. We conclude that 4.5.1 holds.

It remains to show that  $r(A_e) = 4$ . Assume  $r(A_e) = 3$ . Then  $E(M) - \text{cl}(Z) - e \subseteq B_e$ . Let  $Z_B$  be a basis for  $B_e \cap Z$ . Then, for each distinct  $i$  and  $j$  in  $\{1, 2, \dots, m\}$ , the set  $Z_B \cup a_i$  spans  $B_e$  and there is a circuit  $Q_{ji}$  contained in  $Z_B \cup \{a_i, a_j\}$  containing  $a_j$ . By orthogonality,  $Q_{ji}$  must also contain  $a_i$ . Suppose some  $a_k$  is in a triad  $T_k^*$  of  $M$ . Then  $T_k^* - a_k$  is a 2-cocircuit of  $M|Z$ . By orthogonality between  $T_k^*$  and some  $Q_{ki}$ , it follows that  $T_k^* - a_k$  is not  $\{3, 4\}$ ,  $\{3, 5\}$ , or  $\{4, 5\}$ . Hence  $M = M(H_5)$  and  $T_k^* - a_k$

$$\begin{array}{c}
5 \quad x \quad y \\
1 \left[ \begin{array}{ccc} 1 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 1 & 1 & 0 \\ 4 & 1 & 0 & 0 \\ z & 0 & 1 & 1 \\ w & 0 & 1 & 1 \end{array} \right]
\end{array}$$

FIGURE 7.  $A_0$ .

is one of  $\{x, w_1\}$ ,  $\{x, w_2\}$ , or  $\{w_1, w_2\}$ . We deduce that at most two elements of  $\{a_1, a_2, \dots, a_m\}$  are in triads.

We may now assume that  $a_3$  is not in a triad. Then  $M \setminus a_3$  has a 2-separation  $(A_3, B_3)$ . By (i), using  $a_3$  in place of  $e$  in the argument at the start of the proof, we see that may assume that  $A_3 \cap Z = \{3, 4, 5\}$  and  $r(A_3) \in \{3, 4\}$ . If  $r(A_3) = 3$ , then  $Z_B \subseteq B_3$  and  $a_1 \in B_3$ , so  $a_3 \in \text{cl}_M(B_3)$  and  $(A_3, B_3 \cup a_3)$  is a 2-separation of  $M$ , a contradiction. We conclude that  $r(A_3) = 4$ . Again, by interchanging  $e$  and  $a_3$ , we deduce from 4.5.1 that  $m = 3$  and that  $M$  has as circuits  $\{a_1, e\} \cup B_{1e}$ ,  $\{a_2, e\} \cup B_{2e}$ , and  $\{a_1, a_2\} \cup B_{12}$  where  $\{B_{1e}, B_{2e}, B_{12}\} = \{\{3, 4\}, \{3, 5\}, \{4, 5\}\}$ .

Suppose  $a_2$  is not in a triad. Then, by interchanging  $a_2$  and  $a_3$  in the last paragraph, we find that  $M$  has as circuits  $\{a_1, e\} \cup C_{1e}$ ,  $\{a_3, e\} \cup C_{3e}$ , and  $\{a_1, a_3\} \cup C_{13}$  where  $\{C_{1e}, C_{3e}, C_{13}\} = \{\{3, 4\}, \{3, 5\}, \{4, 5\}\}$ . Then  $C_{3e} \in \{B_{1e}, B_{2e}, B_{12}\}$ , so taking the symmetric difference of  $\{a_3, e\} \cup C_{3e}$  with one of  $\{a_1, e\} \cup B_{1e}$ ,  $\{a_2, e\} \cup B_{2e}$ , and  $\{a_1, a_2\} \cup B_{12}$  gives a circuit contained in  $\{a_1, a_2, a_3, e\}$ , a contradiction. We conclude that  $a_2$  is in a triad. But this triad contains two elements of  $\{x, w_1, w_2\}$  and so contradicts orthogonality with the circuit  $\{a_2, e\} \cup B_{2e}$ . We conclude that  $r(A_e) = 4$ . Hence the lemma holds.  $\square$

Let  $A_0$  be the matrix in Figure 7. Then, in Figure 5,  $M[I_6|A_0]$  is the cycle matroid of the graft  $H_7$ , and  $M[I_6|A_0]/w$  is the cycle matroid of the graft  $H_6$ .

**Lemma 4.6.** *Let  $Z$  be a subset of a 3-connected binary matroid  $M$ . Assume that  $M|Z$  is either the cycle matroid of one of the graphs  $H_1$ ,  $H_2$ , or  $H_4$  in Figure 5, or  $M|Z$  is one of  $M[I_6|A_0]$  or  $M[I_6|A_0]/w$ . Assume that  $M \setminus e$  is not 3-connected for all  $e$  in  $E(M) - Z$ , that  $\{x, y, z\}$  is a triad of  $M$ , and that  $r(M) \geq r(Z) + 1$ . Then  $E(M) - Z$  contains a triad of  $M$ .*

*Proof.* We may assume that  $|E(M) - \text{cl}(Z)| \geq 4$ , otherwise the lemma holds. Suppose that  $r(M) = r(Z) + 1$ . Each of the choices for  $M|Z$  has either at most three non-trivial series classes with each series class having size two, or has one non-trivial series class having size three. Suppose  $f$  and  $g$  are distinct elements of  $E(M) - \text{cl}(Z)$  that are in triads. Then, by Lemma 4.4,

either these triads meet  $Z$  in disjoint sets, which are series pairs in  $M|Z$ ; or these triads meet in a single element and  $M|Z$  has a series class of size three. In the latter case,  $M|Z$  has no other non-trivial series class and the triads containing  $f$  and  $g$  are the only triads of  $M$  meeting  $E(M) - \text{cl}(Z)$ . Hence there are at most three elements of  $E(M) - \text{cl}(Z)$  that are in triads of  $M$ . Take an element  $e$  of  $E(M) - \text{cl}(Z)$  that is not in a triad. Then, by Lemma 4.3, as  $M \setminus e$  has a 2-separation  $(A_e, B_e)$ , we have  $r(A_e) \geq 3$  and  $r(B_e) \geq 3$ . Moreover,  $(A_e \cap Z, B_e \cap Z)$  is a 2-separation of  $M|Z$  and neither  $A_e \cap Z$  nor  $B_e \cap Z$  is a series pair of  $M|Z$ . We deduce that  $M|Z = M(H_4)$ , otherwise  $M|Z$  is obtained from a 3-connected matroid,  $M(K_4)$  or  $F_7^*$ , by adding at most one element in series to at most three elements of the matroid, so its only 2-separations have a 2-cocircuit as one side.

By Lemma 4.5, we may now assume that  $A_e \cap Z = \{3, 4, 5\}$ , that  $E(M) - \text{cl}(Z) = \{e, a_1, a_2, a_3\}$ , and that there are circuits  $\{a_1, a_2\} \cup A_{12}$ ,  $\{a_1, a_3\} \cup A_{13}$ , and  $\{a_2, a_3\} \cup A_{23}$  such that  $\{A_{12}, A_{13}, A_{23}\} = \{\{3, 4\}, \{3, 5\}, \{4, 5\}\}$ . Moreover, at most two members of  $\{a_1, a_2, a_3\}$  are in triads of  $M$ . In particular, we may assume that  $a_3$  is not in a triad. Consider  $M \setminus a_3$ . It has a 2-separation  $(A_3, B_3)$ . By Lemma 4.5, we may assume that  $A_3 \cap Z = \{3, 4, 5\}$  and that there are circuits  $\{a_1, a_2\} \cup B_{12}$ ,  $\{a_1, e\} \cup B_{1e}$ , and  $\{a_2, e\} \cup B_{2e}$  such that  $\{B_{12}, B_{1e}, B_{2e}\} = \{\{3, 4\}, \{3, 5\}, \{4, 5\}\}$ . Then  $B_{1e} \in \{A_{12}, A_{13}, A_{23}\}$ . Thus the symmetric difference of  $\{a_1, e\} \cup B_{1e}$  and one of  $\{a_1, a_2\} \cup A_{12}$ ,  $\{a_1, a_3\} \cup A_{13}$ , and  $\{a_2, a_3\} \cup A_{23}$  is a circuit contained in  $\{a_1, a_2, a_3, e\}$ . This contradiction to Lemma 2.4 implies that the lemma holds when  $r(M) = r(Z) + 1$ .

Assume  $r(M) > r(Z) + 1$  and the result holds for smaller values of  $r(M)$  that are at least  $r(Z) + 1$ . Then, as  $|E(M) - \text{cl}(Z)| \geq 4$ , there is an element  $e$  of  $E(M) - \text{cl}(Z)$  that is not in a triad of  $M$ . By Lemma 2.5,  $M/e$  is 3-connected and  $M/e \setminus f$  is not 3-connected for all  $f$  in  $E(M) - (Z \cup e)$ . Then, by the induction assumption,  $E(M) - (Z \cup e)$  contains a triad of  $M/e$  and so contains a triad of  $M$ . We conclude that the lemma holds.  $\square$

**Lemma 4.7.** *Let  $Z$  be a subset of a 3-connected binary matroid  $M$ . Assume that  $M|Z$  is  $M(H_3)$  or  $M(H_5)$  where the graphs  $H_3$  and  $H_5$  are shown in Figure 5. Assume that  $M \setminus e$  is not 3-connected for all  $e$  in  $E(M) - Z$ , that  $\{x, y, z\}$  is a triad of  $M$ , and that  $r(M) \geq r(Z) + 1$ . Then either  $E(M) - Z$  contains a triad of  $M$ , or  $r(M) = r(Z) + 1$  and  $M$  has an odd circuit having more than five elements.*

*Proof.* Assume that  $E(M) - Z$  does not contain a triad of  $M$ . First we show the following.

**4.7.1.** *If  $r(M) = r(Z) + 1$  and  $e \in E(M) - \text{cl}(Z)$ , then  $e$  is in a triad of  $M$ .*

Suppose  $e$  is not in a triad. Let  $(A_e, B_e)$  is a 2-separation of  $M \setminus e$ . By Lemma 4.3,  $(A_e \cap Z, B_e \cap Z)$  is a 2-separation of  $M|Z$ . Since  $M(H_3)$  can be obtained from the 3-connected matroid  $M(K_4)$  by adding four elements in series with existing elements, the only 2-separations of  $M(H_3)$  have a series

$$\begin{array}{c}
1 \quad x' \quad y' \quad e_3 \quad e_4 \quad e_2 \\
2 \quad \left[ \begin{array}{cccccc}
1 & 1 & 0 & 1 & \beta & \gamma \\
1 & 1 & 0 & 0 & \beta + 1 & \gamma + 1 \\
1 & 0 & 1 & \alpha & \delta & 1 \\
1 & 0 & 1 & \alpha & \delta & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1
\end{array} \right] \\
x \\
y \\
z \\
e_1
\end{array}$$

FIGURE 8.  $A_9$ .

pair on one side. Thus, by Lemma 4.3(iii),  $M|Z \neq M(H_3)$ . Thus  $M|Z = M(H_5)$ . Then, by Lemma 4.5, we may assume that  $A_e \cap Z = \{3, 4, 5\}$ , that  $E(M) - \text{cl}(Z) = \{e, a_1, a_2, a_3\}$ , that there are circuits  $\{a_1, a_2\} \cup A_{12}$ ,  $\{a_1, a_3\} \cup A_{13}$ , and  $\{a_2, a_3\} \cup A_{23}$  where  $\{A_{12}, A_{13}, A_{23}\} = \{\{3, 4\}, \{3, 5\}, \{4, 5\}\}$ , and that  $a_3$  is not in a triad of  $M$ . Now  $M \setminus a_3$  has a 2-separation  $(A_3, B_3)$  and, again by Lemma 4.5, we may assume that  $A_3 \cap Z = \{3, 4, 5\}$  and that  $M$  has a circuit  $\{a_1, e\} \cup B_{1e}$  for some  $B_{1e}$  in  $\{\{3, 4\}, \{3, 5\}, \{4, 5\}\}$ . Then  $B_{1e} = A_{ij}$  for some  $i$  and  $j$ . The symmetric difference of  $\{a_1, e\} \cup B_{1e}$  with  $\{a_i, a_j\} \cup A_{ij}$  contains a circuit that is contained in  $\{a_1, a_2, a_3, e\}$ . This contradiction to Lemma 2.4 implies that 4.7.1 holds.

Now, for  $\alpha, \beta, \gamma$ , and  $\delta$  in  $\{0, 1\}$ , let  $A_9$  be as shown in Figure 8. Next we show the following.

**4.7.2.** *If  $r(M) = r(Z) + 1$  and  $M|Z = M(H_3)$ , then  $M = M[I_8|A_9]$ .*

As each element of  $E(M) - \text{cl}(Z)$  is in a triad of  $M$  and this triad must meet  $Z$ , it follows, since  $M(H_3)$  has exactly four non-trivial series classes each of size exactly two, that  $|E(M) - \text{cl}(Z)| = 4$ . Thus  $E(M) - \text{cl}(Z) = \{e_1, e_2, e_3, e_4\}$ , say, and  $M$  has  $\{e_1, 2, 3\}$ ,  $\{e_2, 4, 5\}$ ,  $\{e_3, x, x'\}$ , and  $\{e_4, y, y'\}$  as triads. Now  $M$  has  $\{2, 3, 4, 5, x, y, z, e_1\}$  as a basis  $B$ . For each  $i$  in  $\{2, 3, 4\}$ , let  $D_i$  be  $C(e_i, B)$ , the fundamental circuit of  $e_i$  with respect to  $B$ . By orthogonality,  $D_3$  must contain  $e_1$  and  $x$ , so  $y \notin D_3$  and  $z \in D_3$ . As  $M$  is binary,  $D_3$  contains exactly one of 2 and 3 as well as an even number of members of  $\{4, 5\}$ . Thus, by symmetry, we may assume that  $D_3$  is  $\{2, x, z, e_1, e_3\}$  or  $\{2, 4, 5, x, z, e_1, e_3\}$ .

By orthogonality,  $D_4$  contains  $y$  and hence contains  $z$ . Also  $D_4$  contains  $e_1$ . It contains an even number of elements in  $\{4, 5\}$  and exactly one element in  $\{2, 3\}$ . Thus  $D_4$  is  $\{2, y, z, e_1, e_4\}$ ,  $\{3, y, z, e_1, e_4\}$ ,  $\{2, 4, 5, y, z, e_1, e_4\}$ , or  $\{3, 4, 5, y, z, e_1, e_4\}$ .

By orthogonality again,  $D_2$  avoids  $x$  and  $y$  and so avoids  $z$ . It contains  $e_1$  and exactly one member of each of  $\{2, 3\}$  and  $\{4, 5\}$ . By symmetry, we may assume that  $D_2$  is  $\{4, 2, e_1, e_2\}$  or  $\{4, 3, e_1, e_2\}$ . We deduce that  $M|(Z \cup \{e_1, e_2, e_3, e_4\}) = M[I_8|A_9]$ .

$$\begin{array}{c}
\begin{array}{cccccc}
& 1 & 3 & w_1 & e_2 & e_3 & e_4 \\
2 & \left[ \begin{array}{cccccc}
1 & 0 & 1 & \gamma_1 & \gamma_2 & \gamma_3 \\
0 & 1 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & \alpha & 1 \\
1 & 1 & 0 & \beta_1 & \beta_2 & \beta_3 \\
1 & 1 & 1 & \beta_1 & \beta_2 + \alpha & \beta_3 + 1 \\
0 & 0 & 1 & 0 & \alpha + 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1
\end{array} \right. \\
4 \\
5 \\
x \\
y \\
z \\
w_2 \\
e_1
\end{array}
\end{array}$$

FIGURE 9.  $B_9$ .

Clearly  $M[I_8|A_9]$  is connected. As the rows of  $A_9$  are distinct and each has at least two ones,  $M[I_8|A_9]$  is cosimple. Suppose that  $M[I_8|A_9]$  has a 2-separation  $(X, Y)$ . Then  $|X|, |Y| \geq 3$  and we may assume that each of the triads  $\{e_1, 2, 3\}$ ,  $\{e_2, 4, 5\}$ ,  $\{e_3, x, x'\}$ , and  $\{e_4, y, y'\}$  is contained in  $X$  or  $Y$ . Now  $r(X) + r(Y) = 9$ , so we may assume that  $r(X) \leq 4$ . Then  $X$  contains exactly one of the distinguished triads, so  $r(X) \geq 3$  and  $r(Y) \geq 7$ , a contradiction. Hence  $M[I_8|A_9]$  is 3-connected. Thus 4.7.2 holds.

Next we show the following.

**4.7.3.** *If  $r(M) = r(Z) + 1$  and  $M|Z = M(H_3)$ , then  $M$  has a 7-circuit.*

We have  $C(y', B) \triangle D_2 = \{y', e_2, 5, y, z, e_1, s\}$  where  $s$  is 2 or 3 depending on whether  $\gamma$  is 1 or 0. As  $\{y', e_2, 5, y, z, s\}$  is independent, we deduce that  $M$  has a 7-circuit, that is, 4.7.3 holds.

For  $\alpha, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2$ , and  $\gamma_3$  in  $\{0, 1\}$ , let  $B_9$  be as shown in Figure 9. We now show the following.

**4.7.4.** *If  $r(M) = r(Z) + 1$  and  $M|Z = M(H_5)$ , then  $M = M[I_8|B_9]$ .*

Each element of  $E(M) - \text{cl}(Z)$  is in a triad and this triad must meet  $Z$ . As  $M(H_5)$  has exactly two non-trivial series classes and each of these has size three, it follows by Lemma 4.4 that we may assume that  $M$  has as triads  $\{e_1, 3, 4\}$ ,  $\{e_2, 4, 5\}$ ,  $\{e_3, x, w_2\}$ , and either  $\{e_4, x, w_1\}$  or  $\{e_4, w_1, w_2\}$  where  $\{e_1, e_2, e_3, e_4\} = E(M) - \text{cl}(Z)$ . Call these triads  $T_1^*, T_2^*, T_3^*$ , and  $T_4^*$ , and let  $M_1 = M|(Z \cup \{e_1, e_2, e_3, e_4\})$ . Then  $r(Z) + 1 \leq r(M_1) \leq r(M) \leq r(Z) + 1$ , so  $r(M_1) = r(Z) + 1$  and  $\{e_1, e_2, e_3, e_4\}$  is a cocircuit of  $M_1$ .

Continuing with the proof of 4.7.4, we show next that

**4.7.5.**  *$M_1$  is 3-connected, so  $M_1 = M$ .*

Certainly  $M_1$  is connected. As each of the triads  $T_1^*, T_2^*, T_3^*$ , and  $T_4^*$  of  $M$  is either a triad or a disjoint union of cocircuits of the connected matroid  $M_1$ , we deduce that each  $T_i^*$  is a cocircuit of  $M_1$ . Now  $\{3, 5\}$  is not a cocircuit of  $M_1$ , otherwise  $T_1^* \triangle T_2^* \triangle \{3, 5\}$ , which equals  $\{e_1, e_2\}$ , contains a cocircuit. Likewise, no 2-subset of  $\{x, w_1, w_2\}$  is a cocircuit of  $M_1$ , so  $M_1$  is cosimple.

Assume  $M_1$  has a 2-separation  $(J, K)$ . Then  $r(J) + r(K) = 9$  and  $|J|, |K| \geq 3$ . Suppose  $|J \cap Z| \leq 1$ . Then  $r(K \cap Z) \geq 7$ . But  $r(K) < 8$ , so  $J \supseteq \{e_1, e_2, e_3, e_4\}$ . By Lemma 2.4,  $r(J) \geq 4$ , a contradiction. We deduce that  $|J \cap Z| \geq 2$ . By symmetry,  $|K \cap Z| \geq 2$ . As  $|J|, |K| \geq 3$ , we see, since  $M(H_5)$  has no triangles, that  $r(J), r(K) \geq 3$ . If  $|J \cap Z| = 2$ , then  $r(K \cap Z) \geq 6$ , so  $K - Z = \emptyset$ . Hence  $r(J) \geq 4$ , a contradiction. Thus  $|J \cap Z| \geq 3$ . By symmetry,  $|K \cap Z| \geq 3$ . As  $(J \cap Z, K \cap Z)$  is a 2-separation of  $M|Z$ , without loss of generality,  $J \cap Z$  is  $\{3, 4, 5\}$  or  $\{x, w_1, w_2\}$ . Thus  $r(K \cap Z) = 5$ . If both  $J - Z$  and  $K - Z$  are non-empty, then  $r(J) \geq 4$  and  $r(K) \geq 6$ , a contradiction. Thus  $\{e_1, e_2, e_3, e_4\}$  is contained in  $J$  or  $K$ . The triads  $T_1^*, T_2^*, T_3^*$ , and  $T_4^*$  mean that, while maintaining a 2-separation, we can move  $\{e_1, e_2\}$  or  $\{e_3, e_4\}$  an element at a time so that exactly one such pair is contained in  $J$ , a contradiction. We conclude that 4.7.5 holds.

Clearly  $M$  has  $\{2, 4, 5, x, y, z, w_2, e_1\}$  as a basis  $B$ . It also has as cocircuits  $\{e_1, e_2, e_3, e_4\}$ ,  $\{x, y, z\}$ ,  $\{e_1, 3, 4\}$ ,  $\{e_2, 4, 5\}$ ,  $\{e_3, x, w_2\}$ , and  $T_4^*$  where  $T_4^*$  is  $\{e_4, x, w_1\}$  or  $\{e_4, w_1, w_2\}$ . Using these cocircuits plus our knowledge of  $M|Z$ , it is straightforward to check that  $M$  is represented by the matrix  $[I_8|B_9]$  for some  $\alpha, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2$ , and  $\gamma_3$  in  $\{0, 1\}$ , where  $\alpha$  is 0 when  $\{e_4, x, w_1\}$  is a triad, while  $\alpha = 1$  when  $\{e_4, w_1, w_2\}$  is a triad or, equivalently, when  $\{e_3, e_4, x, w_1\}$  is a cocircuit.

Next we show the following.

**4.7.6.** *If  $r(M) = r(Z) + 1$  and  $M|Z = M(H_5)$ , then  $M$  has a 7-circuit.*

By 4.7.4,  $M = M[I_8|B_9]$ . We may assume that  $\gamma_3 = 1$ , otherwise  $|C(e_4, B)| = 7$ . Then  $C(3, B) \Delta C(e_4, B)$ , which equals  $\{3, e_4, 2, x, u, w_2, e_1\}$ , is a disjoint union of circuits of  $M$  where  $u$  is  $y$  or  $z$  depending on whether  $\beta_3$  is 0 or 1. Thus we may assume that  $\{3, e_4, 2, x, u, w_2, e_1\}$  is the union of a 3-circuit and a 4-circuit. By orthogonality,  $x$  and  $w_2$  are in the same such circuit and this circuit must also contain both  $e_4$  and  $u$ . It follows that  $\{2, 3, e_1\}$  is a circuit of  $M$ , a contradiction. We conclude that 4.7.6 holds.

**4.7.7.** *When  $r(M) \geq r(Z) + 2$ , there is an element  $e$  of  $E(M) - \text{cl}(Z)$  such that  $M/e$  is 3-connected,  $(M/e) \setminus f$  is not 3-connected for all  $f$  in  $E(M) - (Z \cup e)$ , and  $(M/e)|Z$  is  $M|Z$ .*

As  $M$  has no triads avoiding  $Z$ , and  $E(M) - \text{cl}(Z)$  contains at least two cocircuits,  $|E(M) - \text{cl}(Z)| \geq 5$ . By Lemma 2.5, it suffices to show that  $E(M) - \text{cl}(Z)$  contains an element that is not in a triad meeting  $Z$ . Assume that every element of  $E(M) - \text{cl}(Z)$  is in a triad that meets  $Z$ . Two such triads cannot meet  $Z$  in the same 2-element set, otherwise their symmetric difference is a 2-cocircuit of  $M$ . Moreover, if three such triads meet  $Z$  in different 2-element subsets of a 3-element series class of  $M|Z$ , then their symmetric difference is a triad contained in  $E(M) - \text{cl}(Z)$ . Since the non-trivial series classes of  $M|Z$  consist of either four 2-element sets or two 3-element sets, we deduce that  $|E(M) - \text{cl}(Z)| \leq 4$ . This contradiction completes the proof of 4.7.7.

We shall complete the proof of the lemma by establishing the following.

**4.7.8.**  $r(M) = r(Z) + 1$ .

Assume that  $r(M) \geq r(Z) + 2$ . By repeated application of 4.7.7, we get a 3-connected contraction  $M'$  of  $M$  of rank  $r(Z) + 2$  such that  $M'|Z = M|Z$  and  $M' \setminus f$  is not 3-connected for all  $f$  in  $E(M') - Z$ . By 4.7.7,  $E(M') - \text{cl}(Z)$  contains an element  $e$  such that  $M'/e$  is 3-connected and  $M'/e \setminus f$  is not 3-connected for all  $f$  in  $E(M') - (Z \cup e)$ . As  $(M'/e)|Z = M'|Z$ , we deduce, by 4.7.2 and 4.7.4, that  $M'/e$  is  $M[I_8|A_9]$  or  $M[I_8|B_9]$ . Moreover, to obtain a representation of  $M'$ , we adjoin an extra row to  $[I_8|A_9]$  or  $[I_8|B_9]$  corresponding to the element  $e$ . We consider what rows we can add to  $A_9$  or  $B_9$ . As we know the structure of  $M'|Z$ , we deduce that the first three entries in the row corresponding to  $e$  are zero. As  $M'$  is 3-connected and  $M'$  has no triad avoiding  $Z$ , the other three entries of  $e$  must equal one. Thus  $e$  and  $e_1$  are in series in  $M'$ , a contradiction.  $\square$

## 5. AN EXHAUSTIVE SEARCH

In the previous section, we showed that  $M$  has the cycle matroid of one of the graphs and grafts in Figure 5 as a restriction. We also showed that, in each of these cases, either this restriction is spanning, or  $M$  has a triad disjoint from this restriction. These two cases are treated in this section and the next.

The next lemma is proved by exhaustive search, using the matroid functionality of the Sage mathematics package (Version 7.6) [11]. We will show that any 3-connected binary matroid with rank at least six that is spanned by a graph or graft in Figure 5 and has no 7-circuit is represented over  $GF(2)$  by  $[I_6|A]$  where  $A$  is one of the nine matrices from Figure 2.

**Lemma 5.1.** *Let  $M$  be a 3-connected binary matroid of rank at least six having the cycle matroid of a graph or graft  $G$  in Figure 5 as a restriction. Suppose  $r(M) = r(M(G))$ . Then  $M$  has no odd circuit of size exceeding five if and only if  $M$  is  $M[I_6|A]$  where  $A$  is one of the matrices  $A_{21}, A_{22}, A_{23}, A_{24}, A_{25}, A_{72}, A_{76}, A_{77}$ , or  $A_{78}$  from Figure 2.*

*Proof.* Since  $r(M(H_1)) = r(M(H_4)) = r(M(H_6)) = 5$ , it follows that  $G \in \{H_2, H_3, H_5, H_7\}$ . We assume that  $M$  has no odd circuits of size exceeding five. Our exhaustive search will identify precisely the matroids that satisfy both this condition and the hypotheses of the lemma.

Suppose  $M$  is spanned by  $M(H_2)$ . We construct a binary representation  $[I_6|A_2]$  of  $M(H_2)$  whose first six columns are labelled 2, 3, 4, 5,  $w$ , and  $z$ , respectively. Then  $A_2$  is as shown in Figure 10.

We extend  $M[I_6|A_2]$  by adding binary columns, staying simple of rank 6. An exhaustive search shows that every simple, five-element extension has a 7-circuit. Thus  $|E(M)| \leq 13$ . We consider the simple, four-element extensions of  $M[I_6|A_2]$ . Each either contains a 7-circuit or fails to be 3-connected. Hence  $|E(M)| \leq 12$ . Up to isomorphism,  $M[I_6|A_2]$  has





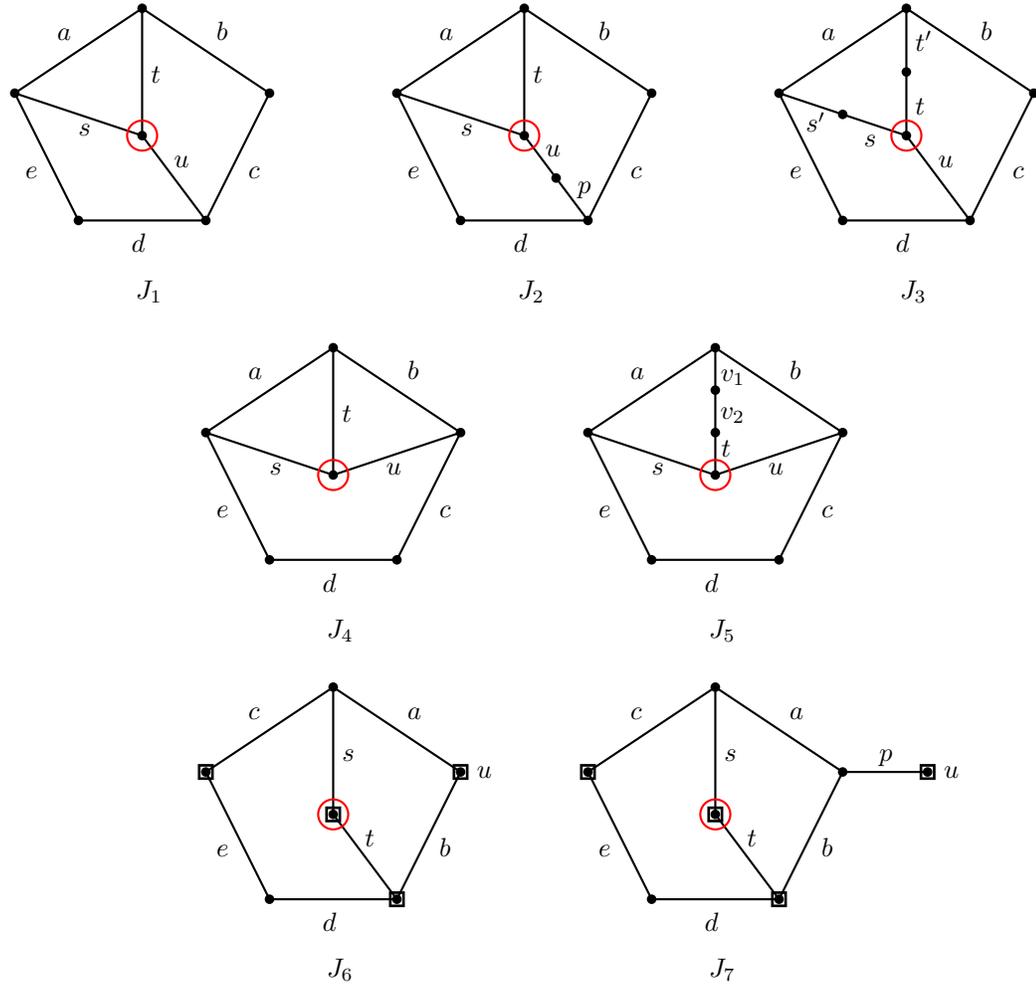


FIGURE 12. Each of  $J_1$  through  $J_7$  is a graph or graft whose cycle matroid is a restriction of  $M$ , where  $\{s, t, u\}$  is a triad of  $M$ .

- (iv)  $\{s, t, u\}$  is a triad of  $M$  that is disjoint from  $Z$ ;
- (v) for all  $e$  in  $E(M)$ , the matroid  $M \setminus e$  is either not 3-connected or has no 5-circuit; and
- (vi)  $M$  has as a restriction to some set  $Z'$  the cycle matroid of one of the graphs or grafts in Figure 12 where  $\{a, b, c, d, e\} = \{1, 2, 3, 4, 5\}$ .

Clearly, as  $\{s, t, u\}$  is a triad of  $M$ ,

$$r(M|(Z \cup \{s, t, u\})) \geq r(Z) + 1.$$

Lemma 6.3 below effectively gives us symmetry between  $(Z, \{1, 2, 3, 4, 5\}, \{x, y, z\})$  and  $(Z', \{a, b, c, d, e\}, \{s, t, u\})$ . The rest of

this section considers the various possibilities for  $M|Z'$  and what each implies about  $M|Z$ . We will use the following preparatory result.

**Lemma 6.2.** *If  $M|Z = M(H_5)$  and  $M|Z' = M(J_5)$ , then*

- (i)  *$M$  has a 7-circuit; or*
- (ii) *without loss of generality,  $(a, b, c, d, e) = (4, 5, 1, 2, 3)$  and  $\{v_1, v_2\}$  avoids  $\{x, y, z\}$ .*

*Proof.* Assume that (i) does not hold. We see that  $\{a, b\}$  avoids  $\{1, 2\}$ , otherwise the symmetric difference of  $\{a, b, s, u\}$  with  $\{1, w_1, w_2, x, y\}$  or  $\{2, w_1, w_2, x, z\}$  is a 7-circuit. By symmetry, we may assume that  $(a, b, c, d, e) = (4, 5, 1, 2, 3)$ . By orthogonality, if  $\{v_1, v_2\}$  meets  $\{x, y, z\}$ , then  $\{v_1, v_2\}$  contains  $x$  and one element of  $\{y, z\}$  as  $\{1, 2, y, z\}$  is not a cycle of  $J_5$ . Thus  $\{1, 2, y, z\} \Delta \{4, s, t, v_1, v_2\}$  is a 7-circuit, a contradiction.  $\square$

**Lemma 6.3.** *Either the sets  $\{x, y, z\}$  and  $Z'$  are disjoint, or  $M$  has an odd circuit of size exceeding five.*

*Proof.* Suppose the lemma fails. Then, by orthogonality,  $|Z' \cap \{x, y, z\}| \geq 2$  and  $Z' \cap \{x, y, z\}$  is a union of cocircuits of  $M|Z'$ . Since  $\{s, t, u\} \cap Z = \emptyset$ , it follows that  $M|Z' = M(J_5)$  and  $\{v_1, v_2\} \subseteq \{x, y, z\}$ . The element of  $\{x, y, z\} - \{v_1, v_2\}$  is not in  $Z'$ .

We now consider the various possibilities for  $M|Z$  noting that, by the last lemma,  $M|Z \neq M(H_5)$ . In each of  $M(H_4)$ ,  $M(H_1)$ , and  $M(H_6)$ , each 2-element subset of  $\{x, y, z\}$  is in a 3- or 4-circuit that is contained in  $\{x, y, z, 1, 2, 3, 4, 5\}$ . As  $M(J_5)$  has no such circuit, we deduce that  $M|Z$  is  $M(H_2)$ ,  $M(H_3)$ , or  $M(H_7)$ .

Suppose  $M|Z = M(H_2)$ . The circuit  $\{1, x, y\}$  implies that  $\{v_1, v_2\} \neq \{x, y\}$ . By symmetry, we may assume that  $\{v_1, v_2\} = \{x, z\}$ . Then the symmetric difference of  $\{1, x, y\}$  with  $\{x, z, t, a, e, d, c, u\}$  or  $\{x, z, t, s, e, d, c, b\}$  is a 7-circuit, a contradiction.

Suppose  $M|Z = M(H_3)$ . Assume  $\{x, y\} = \{v_1, v_2\}$ . Then  $1 \notin \{a, b\}$ , otherwise  $\{a, b, s, u\} \Delta \{1, x, y, x', y'\}$  is a 7-circuit. Thus  $\{x, y, t, s, e, d, c, b\} \Delta \{1, x, y, x', y'\}$  is a 7-circuit, a contradiction. Hence, by symmetry, we may assume that  $\{v_1, v_2\} = \{x, z\}$ . Then  $a \in \{2, 3\}$ , otherwise  $\{x, z, t, s, e, d, c, b\} \Delta \{x, x', z, 3, 2\}$  is a 7-circuit. By symmetry, we may assume that  $a = 2$ . Then  $\{2, x, z, t, s\} \Delta \{2, 3, x, z, x'\}$ , which is  $\{t, s, 3, x'\}$ , is a circuit. Thus  $b = 3$ , otherwise  $\{t, s, 3, x'\} \Delta \{b, x, z, t, u\}$  is a 7-circuit. Then  $\{1, 4, 5, s, u\}$  is a circuit and we obtain the contradiction that  $\{1, 4, 5, s, u\} \Delta \{t, s, 3, x'\}$  is a 7-circuit. Therefore  $M|Z \neq M(H_3)$ .

Finally, suppose that  $M|Z = M(H_7)$ . Then  $\{a, b\} \neq \{2, 3\}$ , otherwise  $\{a, b, s, u\} \Delta \{1, x, 3, w, z\}$  is a 7-circuit. From  $M(H_7)$ , we see that  $\{2, 3, x, y\}$  is a circuit of  $M$ . Since  $\{2, 3, v_1, v_2\}$  is independent in  $M(J_5)$ , we must have that  $z \in \{v_1, v_2\}$ . Then the symmetric difference of  $\{2, 3, x, y\}$  with  $\{a, s, t, v_1, v_2\}$  or  $\{b, u, t, v_1, v_2\}$  is a 7-circuit, a contradiction.  $\square$

At some point in the argument, we will want to switch from  $(Z, \{1, 2, 3, 4, 5\}, \{x, y, z\})$  and  $(Z', \{1, 2, 3, 4, 5\}, \{s, t, u\})$  to another pair of

triples obeying the same hypotheses. The next result, which follows immediately by combining Lemmas 5.1, 6.1, and 6.3, enables us to do this.

**Lemma 6.4.** *Assume that  $M$  is not isomorphic to  $M[I_6|A]$  where  $A$  is one of the matrices in Figure 2. Let  $Y$  be a subset of  $E(M)$  such that, for some  $i$  in  $\{1, 2, \dots, 7\}$ , there is an isomorphism  $\varphi : E(H_i) \rightarrow Y$  between  $M(H_i)$  and  $M|Y$ . Let  $\varphi(\{1, 2, 3, 4, 5\}) = X$  and assume that  $\varphi(\{x, y, z\})$  is a triad  $R$  of  $M$ . Suppose  $M$  has no odd circuits of size exceeding five. Then  $M$  has a triad  $R'$  that is disjoint from  $Y$ , and  $M$  has a subset  $Y'$  that contains  $X \cup R'$  such that  $M|Y'$  is isomorphic to the cycle matroid of a graph or graft in Figure 12 where  $X$  and  $R'$  correspond to  $\{a, b, c, d, e\}$  and  $\{s, t, u\}$ . Moreover,  $R \cap Y' = \emptyset$ . In particular,  $(Z, \{1, 2, 3, 4, 5\}, \{x, y, z\})$  and  $(Z', \{1, 2, 3, 4, 5\}, \{s, t, u\})$  can be replaced by  $(Y, X, R)$  and  $(Y', X, R')$ .*

**Lemma 6.5.** *If  $M|Z'$  has  $\{s, t, a\}$  as a circuit and  $M|Z$  has a 6- or 8-circuit  $C$  containing  $a$ , then  $M$  has an odd circuit of size exceeding five.*

*Proof.* As  $(C - \{a\}) \cup \{s\}$  is independent, and  $\{s, t, a\} \Delta C$  is a disjoint union of circuits, it follows that  $\{s, t, a\} \Delta C$  is a 7- or 9-circuit.  $\square$

**Lemma 6.6.** *If  $M|Z' \in \{M(J_1), M(J_2), M(J_4)\}$  and  $M|Z \in \{M(H_1), M(H_2), M(H_3), M(H_4), M(H_5)\}$ , then either*

- (i)  *$M$  has an odd circuit of size exceeding five; or*
- (ii)  *$M|Z = M(H_1)$  and  $M|Z' = M(J_1)$ , and  $a = 1$  and  $\{\{b, c\}, \{d, e\}\} = \{\{2, 3\}, \{4, 5\}\}$ .*

*Proof.* Clearly  $M|Z'$  has  $\{s, t, a\}$  as a circuit. Unless  $M|Z = M(H_1)$  and  $a = 1$ , the matroid  $M|Z$  has a 6- or 8-circuit containing  $a$ , so (i) holds by Lemma 6.5. In the exceptional case,  $M(H_1)$  has  $\{1, x, y\}$  as a circuit. If  $M|Z' \in \{M(J_2), M(J_4)\}$ , then the symmetric difference of  $\{1, x, y\}$  with a 6- or 8-circuit in  $M|Z'$  is a 7- or 9-circuit and (i) holds. Finally, assume that  $M|Z' = M(J_1)$ . Suppose that  $\{\{b, c\}, \{d, e\}\} \neq \{\{2, 3\}, \{4, 5\}\}$ . We may assume that  $\{b, c\} = \{3, 4\}$ . Then  $\{1, 2, 3, y, z\} \Delta \{t, u, 3, 4\}$  is a 7-circuit.  $\square$

**Lemma 6.7.** *If  $M|Z' = M(J_4)$ , then  $M$  has an odd circuit of size exceeding five.*

*Proof.* By switching the labels of  $s$  and  $u$  if necessary, we may assume that  $a \neq 1$ . By Lemma 6.6,  $M|Z \neq M(H_i)$  for all  $i$  in  $\{1, 2, 3, 4, 5\}$ . If  $M|Z = M(H_7)$ , then the result follows by Lemma 6.5 as  $M$  has  $\{2, 4, 5, w, x, z\}$  and  $\{3, 4, 5, w, y, z\}$  as circuits. Now suppose that  $M|Z = M(H_6)$ . If  $\{a, b\}$  meets both  $\{1, 4, 5\}$  and  $\{2, 3\}$ , then  $\{s, u, a, b\} \Delta \{1, 4, 5, x, y\}$  is a 7-circuit. If  $\{a, b\}$  is  $\{2, 3\}$ ,  $\{1, 4\}$ , or  $\{1, 5\}$ , then  $\{3, 4, 5, y, z\} \Delta \{s, u, a, b\}$  is a 7-circuit. We may now assume that  $\{a, b\} = \{4, 5\}$ . Then  $\{1, x, y, 4, 5\} \Delta \{s, u, 4, 5\}$ , which equals  $\{1, x, y, s, u\}$ , is a circuit. As  $\{1, 2, 3, 4, 5\}$  and  $\{3, 4, 5, y, z\}$  are also 5-circuits, Lemma 2.7 implies that  $M$  is minimally 3-connected. By Theorem 2.2, the triangle  $\{s, t, a\}$  meets two triads. If  $\{a, s\}$  is contained in a triad, then orthogonality implies that this triad contains two elements from

each circuit in  $\{s, u, 1, 2, 3\}$ ,  $\{2, 4, 5, x, z\}$ , and  $\{3, 4, 5, y, z\}$ , a contradiction. Therefore  $\{a, t\}$  is contained in a triad  $T^*$ . By orthogonality with  $\{t, u, b\}$ , the third element of  $T^*$  is contained in  $\{u, b\}$ . Thus  $\lambda(\{4, 5, s, t, u\}) = 1$ , contradicting the fact that  $M$  is 3-connected.  $\square$

**Lemma 6.8.** *If  $M|Z' = M(J_1)$ , then*

- (i)  *$M$  has an odd circuit of size exceeding five; or*
- (ii)  *$M|Z = M(H_1)$  and  $a = 1$  and  $\{\{b, c\}, \{d, e\}\} = \{\{2, 3\}, \{4, 5\}\}$ .*

*Proof.* If  $M|Z = M(H_i)$  for some  $i$  in  $\{1, 2, 3, 4, 5\}$ , then the result holds by Lemma 6.6. Thus we may assume that  $M|Z$  is  $M(H_6)$  or  $M(H_7)$ . Then a circuit in  $\{\{t, u, b, c\}, \{s, u, d, e\}\}$  that meets  $\{1, 4, 5\}$  in a single element has a 7-circuit as its symmetric difference with the circuit  $\{1, 4, 5, x, y\}$ . Without loss of generality, we may assume that  $\{b, c\} = \{2, 3\}$ . Then  $M$  has  $\{2, 4, 5, x, z\} \Delta \{t, u, 2, 3\}$  or  $\{1, 2, w, y, z\} \Delta \{t, u, 2, 3\}$  as a 7-circuit.  $\square$

**Lemma 6.9.** *If  $M|Z' = M(J_2)$ , then  $M$  has an odd circuit of size exceeding five.*

*Proof.* By orthogonality,  $p \notin \{x, y, z\}$ . By Lemma 6.6,  $M|Z \neq M(H_i)$  for all  $i$  in  $\{1, 2, 3, 4, 5\}$ . Thus  $M|Z$  is  $M(H_6)$  or  $M(H_7)$ . If  $\{b, c\}$  or  $\{d, e\}$  contains exactly one element of  $\{2, 3\}$ , then  $\{p, t, u, b, c\} \Delta \{2, 3, x, y\}$  or  $\{p, s, u, d, e\} \Delta \{2, 3, x, y\}$  is a 7-circuit, so we assume not. Without loss of generality,  $\{b, c\} = \{2, 3\}$  and  $\{a, d, e\} = \{1, 4, 5\}$ . When  $M|Z = M(H_6)$ , it follows that  $M$  has  $\{1, 2, y, z\} \Delta \{p, t, u, 2, 3\}$  as a 7-circuit. Thus we may assume that  $M|Z$  is  $M(H_7)$ . If  $a \in \{4, 5\}$ , then  $\{3, 4, 5, w, y, z\} \Delta \{s, t, a\}$  is a 7-circuit, so we may assume that  $a = 1$  and  $\{d, e\} = \{4, 5\}$ . Moreover,  $p \neq w$ , otherwise  $\{2, 4, 5, w, x, z\} \Delta \{p, t, u, 2, 3\}$  is a 7-circuit. Then  $\{2, 4, 5, w, x, z\} \Delta \{p, s, u, 4, 5\}$  is a 7-element set  $S$  that contains the independent set  $\{2, p, s, u, x\}$ . We may assume that  $S$  is not a circuit. Then  $\{2, p, s, u, x\}$  is a basis of  $M|S$ , so  $S$  is the disjoint union of two circuits,  $C_1$  and  $C_2$ , where  $w \in C_1$  and  $z \in C_2$ . By orthogonality,  $x \in C_2$  and  $\{s, u\}$  is contained in  $C_1$  or  $C_2$ . Thus  $C_1$  is  $\{w, 2, p\}$ ,  $\{w, s, u\}$ ,  $\{w, s, u, 2\}$ , or  $\{w, s, u, p\}$ . In the first two cases,  $C_1 \Delta \{s, t, 2, 3, 4, 5\}$  is a 7-circuit. If  $C_1 = \{w, s, u, 2\}$ , then  $C_1 \Delta \{1, 2, 3, 4, 5\}$  is a 7-circuit. Finally, if  $C_1 = \{w, s, u, p\}$ , then  $C_2$  is the independent set  $\{z, x, 2\}$ , a contradiction.  $\square$

As the last three lemmas treat the cases when  $M|Z'$  is  $M(J_4)$ ,  $M(J_1)$ , or  $M(J_2)$ , and we have symmetry between  $M|Z$  and  $M|Z'$ , provided  $M$  has no odd circuit of size exceeding five, we may assume  $M|Z \neq M(H_1)$ .

**Lemma 6.10.** *If  $M|Z \notin \{M(H_1), M(H_2), M(H_4)\}$  and  $M|Z' = M(J_6)$ , then  $M$  has an odd circuit of size exceeding five.*

*Proof.* Assume that the lemma fails. As  $M|Z'$  has each of  $\{s, t, a, b\}$ ,  $\{s, u, b, c\}$ , and  $\{t, u, a, c\}$  as circuits,  $M|Z$  does not have a 5-circuit meeting such a 4-circuit in a single element, otherwise the symmetric difference of these two circuits is a 7-circuit. Therefore  $M|Z \neq M(H_3)$  as

$M(H_3)$  has  $\{2, 3, x, x', z\}$  and  $\{4, 5, y, y', z\}$  as circuits;  $M|Z \neq M(H_6)$  as  $M(H_6)$  has  $\{1, 4, 5, x, y\}$  and  $\{3, 4, 5, y, z\}$  as circuits; and  $M|Z \neq M(H_7)$  as  $M(H_7)$  has  $\{1, 2, w, y, z\}$  and  $\{1, 4, 5, x, y\}$  as circuits. We deduce that  $M|Z = M(H_5)$ . Moreover, the circuits  $\{1, w_1, w_2, x, y\}$  and  $\{2, w_1, w_2, x, z\}$  of the latter imply that  $\{1, 2\}$  avoids  $\{a, b, c\}$ . By symmetry, we may assume that  $(a, b, c) = (3, 4, 5)$ . Using the circuits  $\{s, u, 4, 5\}$  and  $\{s, t, 3, 4\}$  of  $M$ , it is straightforward to check that  $M$  has  $M(J)$  as a restriction where  $J$  is the graft in Figure 13.

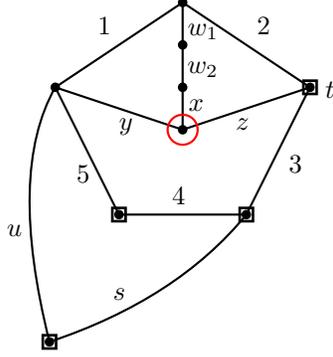


FIGURE 13. The graft  $J$  where  $\{x, y, z\}$  and  $\{s, t, u\}$  are triads of  $M$ .

Since  $\{s, t, u\}$  is a triad of  $M$  that avoids the circuit  $\{1, w_1, w_2, x, y\}$ , Lemma 4.2 implies that  $\{s, t, u\} \cup \{1, w_1, w_2, x, y\}$  is contained in one of the configurations shown in Figure 5, where  $\{s, t, u\}$  and  $\{1, w_1, w_2, x, y\}$  take the place of  $\{x, y, z\}$  and  $\{1, 2, 3, 4, 5\}$ . From Figure 13, no 3-circuit or 4-circuit contains two elements of  $\{s, t, u\}$  and its other elements in  $\{1, w_1, w_2, x, y\}$ . Therefore  $\{s, t, u\} \cup \{1, w_1, w_2, x, y\}$  is contained in a restriction  $K$  of  $M$  that is isomorphic to  $M(H_3)$ . Since no element in  $E(H_5)$  is contained in every 5-circuit of  $M(H_5)$ , Lemma 2.7 implies that  $M$  is minimally 3-connected. By Theorem 2.2, two triads of  $M$  meet  $\{3, 4, 5, y, z\}$ . By orthogonality with the circuits  $\{s, t, 3, 4\}$ ,  $\{t, u, 3, 5\}$ ,  $\{s, u, 4, 5\}$ ,  $\{1, 2, 3, 4, 5\}$  and  $\{1, 2, y, z\}$  of  $M$ , we deduce that one of  $\{3, 4, u\}$ ,  $\{3, 5, s\}$ , or  $\{4, 5, t\}$  is a triad  $T^*$  of  $M$ . Now  $K$  has a 5-circuit  $D$  such that  $D$  contains exactly two elements of  $\{s, t, u\}$ , one of which is in  $T^*$ , and  $D$  contains exactly two elements of  $\{1, w_1, w_2, x, y\}$ . By orthogonality between  $D$  and  $T^*$ , the fifth element of  $D$  is in  $\{3, 4, 5\}$ . But, from the graft  $J$ , we see that  $M$  has no such 5-circuit  $D$ , a contradiction.  $\square$

Before treating the cases when  $M|Z'$  is  $M(J_7)$ ,  $M(J_5)$ , or  $M(J_3)$ , we eliminate the possibility that  $M$  has a triangle meeting  $Z$  when  $M|Z = M(H_3)$ .

**Lemma 6.11.** *Suppose  $M|Z = M(H_3)$ . If  $M$  has a triangle  $T$  meeting  $Z$ , then  $M$  has a 7-circuit or a 9-circuit.*

*Proof.* Assume that the lemma fails. As no element of  $M(H_3)$  is in every 5-circuit,  $M$  is minimally 3-connected. If  $T$  contains two elements in the circuit  $\{2, 3, 4, 5, x, x', y, y'\}$ , then the symmetric difference of this circuit with  $T$  is a 7-circuit. We assume therefore that

**6.11.1.**  $T$  contains at most one element in  $\{2, 3, 4, 5, x, x', y, y'\}$ .

By Theorem 2.2, every element of  $T$  is in a triad of  $M$ . By symmetry, we may assume that  $T$  contains  $x, z, 1, 2$ , or  $x'$ . Suppose  $x \in T$ . By orthogonality,  $T$  contains exactly one element of  $\{y, z\}$ . Then  $T \Delta \{1, 2, 3, y, y', z\}$  is a 7-circuit, a contradiction. Thus  $x \notin T$ . By symmetry,  $y \notin T$ . By orthogonality,  $z \notin T$ .

Suppose  $1 \in T$ . By symmetry and orthogonality,  $\{1, 2, x\}$  or  $\{1, 2, x'\}$  is a triad. Since  $x \notin T$ , by orthogonality,  $\{1, 2\}$  or  $\{1, x'\}$  is in  $T$ . If  $T$  contains  $\{1, x'\}$ , then clearly the third element of  $T$  is not in  $E(H_3)$ . Hence  $T \Delta \{1, 2, 3, y, y', z\}$  is a 7-circuit, a contradiction. Then  $\{1, 2\} \subseteq T$  and  $T \Delta \{1, 4, 5, x, x', z\}$  is a 7-circuit. This contradiction implies that  $1 \notin T$ .

Suppose  $2 \in T$ . By symmetry and orthogonality,  $\{1, 2, x\}$ ,  $\{1, 2, x'\}$ ,  $\{2, 4, z\}$ , or  $\{2, 3, f\}$  is a triad,  $T^*$ , for some element  $f$  that is not in  $E(H_3)$ . As none of  $x, z$ , or  $1$  is in  $T$ , by orthogonality,  $\{2, x'\}$ ,  $\{2, 4\}$ ,  $\{2, 3\}$ , or  $\{2, f\}$  is contained in  $T$ . By 6.11.1, we deduce that  $\{2, f\} \subseteq T$ , so  $T^* = \{2, 3, f\}$ . Let  $T - \{2, f\} = \{g\}$ . Since  $1 \notin T$ , it follows by 6.11.1 that  $g \notin E(H_3) - z$ . Moreover, by orthogonality,  $g \neq z$ , so  $\{f, g\}$  avoids  $E(H_3)$ . As  $M(H_3)$  has an 8-circuit containing  $2$  but has no 3-element series class, it follows by Lemma 2.6 that  $M$  has a 9-circuit, a contradiction.

Finally, suppose  $x' \in T$ . By symmetry and orthogonality,  $\{1, 2, x'\}$ ,  $\{x', y', z\}$ ,  $\{x', y, z\}$ , or  $\{x', x, f\}$  is a triad of  $M$  for some  $f \notin E(H_3)$ . As  $\{x, y, z\}$  is a triad,  $\{x', y, z\}$  is not a triad. Since none of  $x, y, z, 1$ , or  $2$  is in  $T$ , we see that  $\{x', y'\}$  or  $\{x', f\}$  is contained in  $T$ . By 6.11.1,  $\{x', f\} \subseteq T$ . Let  $T - \{x', f\} = \{g\}$ . We know that  $g \notin \{x, y, z, 1, 2, 3, 4, 5, x'\}$ . Thus, by 6.11.1,  $g \notin E(H_3)$ . As for the case when  $2 \in T$ , we use Lemma 2.6 to obtain the contradiction that  $M$  has a 9-circuit.  $\square$

**Lemma 6.12.** *If  $M|Z \notin \{M(H_1), M(H_2), M(H_4), M(H_6)\}$  and  $M|Z' = M(J_7)$ , then  $M$  has an odd circuit of size exceeding five.*

*Proof.* Assume that the lemma fails. By Lemma 6.3,  $p \notin \{x, y, z\}$ . Suppose  $M|Z = M(H_5)$ . As no element of  $M(H_5)$  is in every 5-circuit, by Lemma 2.7,  $M$  is minimally 3-connected. Observe that  $\{a, b\}$  avoids  $\{1, 2\}$ , otherwise  $\{s, t, a, b\} \Delta \{1, w_1, w_2, x, y\}$  or  $\{s, t, a, b\} \Delta \{2, w_1, w_2, x, z\}$  is a 7-circuit. Without loss of generality,  $(a, b) = (3, 4)$ . Now  $c \notin \{1, 2\}$ , otherwise  $\{1, 2, y, z\} \Delta \{p, 3, c, t, u\}$  is a 7-circuit. Then  $c = 5$  and, without loss of generality,  $(d, e) = (1, 2)$ . Next note that  $p \notin \{w_1, w_2\}$ , otherwise  $\{1, 2, 4, p, t, u\} \Delta \{1, w_1, w_2, x, y\}$  is a 7-circuit. Then  $E(H_5) \cap E(J_7) = \{1, 2, 3, 4, 5\}$  and  $\{2, 3, 4, 5, w_1, w_2, x, y\} \Delta \{4, 5, p, s, u\}$ , which is  $\{2, 3, p, s, u, w_1, w_2, x, y\}$ , is a disjoint union of circuits. As  $\{2, 3, s, w_1, w_2, x, y\}$  is an independent set and  $\{2, 3, p, s, u, w_1, w_2, x, y\}$  is

not a circuit, the last set is the disjoint union of circuits  $C_1$  and  $C_2$  where  $p \in C_1$  and  $u \in C_2$ . By orthogonality,  $s \in C_2$ . Neither  $C_1$  nor  $C_2$  is contained in  $E(J_7)$ , so both meet  $E(H_5)$ . Observe that  $|C_2| > 3$ , otherwise the element of  $C_2 - \{s, u\}$  is in  $\{2, 3, w_1, w_2, x, y\}$  and  $C_2 \Delta \{2, 3, 4, 5, x, y, w_1, w_2\}$  is a 9-circuit. Suppose  $|C_2| = 4$ . Then  $2 \notin C_2$ , otherwise the fourth element of  $C_2$  is in  $\{w_1, w_2\}$  and  $C_2 \Delta \{1, w_1, w_2, x, y\}$  is a 7-circuit. It follows that  $C_2 \Delta \{1, 2, 5, s, t\}$  is a 7-circuit, a contradiction. We deduce that  $|C_2| > 4$ . If  $|C_1| = 4$ , then  $p \in C_1 \subseteq \{2, 3, p, w_1, w_2, x, y\}$ , and  $C_1 \Delta \{4, 5, p, s, u\}$  is a 7-circuit. Thus  $|C_1| = 3$  and, by orthogonality,  $C_1 = \{p, x, y\}$  or  $C_1 - p \subseteq \{w_1, w_2, 2, 3\}$ . In both cases, we get the contradiction that  $C_1 \Delta \{2, 3, w_1, w_2, 4, 5, x, y\}$  is a 7-circuit.

Suppose  $M|Z = M(H_3)$ . By Lemma 2.7,  $M$  is minimally 3-connected. Now  $\{a, b\}$  does not contain exactly one element in  $\{1\}$ , in  $\{2, 3\}$  or in  $\{4, 5\}$ , otherwise the symmetric difference of  $\{s, t, a, b\}$  with  $\{1, x, x', y, y'\}$ ,  $\{2, 3, x, x', z\}$ , or  $\{4, 5, y, y', z\}$  is a 7-circuit. By symmetry, we may assume that  $(a, b) = (2, 3)$ . The elements in  $E(H_3) - \{1, 2, 3, 4, 5\}$  and  $E(J_7) - \{1, 2, 3, 4, 5\}$  are all distinct except that  $p$  may be in  $\{x', y'\}$ . But  $p \neq x'$ , otherwise  $\{1, 4, 5, x, p, z\} \Delta \{3, c, p, s, u\}$  is a 7-circuit. Moreover,  $p \neq y'$ , otherwise either  $1 \in \{d, e\}$  and  $\{3, d, e, p, t, u\} \Delta \{4, 5, y, p, z\}$  is a 7-circuit, or  $c = 1$  and  $\{2, 3, 4, 5, x, x', y, p\} \Delta \{1, 3, p, s, u\}$  is a 9-circuit. Thus  $E(H_3) \cap E(J_7) = \{1, 2, 3, 4, 5\}$ . As  $c \in \{1, 4, 5\}$ , the set  $\{3, c, p, s, u\} \Delta \{1, 4, 5, x, x', z\}$ , which contains the independent set  $\{3, s, x, x', z\} \cup (\{1, 4, 5\} - c)$ , is the disjoint union of circuits  $C_1$  and  $C_2$  where  $p \in C_1$  and  $u \in C_2$ . By orthogonality,  $s \in C_2$ . Neither  $C_1$  nor  $C_2$  is contained in  $E(J_7)$ , so Lemma 6.11 implies that  $|C_1| > 3$  and  $|C_2| > 3$ . If  $|C_2| = 4$ , then  $C_2 = \{s, u, x, z\}$  or  $\{s, u\} \subseteq C_2 \subseteq \{3, s, u, x'\} \cup (\{1, 4, 5\} - c)$ . The former implies that  $C_2 \Delta \{1, 4, 5, s, t\}$  is a 7-circuit so the latter holds. Then  $x' \in C_2$ , since  $C_2 \not\subseteq E(J_7)$ , so  $C_2$  is  $\{s, u, x', \alpha\}$  for some  $\alpha$  in  $\{1, 3, 4, 5\}$ . Then  $C_2 \Delta \{1, 2, 3, 4, 5\}$  is a 7-circuit. We deduce that  $|C_2| > 4$ . Then  $|C_1| = 4$ . If  $\{p, x, z\} \subseteq C_1$ , then  $x' \in C_2$ , since  $C_2 \not\subseteq E(J_7)$ . Then  $C_1$  is  $\{p, x, z, \beta\}$  for some  $\beta$  in  $\{1, 3, 4, 5\}$ , and the symmetric difference of  $C_1$  with  $\{1, 2, 3, 4, 5\}$  is a 7-circuit, a contradiction. It follows by orthogonality that  $C_1 \subseteq \{3, p, x'\} \cup (\{1, 4, 5\} - c)$ . Moreover,  $x' \in C_1$  since  $C_1 \not\subseteq E(J_7)$ . Then  $C_1$  is the union of  $\{p, x'\}$  with two elements of  $\{1, 3, 4, 5\}$ . Thus the symmetric difference of  $C_1$  with  $\{1, x, x', y, y'\}$  or  $\{p, u, 2, c, t\}$  is a 7-circuit, a contradiction.

Suppose  $M|Z = M(H_7)$ . Now  $\{a, b\}$  does not contain exactly one element in  $\{1, 4, 5\}$  or in  $\{1, 3\}$ , otherwise  $\{s, t, a, b\} \Delta \{1, 4, 5, x, y\}$  or  $\{s, t, a, b\} \Delta \{1, 3, w, x, z\}$  is a 7-circuit. By symmetry, we may assume that  $(a, b) = (4, 5)$ . If  $c \in \{2, 3\}$ , then  $\{2, 3, x, y\} \Delta \{5, c, p, s, u\}$  is a 7-circuit. Thus  $c = 1$  and  $\{d, e\} = \{2, 3\}$ . Now  $p \neq w$ , otherwise  $\{1, 3, w, x, z\} \Delta \{2, 3, 5, p, t, u\}$  is a 7-circuit. Then  $\{1, 3, w, x, z\} \Delta \{2, 3, 5, p, t, u\}$ , which contains the independent set  $\{1, 2, 5, t, w, x, z\}$ , is the disjoint union of circuits  $C_1$  and  $C_2$ , where  $p \in C_1$  and  $u \in C_2$ . By orthogonality,  $t \in C_2$ . If  $|C_2| = 3$ , then

orthogonality implies that  $C_2$  avoids  $\{x, z\}$ . Since  $M(J_7)$  contains no 3-circuit,  $C_2 = \{t, u, w\}$  and  $C_2 \Delta \{3, 4, 5, w, y, z\}$  is a 7-circuit, a contradiction. Thus  $|C_2| > 3$ . If  $|C_2| = 4$ , then  $C_2 \subseteq \{1, 2, 5, t, u, w\}$ , otherwise  $C_2 = \{t, u, x, z\}$  and  $C_2 \Delta \{1, 4, 5, x, y\}$  is a 7-circuit. Since  $\{1, 2, 5, t, u\}$  is an independent set,  $w \in C_2$  and  $C_2$  has a single element in  $\{1, 2, 5\}$ , so  $C_2 \Delta \{1, 2, 3, 4, 5\}$  is a 7-circuit. We deduce that  $|C_2| > 4$ . Hence  $|C_1| < 5$ . Now  $C_1 \neq \{p, x, z\}$ , otherwise  $C_1 \Delta \{3, 4, 5, w, y, z\}$  is a 7-circuit. Thus if  $|C_1| = 3$ , then  $C_1 \subseteq \{1, 2, 5, p, w\}$ . Since  $\{1, 2, 5, p\}$  is an independent set,  $w \in C_1$ . Thus  $C_1$  is  $\{p, w, 1\}$ ,  $\{p, w, 2\}$ , or  $\{p, w, 5\}$ , and  $C_1 \Delta \{3, 4, 5, w, y, z\}$  or  $C_1 \Delta \{2, 3, 4, p, s, u\}$  is a 7-circuit, a contradiction. Hence  $|C_1| = 4$ . If  $\{x, z\} \subseteq C_1$ , then  $C_1$  is  $\{p, x, z, \alpha\}$  for some  $\alpha$  in  $\{1, 2, 5, w\}$ , and  $C_1 \Delta \{1, 2, 3, 4, 5\}$  or  $C_1 \Delta \{1, 5, p, s, u\}$  is a 7-circuit. We deduce by orthogonality that  $C_1 \subseteq \{1, 2, 5, p, w\}$ . Since  $\{1, 2, 5, p\}$  is an independent set,  $w \in C_1$  and  $C_1$  is  $\{1, 2, p, w\}$ ,  $\{1, 5, p, w\}$ , or  $\{2, 5, p, w\}$ . Then  $C_1 \Delta \{1, 4, 5, x, y\}$  or  $C_1 \Delta \{1, 2, 3, s, t\}$  is a 7-circuit, a contradiction.  $\square$

**Lemma 6.13.** *Suppose that  $M$  has no non-spanning restriction isomorphic to a matroid in  $\{M(H_2), M(H_4), M(H_6), M(H_7)\}$  where the elements corresponding to  $\{x, y, z\}$  form a triad of  $M$ . If  $M|Z \neq M(H_1)$  and  $M|Z' = M(J_5)$ , then  $M$  has an odd circuit of size exceeding five.*

*Proof.* Assume that the lemma fails. Let  $M|Z = M(H_5)$ . Then  $r(M) \geq 8$ . By Lemma 6.2, without loss of generality,  $(a, b, c, d, e) = (4, 5, 1, 2, 3)$  and  $\{v_1, v_2\}$  avoids  $\{x, y, z\}$ . Note that  $\{s, t, u\}$  is a triad that avoids the 5-circuit  $\{1, w_1, w_2, x, y\}$ . By Lemma 4.2, this triad and 5-circuit are contained in one of the configurations in Figure 5, with  $\{s, t, u\}$  and  $\{1, w_1, w_2, x, y\}$  taking the places of  $\{x, y, z\}$  and  $\{1, 2, 3, 4, 5\}$ . For each  $i$ , let  $L_i$  be the resulting relabelled version of  $H_i$ . Then  $E(M)$  has a subset  $Z''$  containing  $\{s, t, u, 1, w_1, w_2, x, y\}$  such that  $M|Z'' = M(L_i)$  for some  $i$  in  $\{1, 3, 5\}$ .

First we note that  $M|Z''$  is not  $M(L_1)$ , otherwise  $M$  has a triangle  $T$  containing two elements of  $\{s, t, u\}$  and one element of  $\{1, w_1, w_2, x, y\}$ , so the symmetric difference of  $T$  with one of  $\{1, w_1, w_2, x, z, 3, 4, 5\}$  or  $\{2, w_1, w_2, x, y, 3, 4, 5\}$  is a 9-circuit. Next, we note, from  $M(J_5)$ , that  $M$  has  $\{s, u, 4, 5\}$  as a circuit. Suppose  $M|Z'' = M(L_5)$ . Then  $M$  has a 4-circuit  $C$  that contains two elements of  $\{s, t, u\}$  and two elements of  $\{1, w_1, w_2, x, y\}$ . By orthogonality and symmetry,  $C \cap \{1, w_1, w_2, x, y\}$  is  $\{x, y\}$ ,  $\{1, w_1\}$ , or  $\{w_1, w_2\}$ . Thus  $C \cap \{s, t, u\} \neq \{s, u\}$ , otherwise  $C \Delta \{s, u, 4, 5\}$  is dependent, which, from  $M(H_5)$ , is not so. Then  $t \in C$  and  $\{1, w_1, w_2, x, y\} \Delta C \Delta \{s, u, 4, 5\}$  is a 7-circuit since it contains a 6-element independent set. This contradiction implies that  $M|Z'' = M(L_3)$ .

Since  $M|Z''$  is the cycle matroid of the graph in Figure 14, we may assume by symmetry that  $\{g, i\} \neq \{s, u\}$ . Thus  $t \in \{g, i\}$ . Assume  $g' \notin \{4, 5\}$ . Then  $\{s, u, 4, 5\} \Delta \{\beta, \gamma, g, i, g'\}$  is the disjoint union  $D$  of a 3-circuit  $C_1$  and a 4-circuit  $C_2$ . Now  $\{\beta, \gamma, 4, 5, t\}$  spans  $D$ . Let  $\{s, u\} \Delta \{g, i\} = \{t, g''\}$ . Then  $g'' \in \{s, u\}$ . We see that one of  $g''$  and  $g'$  is in  $C_1$  while the other is in  $C_2$ . Suppose  $g'' \in C_1$ . Then, by orthogonality,  $t \in C_1$ .

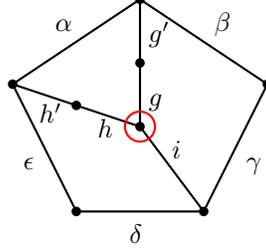


FIGURE 14.  $M|Z''$  is the cycle matroid of this graph where  $\{g, h, i\} = \{s, t, u\}$  and  $\{\alpha, \beta, \gamma, \delta, \epsilon\} = \{1, w_1, w_2, x, y\}$ .

Thus  $C_1$  is a triangle containing two elements of  $\{s, t, u\}$ . Its third element is in  $\{1, w_1, w_2, x, y\}$ . Taking the symmetric difference of  $C_1$  with  $\{1, w_1, w_2, x, z, 3, 4, 5\}$  or  $\{2, w_1, w_2, x, y, 3, 4, 5\}$  is a 9-circuit, a contradiction. Thus  $g'' \notin C_1$ , so  $g' \in C_1$ . The remaining two elements of  $C_1$  are in  $\{\beta, \gamma, 4, 5\}$  and so are in  $\{1, w_1, w_2, x, y, 4, 5\}$ . Every 2-element subset of the last set except  $\{1, y\}$  is contained in an 8-circuit  $D'$  where  $D'$  is  $\{1, w_1, w_2, x, z, 3, 4, 5\}$  or  $\{2, w_1, w_2, x, y, 3, 4, 5\}$ . By orthogonality,  $C_1 \neq \{g', 1, y\}$ . Then  $C_1 \Delta D'$  is a 7-circuit, a contradiction. We deduce that  $g' \in \{4, 5\}$ .

If  $h' = z$ , then orthogonality implies that  $\alpha \in \{x, y\}$ , and  $\{\alpha, \beta, \gamma, z, h, i\} \Delta \{3, 4, 5, y, z\}$  is a 7-circuit or a 9-circuit, a contradiction. Thus  $h' \neq z$ . By orthogonality,  $\{x, y\}$  is  $\{\beta, \gamma\}$  or  $\{\delta, \epsilon\}$ . In the former case,  $\{g', g, i, \beta, \gamma\} \Delta \{1, 2, y, z\}$  is a 7-circuit, a contradiction. Thus  $\{x, y\} = \{\delta, \epsilon\}$ . If  $1 \in \{\beta, \gamma\}$ , then, again,  $\{g', g, i, \beta, \gamma\} \Delta \{1, 2, y, z\}$  is a 7-circuit. Thus  $1 = \alpha$ . Then  $\{1, g', g, i, x, y\} \Delta \{1, 2, 3, 4, 5\}$  is a 7-circuit. We conclude that  $M|Z'' \neq M(L_3)$ . This completes the argument when  $M|Z = M(H_5)$ .

We may now assume that  $M|Z = M(H_3)$ . Then  $\{a, b\}$  does not contain exactly one element in  $\{1\}$ , in  $\{2, 3\}$ , or in  $\{4, 5\}$ , otherwise the symmetric difference of  $\{a, b, s, u\}$  with  $\{1, x, x', y, y'\}$ ,  $\{2, 3, x, x', z\}$ , or  $\{4, 5, y, y', z\}$  is a 7-circuit. By symmetry, we may assume that  $(1, 2, 3, 4, 5) = (e, a, b, c, d)$ . Let  $S = \{1, 2, 4, 5, t, u, v_1, v_2\} \Delta \{4, 5, y, y', z\}$ . Now  $y' \notin \{v_1, v_2\}$ , otherwise  $S$  is a 7-circuit. Thus, by Lemma 6.3,  $|S| = 9$ . Then  $S$  is a disjoint union of circuits,  $\{1, 2, t, u, v_1, v_2, y, y', z\}$ . Using  $M(J_5)$ , we see that  $\{1, 2, t, u, v_1, v_2, y\}$  is independent, so  $S$  is the union of two circuits,  $C_1$  and  $C_2$ , where  $y' \in C_1$  and  $z \in C_2$ . As both  $C_1$  and  $C_2$  meet  $M(H_3)$ , Lemma 6.11 implies that each has at least four elements. By orthogonality,  $y \in C_2$  and  $\{t, u\}$  is contained in  $C_1$  or  $C_2$ .

Suppose  $\{t, u\} \subseteq C_2$ . Then  $C_2 \supseteq \{t, u, y, z\}$ , otherwise  $C_2 \Delta \{2, s, t, v_1, v_2\}$  is a 7-circuit. Thus  $|C_2| = 5$ . Then  $2 \in C_2$ , otherwise  $C_2 \Delta \{2, 3, s, u\}$  is a 7-circuit. Thus  $C_2 \Delta \{1, 4, 5, x, x', z\}$  is a 9-circuit, a contradiction. We deduce that  $\{t, u\} \subseteq C_1$ . Then  $C_1 \neq \{y', t, u, \tau\}$  where  $\tau \in \{1, 2, v_1, v_2\}$ , otherwise  $C_1 \Delta \{4, 5, y, y', z\}$  is a 7-circuit. We deduce that  $|C_1| = 5$  and  $|C_2| = 4$ . As  $\{1, 2, y, z\}$  is not a circuit, by symmetry,  $C_2$  is  $\{y, z, 1, v_1\}$ ,

$\{y, z, 2, v_1\}$ , or  $\{y, z, v_1, v_2\}$ . In the first two cases,  $C_2 \Delta \{1, 2, 3, 4, 5\}$  is a 7-circuit. Thus  $C_2 = \{y, z, v_1, v_2\}$ . Observe that  $x' \notin \{v_1, v_2\}$ , otherwise  $\{1, 3, 4, 5, s, t, v_1, v_2\} \Delta \{2, 3, x, x', z\}$  is a 9-circuit. Then  $C_2 \Delta \{2, 3, x, x', z\} = \{2, 3, v_1, v_2, x, x', y\}$ . As the last set is not a circuit, it contains a 3-circuit. By Lemma 6.11, this 3-circuit avoids  $E(H_3)$  and so it is contained in  $\{v_1, v_2\}$ , a contradiction.  $\square$

**Lemma 6.14.** *Suppose that  $M$  has no non-spanning restriction isomorphic to a matroid in  $\{M(H_2), M(H_4), M(H_5), M(H_6), M(H_7)\}$  where the elements corresponding to  $\{x, y, z\}$  form a triad of  $M$ . Suppose  $M|Z \neq M(H_1)$ . If  $M|Z = M(H_3)$  and  $M|Z' = M(J_3)$ , then  $M$  has an odd circuit of size exceeding five.*

*Proof.* Assume that the lemma fails. Clearly  $r(M) \geq 8$ . Since no element of  $M(H_3)$  is in all of its 5-circuits,  $M$  is minimally 3-connected. As  $\{s, t, u\}$  avoids  $\{2, 3, x, x', z\}$ , by Lemma 4.2, there is a set  $Z'' \subseteq E(M)$  such that  $M|Z''$  is isomorphic to  $M(H_1)$  or  $M(H_3)$  with  $\{s, t, u\}$  and  $\{2, 3, x, x', z\}$  taking the places of  $\{x, y, z\}$  and  $\{1, 2, 3, 4, 5\}$ . If  $M$  has a triangle  $T$  containing two elements of  $\{s, t, u\}$  and one element of  $\{2, 3, x, x', z\}$ , then the symmetric difference of  $T$  with  $\{1, x, x', z, 4, 5\}$  or  $\{2, 3, z, y, y', 1\}$  is a 7-circuit. Thus  $M|Z'' = M(H)$  for the graph  $H$  in Figure 14 where  $\{g, h, i\} = \{s, t, u\}$  and  $\{\alpha, \beta, \gamma, \delta, \epsilon\} = \{2, 3, x, x', z\}$ . Suppose  $y \in Z''$ . Then  $y \in \{g', h'\}$  and we assume  $y = h'$  by symmetry. Orthogonality implies that  $\alpha \in \{x, z\}$ , and  $\{\alpha, \beta, \gamma, y, h, i\} \Delta \{1, 2, 3, 4, 5\}$  is a 7-circuit or a 9-circuit, a contradiction. Therefore  $y \notin Z''$ . By orthogonality and symmetry,  $\{\beta, \gamma\} = \{x, z\}$ . Thus  $M$  has  $\{2, 3, x', g, g', i\}$  as a circuit. Assume that  $g' \notin \{1, 4, 5\}$ . Then  $\{2, 3, x', g, g', i\} \Delta \{1, 2, 3, 4, 5\}$  is a 7-circuit as, by Lemma 6.11,  $M$  has no triangle meeting  $E(H_3)$ . Thus  $g' \in \{1, 4, 5\}$ . Then the symmetric difference of  $\{2, 3, x', g, g', i\}$  with  $\{1, x, x', y, y'\}$  is a 7- or 9-circuit, a contradiction.  $\square$

The next lemma summarizes the results of this section.

**Lemma 6.15.** *Let  $M$  be a 3-connected binary matroid having  $\{1, 2, 3, 4, 5\}$  as a circuit and  $\{x, y, z\}$  as a disjoint triad such that  $M$  is not isomorphic to  $M[I_6|A]$  for any matrix  $A$  in Figure 2. Assume the following.*

- (I) *For some subset  $Z$  of  $E(M)$  containing  $\{1, 2, 3, 4, 5, x, y, z\}$ , the matroid  $M|Z$  is  $M(H_i)$  for one of the graphs or grafts  $H_i$  in Figure 5;*
- (II) *for a triad  $\{s, t, u\}$  of  $M$  that is disjoint from  $Z$ , there is a subset  $Z'$  of  $E(M)$  that contains  $\{1, 2, 3, 4, 5, s, t, u\}$  such that  $M|Z'$  is  $M(J_i)$  for one of the graphs or grafts  $J_i$  in Figure 5 where  $\{a, b, c, d, e\} = \{1, 2, 3, 4, 5\}$ ; and*
- (III) *for all  $e$  in  $E(M)$ , either  $M \setminus e$  is not 3-connected or it has no 5-circuit.*

Then either

- (i)  *$M$  has an odd circuit of size exceeding five; or*
- (ii) (a)  *$\{x, y, z\}$  and  $Z'$  are disjoint; and*



$$\begin{array}{c}
\begin{array}{cccccccc}
& g & e_1 & f_1 & e_2 & f_2 & \dots & e_n & f_n \\
a & 1 & 1 & 1 & 1 & 1 & \dots & 1 & 1 \\
b & 1 & 1 & 0 & 1 & 0 & \dots & 1 & 0 \\
c & 0 & 0 & 1 & 0 & 1 & \dots & 0 & 1 \\
d_1 & 0 & 1 & 1 & 0 & 0 & \dots & 0 & 0 \\
d_2 & 0 & 0 & 0 & 1 & 1 & \dots & 0 & 0 \\
d_3 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
d_{n-1} & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
d_n & 0 & 0 & 0 & 0 & 0 & \dots & 1 & 1
\end{array} \\
\left. \vphantom{\begin{array}{c} a \\ b \\ c \\ d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_{n-1} \\ d_n \end{array}} \right\} .
\end{array}$$

FIGURE 16.  $A$ .

$\{a, b, c, h_0\}, \{d_1, e_1, f_1, h_1\}, \dots, \{d_n, e_n, f_n, h_n\}$  are circuits. If  $E(M) \subseteq E(G) \cup \{g', g'', h_0, h_1, \dots, h_n\}$ , then  $M$  is a book with  $n$  pages.

Note that every triangle in  $G$  contains  $g$ . Now  $M(G)$  has  $\{a, b, c, d_1, d_2, \dots, d_n\}$  as a basis,  $B$ , and is represented over  $GF(2)$  by  $[I_{n+3}|A]$  where  $A$  is shown in Figure 16. To obtain a binary representation of  $M$ , we add columns to  $A$ . Since  $M$  is simple, we can identify the elements of  $M$  with the columns of the matrix that they label. Let  $h$  be a column that is added to  $A$  where  $h \notin E(G) \cup \{g', g'', h_0, h_1, \dots, h_n\}$ . Let  $k$  be the number of ones in column  $h$ .

Suppose  $k$  is even. Then the fundamental circuit  $C(h, B)$  contains  $k + 1$  elements. Since  $k + 1$  is odd,  $k \leq 4$ . Hence  $k \in \{2, 4\}$ .

#### 7.1.1. $k \neq 2$ .

Suppose that  $k = 2$ . Then at least one of the ones in  $h$  is not in the first three rows, otherwise  $h \in \{g, g', g''\}$ . If exactly one of the ones in  $h$  is in the first three rows, then, without loss of generality, the second of the ones is in the fourth row, and  $\{a, d_1, h\}, \{b, d_1, h\}$ , or  $\{c, d_1, h\}$  is a triangle. The symmetric difference of this triangle with one of the circuits  $\{a, c, d_2, e_2, e_3, f_3\}$  or  $\{a, b, d_3, e_2, f_2, f_3\}$  is a 7-circuit, a contradiction. We deduce that neither of the ones in  $h$  is contained in the first three rows. Then, without loss of generality,  $\{d_1, d_2, h\}$  is a triangle. The symmetric difference of this triangle with the circuit  $\{a, c, d_2, e_2, e_3, f_3\}$  is the 7-circuit  $\{a, c, d_1, e_2, e_3, f_3, h\}$ . This contradiction implies that 7.1.1 holds.

Next we show that

#### 7.1.2. $k \neq 4$ .

Suppose  $k = 4$ . If the first three rows of  $h$  contain ones, then, without loss of generality,  $C(h, B) = \{a, b, c, d_1, h\}$  and its symmetric difference with the circuit  $\{b, c, d_2, d_3, e_2, f_3\}$  is the 7-circuit  $\{a, d_1, d_2, d_3, e_2, f_3, h\}$ , a contradiction. If the first three rows of  $h$  contain exactly two ones, then, we may assume that  $C(h, B)$  is  $\{a, b, d_1, d_2, h\}$ ,  $\{a, c, d_1, d_2, h\}$ , or

$\{b, c, d_1, d_2, h\}$  and the symmetric difference of this circuit with  $\{a, c, d_3, f_3\}$ ,  $\{a, b, d_3, e_3\}$ , or  $\{d_2, d_3, f_2, f_3\}$ , respectively, is a 7-circuit, a contradiction. If the first three rows of  $h$  contain a single one, then, by symmetry,  $C(h, B)$  is  $\{a, d_1, d_2, d_3, h\}$ ,  $\{b, d_1, d_2, d_3, h\}$ , or  $\{c, d_1, d_2, d_3, h\}$ , and its symmetric difference with  $\{a, b, d_2, e_1, f_1, f_2\}$ ,  $\{b, c, e_3, f_3\}$ , or  $\{b, c, e_1, f_1\}$ , respectively, is a 7-circuit, a contradiction. Finally, if the first three rows of  $h$  contain no ones, then  $\{d_1, d_2, d_3, d_4, h\}$  is a circuit whose symmetric difference with the circuit  $\{a, c, d_1, f_1\}$  is a 7-circuit, a contradiction. Thus 7.1.2 holds.

We now know that  $k$  is odd. As  $M$  is simple,  $k \geq 3$ . We show next that

**7.1.3.** *the support of  $h$  is disjoint from the support of  $g$ .*

Suppose not. Then the support of  $h$  meets the support of  $g$  in one or two elements. The symmetric difference of  $C(h, B)$  and  $C(g, B)$  is a circuit containing  $\{g, h\}$  and either  $k$  or  $k - 2$  elements of  $B$ . In the first case,  $k = 3$ . In the second case,  $k \in \{3, 5\}$ . Without loss of generality, we may assume that  $C(h, B)$  is  $\{a, c, d_1, h\}$ ,  $\{a, d_1, d_2, h\}$ ,  $\{b, c, d_1, h\}$ ,  $\{b, d_1, d_2, h\}$ ,  $\{a, b, c, h\}$ ,  $\{a, b, d_1, h\}$ ,  $\{a, b, c, d_1, d_2, h\}$ , or  $\{a, b, d_1, d_2, d_3, h\}$ . As  $C(f_1, B) = \{a, c, d_1, f_1\}$  and  $C(e_1, B) = \{a, b, d_1, e_1\}$  but  $h$  differs from  $f_1$  and  $e_1$ , we deduce that  $C(h, B)$  is not  $\{a, c, d_1, h\}$  or  $\{a, b, d_1, h\}$ . Moreover, as  $h \neq h_0$ , we see that  $C(h, B)$  is not  $\{a, b, c, h\}$ . If  $C(h, B) = \{b, c, d_1, h\}$ , then  $\{b, c, d_1, h\} \triangle \{b, c, e_1, f_1\}$  is the circuit  $\{d_1, e_1, f_1, h\}$ , so  $h = h_1$ , a contradiction. This leaves the following four possibilities for  $C(h, B)$ :  $\{a, d_1, d_2, h\}$ ,  $\{b, d_1, d_2, h\}$ ,  $\{a, b, c, d_1, d_2, h\}$ , and  $\{a, b, d_1, d_2, d_3, h\}$ . In these cases, we see that  $M$  has the following 7-circuits:  $\{h, b, g, e_2, e_3, f_1, f_3\}$ ,  $\{h, a, g, e_2, e_3, f_1, f_3\}$ ,  $\{h, a, d_1, d_2, d_3, g, f_3\}$ , and  $\{h, a, c, d_2, d_3, f_1, g\}$ , respectively. This contradiction completes the proof of 7.1.3.

As  $k \geq 3$  and  $h$  has no ones in its first two rows by 7.1.3, without loss of generality,  $h$  has a one in its fourth row, and the symmetric difference of the supports of  $g, h$ , and  $e_1$  contains  $k - 1$  ones, none of which is in the first two rows. Then  $\{e_1, g, h\}$  is contained in a circuit that is contained in  $\{e_1, g, h\} \cup (B - \{a, b\})$  and that has  $k + 2$  elements. Thus  $k = 3$ . By symmetry,  $C(h, B)$  is  $\{c, d_1, d_2, h\}$  or  $\{d_1, d_2, d_3, h\}$ . Then  $\{h, a, d_1, d_3, g, e_2, f_3\}$  or  $\{h, b, c, g, e_1, e_2, f_3\}$ , respectively, is a 7-circuit, a contradiction.  $\square$

We now prove our main result, which we restate here for convenience.

**Theorem 7.2.** *A 3-connected binary matroid  $M$  has no odd circuits of size exceeding five if and only if*

- (i)  *$M$  is affine; or*
- (ii)  *$r(M) \leq 5$ ; or*
- (iii)  *$M$  is obtained from an  $n$ -page book for some  $n \geq 4$  by deleting up to two elements of its spine; or*
- (iv)  *$r(M) = 6$  and  $M$  is isomorphic to  $M[I_6|A]$  where  $A$  is one of the matrices  $A_{21}, A_{22}, A_{23}, A_{24}, A_{25}, A_{72}, A_{76}, A_{77}$ , or  $A_{78}$  in Figure 2.*

*Proof.* If  $M$  satisfies (i), (ii), or (iv), then, by Lemma 5.1,  $M$  has no odd circuits of size exceeding five. Now let  $N$  be the book  $B_{n,T}$  in which every

page,  $M_i$ , is isomorphic to  $F_7$ . Let  $C$  be a circuit of  $N$  having more than four elements. Assume that  $C \cap (E(M_i) - T)$  is non-empty exactly when  $i \in \{1, 2, \dots, s\}$  relabelling  $C \cap (E(M_i) - T)$  as  $C_i$  for such  $i$ . As  $|C| > 4$ , we see that  $s \geq 2$ . Clearly, for each  $i$ , the set  $C_i$  spans at least one element of  $T$  and  $|C_i| \in \{2, 3\}$ . If  $C_i$  and  $C_j$  span a common element of  $T$ , then  $C_i \cup C_j$  is dependent and so equals  $C$ . In that case,  $|C| \leq 5$ . It follows that  $s \leq 3$  and, if  $s = 3$ , then  $|C| = 6$ . If  $s = 2$ , then  $C_1$  and  $C_2$  span distinct elements of  $T$  and  $|C \cap T| = 1$ , so  $|C| = 5$ . As every book with  $n$  pages is a restriction of  $N$ , we see that if (iii) holds, then  $M$  has no circuit of size exceeding six.

Conversely, suppose  $M$  has no odd circuits of size exceeding five and that  $r(M) \geq 6$  but that none of (i), (ii), or (iv) holds. As  $M$  is not affine, it has an odd circuit. By Theorem 1.2,  $M$  has a 5-circuit. We may assume that  $M$  does not have a spanning restriction isomorphic to the cycle matroid of a graph or graft in Figure 5, otherwise Lemma 5.1 implies that (iv) holds.

Suppose that, for all  $e \in E(M)$ , the matroid  $M \setminus e$  is either not 3-connected or contains no 5-circuits. By Lemma 6.1,  $E(M)$  has a subset  $Z$  such that  $M|Z$  is  $M(H_i)$  for one of the graphs or grafts  $H_i$  in Figure 5. Furthermore, as  $r(M) > r(Z)$ , the matroid  $M$  has a triad  $\{s, t, u\} \subseteq E(M) - Z$  such that  $\{s, t, u\} \cup \{1, 2, 3, 4, 5\}$  is contained in a set  $Z'$  such that  $M|Z'$  is  $M(J_j)$  for one of the graphs or grafts  $J_j$  in Figure 12, where  $\{a, b, c, d, e\} = \{1, 2, 3, 4, 5\}$ .

By Lemma 6.15,  $M|Z = M(H_1)$  and  $M|Z' = M(J_1)$ . Also  $a = 1$  and  $\{\{b, c\}, \{d, e\}\} = \{\{2, 3\}, \{4, 5\}\}$ . By symmetry, we may assume that  $(b, c, d, e) = (2, 3, 4, 5)$ . Next we show that

**7.2.1.**  *$M$  has triads  $\{2, 3, f\}$  and  $\{4, 5, g\}$  where  $\{f, g\}$  avoids  $Z \cup \{s, t, u\}$ .*

As the circuit  $\{2, 3, x, z\}$  avoids the circuit  $\{1, 4, 5, t, u\}$ , Theorem 2.2 implies that  $\{2, 3, x, z\}$  meets at least two triads of  $M$ . By orthogonality with the circuits in  $M|Z'$ , neither 2 nor 3 is in a triad with  $x$  or  $z$ . Hence  $\{2, 3\}$  is contained in a triad of  $M$ . Again, by orthogonality with the circuits of  $M|Z$  and  $M|Z'$ , the set  $\{2, 3, f\}$  is a triad of  $M$  for some element  $f \notin Z \cup \{s, t, u\}$ ; 7.2.1 follows by symmetry.

As the triad  $\{2, 3, f\}$  avoids the circuit  $\{1, 4, 5, x, z\}$ , by Lemma 6.1,  $E(M)$  has a subset  $Z''$  that contains  $\{2, 3, f, 1, 4, 5, x, z\}$  such that  $M|Z''$  is isomorphic to the cycle matroid of one of the graphs or grafts in Figure 5 where  $\{2, 3, f\}$  and  $\{1, 4, 5, x, z\}$  take the places of  $\{x, y, z\}$  and  $\{1, 2, 3, 4, 5\}$ . Let  $L_i$  be the resulting relabelled version of  $H_i$ . By Lemmas 6.4 and 6.15,  $M|Z'' = L_1$ . Note that  $\{2, 3, x, z\}$  is a circuit of  $M$ . Since  $M$  has a restriction isomorphic to  $M(H_1)$  where  $1, x, z, 4, 5, t, s$ , and  $u$  take the places of  $1, 2, 3, 4, 5, x, y$ , and  $z$ , respectively, Lemma 6.6 implies that  $M$  has as circuits either  $\{\{1, 2, f\}, \{3, 4, 5, f\}\}$  or  $\{\{1, 3, f\}, \{2, 4, 5, f\}\}$ . By the symmetry between 2 and 3, we may assume  $\{1, 2, f\}$  and  $\{3, 4, 5, f\}$  are circuits.

Replacing  $\{2, 3, f\}$  by  $\{4, 5, g\}$  in the argument in the last paragraph, we deduce, by symmetry, that  $M$  has  $\{1, 5, g\}$  as a circuit. Moreover, as  $M$  is simple,  $f$  and  $g$  are distinct. Since  $M$  is binary, it follows that  $M$  contains as a restriction the cycle matroid of the graph  $K$  in Figure 17, where solid

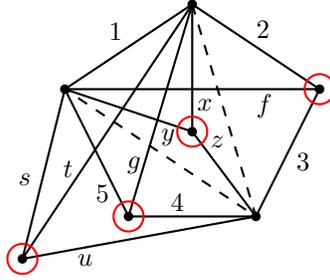


FIGURE 17. The cycle matroid of this graph,  $K$ , is a 4-page book.

lines represent edges that are elements of  $M$  and circled vertices indicate known triads of  $M$ . Adding the dashed edges gives a book with four pages, where the spine of the book is the edge 1 together with the two dashed edges. Hence  $M(K)$  can be obtained from a book with four pages by removing two elements from its spine. We also see that  $M(K)$  is 3-connected.

Let  $N_1, N_2, N_3$ , and  $N_4$  be the pages of  $M(K)$  where  $E(N_1) - T, E(N_2) - T, E(N_3) - T$ , and  $E(N_4) - T$  are the triads  $\{x, y, z\}, \{s, t, u\}, \{2, 3, f\}$ , and  $\{4, 5, g\}$ . Take  $k$  to be maximal for which  $N$  is a  $k$ -page book that is a restriction of  $M$  such that  $T$  is the spine of  $N$ , its pages include  $N_1, N_2, N_3$ , and  $N_4$ , and  $E(N_i) - T$  is a triad of  $M$  for all pages  $N_i$  of  $N$ .

Let  $E(N) = Z_0$ . We show next that

**7.2.2.**  $M$  has no triad avoiding  $Z_0$ .

Suppose that  $\{s', t', u'\}$  is a triad of  $M$  avoiding  $Z_0$ . Since  $\{s', t', u'\}$  avoids the circuit  $\{1, 2, 3, 4, 5\}$ , we can use  $\{s', t', u'\}$  in place of  $\{s, t, u\}$  to get, using Lemma 6.15, that  $E(M)$  has a subset  $Z_1$  such that  $M|Z_1$  is isomorphic to  $M|Z'$  and has  $\{1, s', t'\}, \{2, 3, t', u'\}$ , and  $\{4, 5, u', s'\}$  as circuits. It follows that we can adjoin a new page  $N_{k+1}$  to  $N$  with  $E(N_{k+1}) - T = \{s', t', u'\}$ . This contradiction to the maximality of  $k$  implies that 7.2.2 holds.

Next we note that

**7.2.3.**  $N$  is 3-connected.

We know that  $M(K)$  is 3-connected. If we replace the page  $N_4$  in  $M(K)$  by  $N_5$ , we get another 3-connected matroid. As these two 3-connected matroids have at least ten common elements, the restriction of  $N$  to its first five pages is 3-connected. Extending this argument establishes 7.2.3.

We now show the following.

**7.2.4.** If  $e \in \text{cl}(Z_0) - Z_0$ , then  $e \in T$ .

The set  $\{x, y, z\} \cup B' \cup e$  contains a circuit  $C$  of  $M$  where  $B'$  contains a unique element of  $E(N_i) - T$  for all  $i \geq 2$ . By orthogonality,  $C$  avoids  $B'$ , and  $C$  contains exactly two elements of  $\{x, y, z\}$ . Thus  $C$  is  $\{x, z, e\}$  or  $\{y, z, e\}$ , so  $e$  is on the spine  $T$  of  $N$ . Thus 7.2.4 holds.

**7.2.5.**  $Z_0$  spans  $M$ .

Assume that  $E(M) \neq \text{cl}(Z_0)$ . By 7.2.2 and Lemma 2.4,  $E(M) - \text{cl}(Z_0)$  is independent. Take  $e$  in this set. Let  $(X, Y)$  be a 2-separation of  $M \setminus e$ . As  $M \setminus \text{cl}(Z_0)$  is 3-connected, we may assume that  $\text{cl}(Z_0) \subseteq X$ . Then  $Y \subseteq E(M) - \text{cl}(Z_0)$ , so  $Y$  is independent in  $M \setminus e$ . Now  $M \setminus e$  is the 2-sum with basepoint  $p$  of connected matroids  $M_X$  and  $M_Y$  having ground sets  $X \cup p$  and  $Y \cup p$ . Thus  $M_Y$  is a circuit, so every 2-element subset of  $Y$  is a cocircuit of  $M \setminus e$ , and  $e$  is in a triad of  $M$ , a contradiction to 7.2.3. Thus 7.2.5 holds.

The minimality assumption on  $M$  implies that  $N = M$ . Dropping this minimality assumption allows the addition of extra elements to  $N$ . But, by Lemma 7.1, such elements still produce a book, so the theorem is proved.  $\square$

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