

A Characterization of the Ternary Matroids with No $M(K_4)$ -Minor

JAMES G. OXLEY*

*Mathematics Department, Louisiana State University,
Baton Rouge, Louisiana 70803*

Communicated by P. D. Seymour

Received May 30, 1985

Brylawski identified the class of binary matroids with no minor isomorphic to $M(K_4)$ as being the class of series-parallel networks. From this he deduced that, for all such matroids M , the critical exponent $c(M; 2)$ is at most 2. He also conjectured that a similar result is true over all finite fields $GF(q)$. This paper examines the classes of ternary and $GF(4)$ -representable matroids with no $M(K_4)$ -minor. The main result characterizes the former class by showing that, with one exception, the only non-trivial 3-connected members of this class are whirls or minors of the Steiner system $S(5, 6, 12)$. This characterization is then used to show that, for all ternary matroids M with no $M(K_4)$ -minor, $c(M; 3) \leq 2$, thereby verifying Brylawski's conjecture in the case that $q = 3$. The characterization is also used to give excluded-minor descriptions for the class of ternary gammoids and two other related classes. The first of these results answers a question of Ingleton and verifies another conjecture of Brylawski. © 1987 Academic Press, Inc.

1. INTRODUCTION

In this paper we shall study the classes of ternary and $GF(4)$ -representable matroids with no minor isomorphic to $M(K_4)$, the cycle matroid of the complete graph on 4 vertices. The class of binary matroids with no $M(K_4)$ -minor was identified by Brylawski [5] as being the class of series-parallel networks. Thus the only 3-connected binary matroids with no $M(K_4)$ -minor have three or fewer elements. The main result of this paper is that if M is a 3-connected ternary matroid with no $M(K_4)$ -minor and $|E(M)| \geq 4$, then M is a whirl, M is a certain self-dual rank-4 matroid J , or M is one of 15 non-isomorphic minors of the Steiner system $S(5, 6, 12)$. A consequence of this result is that J and $S(5, 6, 12)$ are the only 3-connected splitters [35] for the class of ternary matroids with no $M(K_4)$ -minor.

* This research was partially supported by National Science Foundation Grant DMS-8500494.

From our characterization of the 3-connected ternary matroids with no $M(K_4)$ -minor, we deduce that every loopless ternary matroid with no $M(K_4)$ -minor has critical exponent at most two. This verifies a special case of a conjecture of Brylawski [6].

In [19], Ingleton asked for an excluded-minor description of the class of ternary gammoids, while in [6], Brylawski conjectured that a gammoid is ternary if and only if it has no minor isomorphic to $U_{2,5}$ or $U_{3,5}$. Using our main result we are able to deduce excluded-minor descriptions for the classes of ternary gammoids, ternary base-orderable matroids and ternary strongly base-orderable matroids. From the first of these, the truth of Brylawski's second conjecture follows.

The matroid terminology used here will in general follow Welsh [38]. The ground set and rank of the matroid M will be denoted by $E(M)$ and $\text{rk } M$ respectively. If $T \subseteq E(M)$, then $\text{rk } T$ and \bar{T} will denote the rank and closure of T , respectively. The deletion of T from M will be denoted by $M \setminus T$ or $M|(E(M) - T)$, and the contraction of T from M by M/T or $M \cdot (E(M) - T)$. Flats of M of ranks one and two will be called *points* and *lines*. If X is an n -element circuit of M , then we shall call X an n -circuit. The $(r+1)$ -vertex wheel, the whirl of rank r [38, pp. 80–81] and the uniform matroid of rank k on an n -element set will be denoted by \mathcal{W}_r , \mathcal{W}^r , and $U_{k,n}$, respectively. Whirls are often defined to have rank at least 3. However, the usual construction of a whirl from a wheel remains valid for $r=2$ and the resulting matroid \mathcal{W}^2 is isomorphic to $U_{2,4}$.

A matroid M is *3-connected* if it is connected and $E(M)$ cannot be partitioned into subsets X and Y each having at least two elements such that $\text{rk } X + \text{rk } Y - \text{rk } M = 1$. It is routine to verify that M is 3-connected if and only if its dual M^* is 3-connected. We call M *minimally 3-connected* if M is 3-connected and, for all elements e of M , $M \setminus e$ is not 3-connected.

If M_1 and M_2 are matroids on the sets S and $S \cup e$ where $e \notin S$, then M_2 is an *extension* of M_1 if $M_2 \setminus e = M_1$, and M_2 is a *lift* of M_1 if M_2^* is an extension of M_1^* . We call M_2 a *non-trivial extension* of M_1 if e is neither a loop nor a coloop of M_2 and e is not in a 2-circuit of M_2 . Likewise, M_2 is a *non-trivial lift* of M_1 if M_2^* is a non-trivial extension of M_1^* . The following result is well known (see, for example, [28, Lemma 2.1]).

(1.1) LEMMA. *Let N be a 3-connected matroid having at least three elements and M be an extension of N . Then M is 3-connected if and only if M is a non-trivial extension of N .*

We shall assume familiarity with the operations of series and parallel connection of matroids; a detailed discussion of these operations and their properties can be found in [5]. For matroids M_1 and M_2 such that $E(M_1) \cap E(M_2) = \{p\}$, we shall denote the series and parallel connections

of M_1 and M_2 with respect to the basepoint p by $S((M_1, p), (M_2, p))$ and $P((M_1, p), (M_2, p))$, respectively. We note here that $S((M_1, p), (M_2, p))/p = P((M_1, p), (M_2, p)) \setminus p$ [5, Corollary 5.3]. The following basic link between 3-connection and series and parallel connections was proved by Seymour [35, (2.6)].

(1.2) THEOREM. *A connected matroid M is not 3-connected if and only if there are matroids M_1 and M_2 each of which has at least three elements and is isomorphic to a minor of M such that $M = P((M_1, p), (M_2, p)) \setminus p = S((M_1, p), (M_2, p))/p$, where p is not a loop or a coloop of M_1 or M_2 .*

When M decomposes as in this theorem, it is called the 2-sum of M_1 and M_2 .

If $\{x, y\}$ is a circuit of the matroid M , we say that x and y are *in parallel* in M . If, instead, $\{x, y\}$ is a cocircuit of M , then x and y are *in series* in M . The matroid M' is a *series extension* of M if $M = M'/T$ and every element of T is in series with some element of M' not in T . *Parallel extensions* are defined analogously. We call M'' a *series-parallel extension* of M if M'' is obtained from M by a sequence of operations each of which is either a series or parallel extension. A matroid in which each connected component is a series-parallel extension of a single-element matroid is called a *series-parallel network*. A detailed investigation of the properties of such matroids can be found in [5].

Given a matrix A with entries from a field F , the *dependence matroid* $D(A)$ of A is the matroid whose ground set is the set S of columns of A and whose independent sets are the subsets of S which are linearly independent over F . If $M \cong D(A)$, we say that A *represents* M or is a *representation* for M . Now suppose we adjoin a new column e to A . Then $A + e$ will denote the resulting matrix and, if $M = D(A)$, we shall sometimes write $M + e$ for the matroid $D(A + e)$. If $M + e$ is a non-trivial extension of M , we shall say that e has been *added non-trivially* to A . A matroid is *ternary* if it is isomorphic to the linear dependence matroid of a matrix over $GF(3)$. The next result characterizes the class of ternary matroids in terms of excluded minors. The Fano matroid will be denoted by F_7 .

(1.3) THEOREM [1, 34]. *A matroid is ternary if and only if it has no minor isomorphic to any of $U_{2,5}$, $U_{3,5}$, F_7 , or F_7^* .*

In Section 2 of this paper we state and prove the main theorem of the paper, a characterization of the ternary 3-connected matroids with no $M(K_4)$ -minor. The proof of this theorem will rely heavily on the next two results. The first of these is an easy consequence of Seymour's splitter theorem [35, (7.3)] (see, for example, [30, Theorem 2.2]).

(1.4) THEOREM. *Let M and N be 3-connected matroids such that N is a minor of M , $|E(N)| \geq 4$, and if $N \cong M(\mathcal{W}_k)$, M has no $M(\mathcal{W}_{k+1})$ -minor, while if $N \cong \mathcal{W}^k$, M has no \mathcal{W}^{k+1} -minor. Then there is a sequence $M_0, M_1, M_2, \dots, M_n$ of 3-connected matroids such that $M_0 \cong N$, $M_n = M$ and, for all i in $\{1, 2, \dots, n\}$, M_i is an extension or lift of M_{i-1} .*

The second result that we shall make extensive use of is the theorem of Brylawski and Lucas that ternary matroids are uniquely representable. To state this precisely, we shall require some further definitions. Given a matrix A over a field F , a *projective operation* on A consists of either an elementary row operation, that is, adding a scalar multiple of one row to another, interchanging two rows, or multiplying a row by a nonzero scalar; multiplication of a column by a nonzero scalar; or replacement of each entry of A by its image under some automorphism of F . The matrices A and A' are *projectively equivalent* if A' can be obtained from A by a sequence of projective operations.

(1.5) THEOREM [8, Corollary 3.3]. *Let A and A' be $r \times n$ matrices over $GF(3)$ such that the map which, for all i in $\{1, 2, \dots, n\}$, takes the i th column of A to the i th column of A' is an isomorphism from $D(A)$ to $D(A')$. Then A and A' are projectively equivalent.*

The well-known Steiner system $S(5, 6, 12)$ plays a fundamental role in our main theorem and we shall make extensive use of the properties of this system in the proof of the theorem. Recall that a *Steiner system* $S(t, k, v)$ is a pair (S, \mathcal{D}) , where S is a v -element set and \mathcal{D} is a collection of k -element subsets of S called *blocks* such that every t -element subset of S is contained in exactly one block. The matroid associated with the Steiner system (S, \mathcal{D}) has S as its ground set and \mathcal{D} as its set of hyperplanes. Its rank is $t + 1$ and every subset of S with fewer than t elements is an independent flat (see [38, Chap. 12]). If $x \in S$, then the contraction of x from the matroid associated with (S, \mathcal{D}) has $S - x$ as its ground set; its set \mathcal{D}' of hyperplanes consists of all sets of the form $H - x$, where H is a member of \mathcal{D} containing x . Evidently the pair $(S - x, \mathcal{D}')$ is an $S(t - 1, k - 1, v - 1)$. This construction may be repeated to give further Steiner systems; the systems obtained in this way are said to be *derived* from the original system.

In general, a Steiner system is not uniquely determined by its parameters, t , k , and v . However, the Steiner system $S(5, 6, 12)$ and its derived systems, $S(4, 5, 11)$, $S(3, 4, 10)$, and $S(2, 3, 9)$, are unique [40]. We observe here that $S(2, 3, 9)$ is the ternary affine plane, $AG(2, 3)$, while $S(3, 4, 10)$ is the ternary inversive plane [14]. There are numerous constructions known for $S(5, 6, 12)$, many of which have been described by Cameron [9] (see also [10]). From the point of view of this paper, the

most convenient description of $S(5, 6, 12)$ is as follows [11]. Let $S = \{s_1, s_2, \dots, s_{12}\}$ be the set of columns of the following matrix over $GF(3)$:

$$X = \begin{bmatrix} & & & & & & & & & & & \\ & & & & & & & & & & & \\ & & & & & & & & & & & \\ & & & & & & & & & & & \\ & & & & & & & & & & & \\ & & & & & & & & & & & \\ & & & & & & & & & & & \\ & & & & & & & & & & & \\ & & & & & & & & & & & \\ & & & & & & & & & & & \\ & & & & & & & & & & & \\ & & & & & & & & & & & \end{bmatrix} \begin{array}{c} s_1 \quad s_2 \quad s_3 \quad s_4 \quad s_5 \quad s_6 \quad s_7 \quad s_8 \quad s_9 \quad s_{10} \quad s_{11} \quad s_{12} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array}$$

FIGURE 1

The set of hyperplanes of $D(X)$ is the set of blocks of an $S(5, 6, 12)$ on S . Since a matroid is uniquely determined by its set of hyperplanes, the automorphism group of $S(5, 6, 12)$ as a matroid is the same as its automorphism group as a Steiner system. The latter is well known to be the Mathieu group M_{12} [39], which is 5-transitive [25].

Throughout this paper when we refer to a Steiner system, we shall in general mean the matroid associated with the system. As the matrix X has the form $[I|A]$ where A is symmetric, $S(5, 6, 12)$ is self-dual. In fact, $S(5, 6, 12)$ is identically self-dual, that is, its sets of circuits and cocircuits coincide. The last observation follows from the fact that the complement of every block of $S(5, 6, 12)$ is also a block [11]. Thus the set of cocircuits of $S(5, 6, 12)$ equals its set of blocks. Since every block is also a circuit and $S(5, 6, 12)$ is self-dual, the set of circuits also equals the set of blocks and we conclude that the sets of circuits, cocircuits, and blocks coincide.

In the third section of this paper, we shall use our main theorem to solve the *critical problem* [12, Chap. 16] for ternary matroids with no $M(K_4)$ -minor. Let M be a rank- r loopless matroid that is isomorphic to the dependence matroid $D(A)$ of an $r \times n$ matrix A over $GF(q)$. Then the set S of distinct columns of A is a subset of $V(r, q)$ and the *critical exponent* $c(M; q)$ of M is the least number k of hyperplanes H_1, H_2, \dots, H_k of $V(r, q)$ such that $(\bigcap_{i=1}^k H_i) \cap S = \emptyset$. The fact that $c(M; q)$ does not depend on the particular matrix A was proved by Crapo and Rota. To state their result, we require a further definition. The *chromatic polynomial* $P(N; \lambda)$ of an arbitrary matroid N is the polynomial $\sum_{X \subseteq E(N)} (-1)^{|X|} \lambda^{\text{rk } N - \text{rk } X}$. Crapo and Rota [12, p. 16.4] proved that, for a loopless matroid M representable over $GF(q)$, $P(M; q^j) \geq 0$ for all positive integers j , and moreover,

$$\begin{aligned} c(M; q) &= \min\{j \in \mathbb{Z}^+ : P(M; q^j) > 0\} \\ &= \min\{j \in \mathbb{Z}^+ : P(M; q^{j+i}) > 0 \text{ for all } i \text{ in } \mathbb{Z}^+ \cup \{0\}\}. \end{aligned}$$

As a simple rank- r matroid has critical exponent 1 if and only if it is

isomorphic to a restriction of $AG(r-1, q)$, a loopless matroid is called *affine* if its critical exponent is one. If $N \cong M(G)$ for a loopless connected graph G , then the chromatic polynomial $P(G; \lambda)$ of the graph G equals $\lambda P(N; \lambda)$. This close link with graphs reflects the fact that a major part of the initial motivation for the critical problem came from colouring and flow problems in graphs (see [23] for details).

Section 4 of this paper is concerned with certain classes of matroids that are related to the class of transversal matroids. It is well known that this last class is not closed under contraction (see, for example, [38, p. 105]). A *gammoid* is a matroid that is isomorphic to a transversal matroid or a contraction of a transversal matroid. The class of gammoids is closed under both minors and duality [20]. Two other related classes introduced by Brualdi [3] and studied by several authors [2, 13, 15, 18, 19, 24] are the classes of base-orderable and strongly base-orderable matroids. A matroid M is *base-orderable* if, given any two bases B_1 and B_2 , there is a bijection $\psi: B_1 \rightarrow B_2$ such that, for every element x of B_1 , both $(B_1 - x) \cup \psi(x)$ and $(B_2 - \psi(x)) \cup x$ are bases of M . If, given any two bases B_1 and B_2 , there is a bijection $\psi: B_1 \rightarrow B_2$ such that, for every subset X of B_1 , both $(B_1 - X) \cup \psi(X)$ and $(B_2 - \psi(X)) \cup X$ are bases, then M is called *strongly base-orderable*. The classes of base-orderable and strongly base-orderable matroids are both closed under minors and duality. Moreover, in general, the class of base-orderable matroids properly contains the class of strongly base-orderable matroids which, in turn, properly contains the class of gammoids. However, within the class of binary matroids, these three classes coincide. In Section 4, we use our main theorem to determine the relationship between these classes within the class of ternary matroids. A characterization of the class of ternary transversal matroids, which also follows from the main theorem, will appear elsewhere [32].

In Section 5, we prove another consequence of the main theorem, a best-possible upper bound on the number of elements in a rank- r simple ternary matroid with no $M(K_4)$ -minor. Just like the corresponding bound for binary matroids, this bound is linear in r . The paper concludes with an addendum in which the class of $GF(4)$ -representable matroids having no $M(K_4)$ -minor is considered.

2. THE TERNARY MATROIDS WITH NO $M(K_4)$ -MINOR

In this section we state and prove the main result of this paper, a characterization of all ternary 3-connected matroids having no $M(K_4)$ -minor. We shall denote by Ω the class of all such matroids. In addition, we shall let J be the simple rank-4 matroid for which a Euclidean representation is shown in Fig. 2. The points of J are the 8 solid dots in this figure.

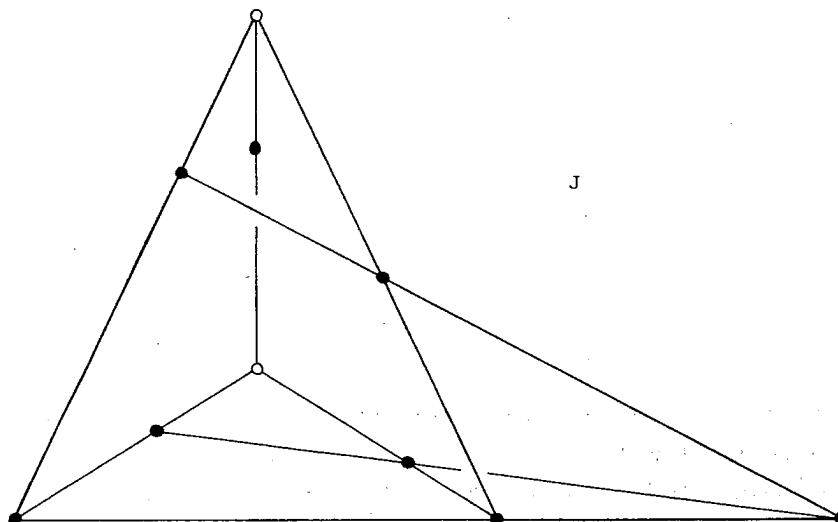


FIGURE 2

(2.1) THEOREM. *A matroid M is 3-connected, ternary and has no $M(K_4)$ -minor if and only if*

- (i) $M \cong \mathcal{W}^r$ for some $r \geq 2$;
- (ii) $M \cong J$; or
- (iii) M is isomorphic to a 3-connected minor of $S(5, 6, 12)$.

By Theorem 1.3, this theorem is precisely a characterization of the class of 3-connected matroids having no minor isomorphic to $U_{2,5}$, $U_{3,5}$, or $M(K_4)$. The proof of Theorem 2.1 is long since it involves building up to J and $S(5, 6, 12)$ from \mathcal{W}^3 by extensions and lifts. A complete list of the members of Ω with at least 4 elements appears in Table II near the end of this section.

Proof of Theorem 2.1. Assume that M is 3-connected, ternary, and has no $M(K_4)$ -minor. We suppose first that M is binary. Then, as M has no $M(K_4)$ -minor, it is a series-parallel network [5, Theorem 7.6]. But M is 3-connected and so M is isomorphic to one of $U_{0,1}$, $U_{1,1}$, $U_{1,2}$, $U_{1,3}$, or $U_{2,3}$. It is easy to check that each of these matroids is a minor of $S(5, 6, 12)$, so the theorem holds if M is binary.

We now assume that M is non-binary. Then both the rank and corank of M are at least two. If equality holds in either case, then $M \cong U_{2,4} \cong \mathcal{W}^2$. We may therefore suppose that both the rank and corank of M exceed two. Then, by [30, Theorem 3.1], M has a minor isomorphic to one of \mathcal{W}^3 , $U_{3,6}$, P_6 , or Q_6 , where Euclidean representations for the last two matroids are shown in Fig. 3. As each of $U_{3,6}$, P_6 , and Q_6 has $U_{2,5}$ as a minor, each is non-ternary. Thus M has \mathcal{W}^3 as a minor.

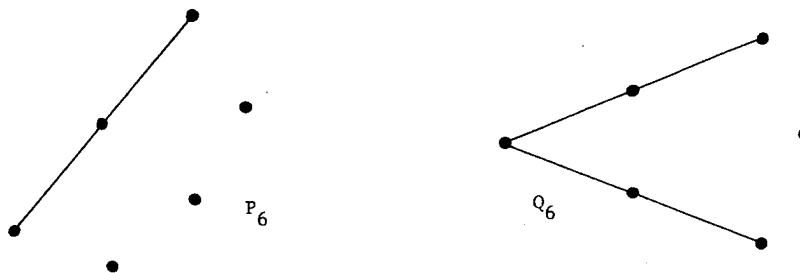


FIGURE 3

The next four lemmas combine to show that if M has \mathcal{W}^4 as a minor, then M is isomorphic to a whirl. We observe here that all whirls are ternary and 3-connected and none has an $M(K_4)$ -minor. The last observation follows from the easily verified fact that if N is a 3-connected minor of the whirl \mathcal{W}^r and $|E(N)| \geq 4$, then $N \cong \mathcal{W}^k$ for some $k \leq r$.

Euclidean representations for the matroids P_7 , Q_7 , and F_7^- , which appear in the next lemma, are shown in Fig. 4. In each case we have marked a special element x that will be important in the lemma. Let the ground set of $M(\mathcal{W}_r)$ and hence of \mathcal{W}^r be labelled as in Fig. 5.

(2.2) LEMMA. *Let r be an integer exceeding two and N be a non-trivial extension of the whirl \mathcal{W}^r by the element x . Then either, for some i in $\{0, 1, \dots, r-1\}$, $N/b_i \setminus a_i$ is a non-trivial extension of \mathcal{W}^{r-1} , or $r=3$ and N is isomorphic to P_7 , Q_7 , or F_7^- , with x being as shown.*

Proof. Since for all i , $\mathcal{W}^r/b_i \setminus a_i \cong \mathcal{W}^{r-1}$, $N/b_i \setminus a_i$ is an extension of \mathcal{W}^{r-1} . This extension is non-trivial and hence the lemma is proved provided that x is not in a 2-circuit of $N/b_i \setminus a_i$. Assume then that, for all i , x is in a 2-circuit $\{x, c_i\}$ of $N/b_i \setminus a_i$. Then, for all i , $\{x, b_i, c_i\}$ is a circuit of N . In particular, $\{x, b_0, c_0\}$ and $\{x, b_1, c_1\}$ are circuits of N for some elements c_0 and c_1 of \mathcal{W}^r .

Now either (i) $\{x, b_0, b_1\}$ is a circuit of N , or (ii) neither $\{x, b_0, c_0\}$ nor $\{x, b_1, c_1\}$ equals $\{x, b_0, b_1\}$. In case (i), perform exchange about the common element x using the circuits $\{x, b_0, b_1\}$ and $\{x, b_2, c_2\}$ to get a circuit

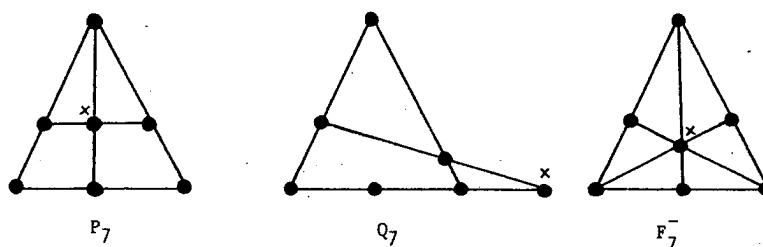


FIGURE 4

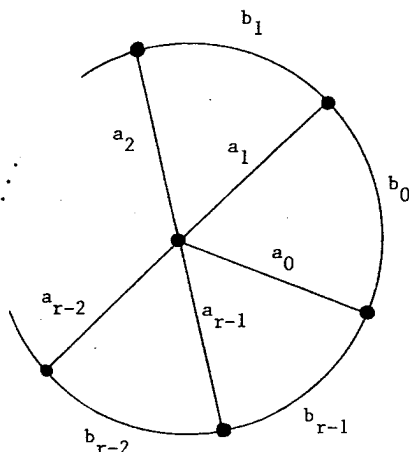


FIGURE 5

of \mathcal{W}^r contained in $\{b_0, b_1, b_2, c_2\}$. As \mathcal{W}^r has no such circuit unless $r=3$, it follows that (i) does not hold for $r>3$. If $r=3$, then N is one of the matroids shown in Fig. 6, hence N is isomorphic to one of the matroids P_7 and Q_7 in Fig. 4.

In case (ii), we perform exchange about x using $\{x, b_0, c_0\}$ and $\{x, b_1, c_1\}$ to get that \mathcal{W}^r has circuits C_0 and C_1 containing b_0 and b_1 respectively so that each of C_0 and C_1 is contained in $\{b_0, b_1, c_0, c_1\}$. As \mathcal{W}^r has no 3-circuits containing $\{b_0, b_1\}$, the elements c_0 and c_1 are distinct. Moreover, if $|C_0|=3$, then $C_0=\{b_0, c_0, c_1\}$ and $C_1=\{b_1, c_0, c_1\}$. But now, performing exchange about c_0 using C_0 and C_1 gives a 3-circuit of \mathcal{W}^r containing $\{b_0, b_1\}$, a contradiction. Thus $|C_0|>3$ and so $C_0=C_1=\{b_0, b_1, c_0, c_1\}$. Now, the only 4-circuit of \mathcal{W}^r containing $\{b_0, b_1\}$ is $\{b_0, b_1, a_0, a_2\}$ unless $r=3$. In the exceptional case, if $\{b_0, b_1, c_0, c_1\} \neq \{b_0, b_1, a_0, a_2\}$, it is not difficult to show that N is isomorphic to P_7 or Q_7 . Thus we may assume that, for all $r \geq 3$, $\{b_0, b_1, c_0, c_1\} = \{b_0, b_1, a_0, a_2\}$. Hence, $\{a_0, a_2\} = \{c_0, c_1\}$. If we now repeat the above argument using $\{x, b_1, c_1\}$ and $\{x, b_2, c_2\}$ in place of $\{x, b_0, c_0\}$ and $\{x, b_1, c_1\}$, we get that either N is isomorphic to P_7 or Q_7 , or $\{a_1, a_3\} = \{c_1, c_2\}$. In the latter case, $\{a_1, a_3\} \cap \{a_0, a_2\} \neq \emptyset$ and therefore $a_0=a_3$. Thus $r=3$, and

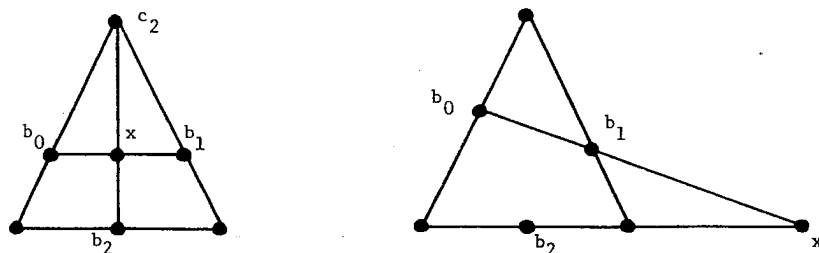


FIGURE 6

$c_1 = a_0$, $c_2 = a_1$, and $c_0 = a_2$. Hence N is as shown in Fig. 7, so N is isomorphic to the non-Fano matroid F_7^- . ■

We noted earlier that M has \mathcal{W}^3 as a minor. Therefore, by Theorem 1.4, there is a sequence $M_0, M_1, M_2, \dots, M_n$ of ternary 3-connected matroids such that $M_0 \cong \mathcal{W}^r$ for some $r \geq 3$, $M_n = M$ and, for all i in $\{1, 2, \dots, n\}$, M_i is an extension or lift of M_{i-1} .

Next we shall list all the rank-3 ternary 3-connected matroids with no $M(K_4)$ -minor. First, however, we shall need a preliminary lemma. We shall denote by $AG(2, 3) - p$ the unique matroid obtained from the affine geometry $AG(2, 3)$ by deleting a single point.

(2.3) LEMMA. *Let N be a rank-3 ternary matroid having a 4-point line L as a restriction. If N has at least 7 points, then N has an $M(K_4)$ -minor unless N is the parallel connection of two 4-point lines.*

Proof. Assume that N is not the parallel connection of two 4-point lines and let a, b , and c be three non-collinear points of $E(N) - L$. As L is a full line of $PG(2, 3)$, each of the lines $\{a, b\}$, $\{a, c\}$, and $\{b, c\}$ contains a point of L . If these points are d, e , and f , respectively, then d, e , and f are distinct and $M| \{a, b, c, d, e, f\} \cong M(K_4)$. ■

(2.4) LEMMA. *The only rank-3 ternary 3-connected matroids having no $M(K_4)$ -minor are \mathcal{W}^3 , P_7 , $AG(2, 3) - p$, and $AG(2, 3)$, each matroid in this list being an extension of its predecessor.*

Proof. Let N be a rank-3 ternary 3-connected matroid having no $M(K_4)$ -minor. Then we can view N as a restriction of $PG(2, 3)$ (see Fig. 8). Moreover, as $PG(2, 3)| \{a, b, c, d, e, f\} \cong \mathcal{W}^3$ and N has a \mathcal{W}^3 -minor, we can assume, by Theorem 1.5, that $E(N) \supseteq \{a, b, c, d, e, f\}$.

If $E(N)$ contains 1, 2, or 4, then N contains a 4-point line and so, by Lemma 2.3, has an $M(K_4)$ -minor, a contradiction. If $E(N)$ contains 3, then $N| \{a, b, c, e, f, 3\} \cong M(K_4)$, a contradiction. We conclude that $E(N) \subseteq E(PG(2, 3)) - \{1, 2, 3, 4\}$. But, as $PG(2, 3) \setminus \{1, 2, 3, 4\} \cong AG(2, 3)$, it

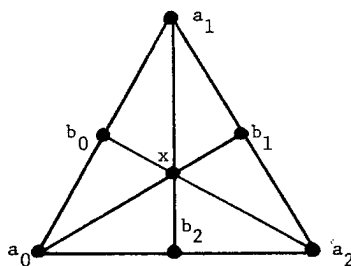


FIGURE 7

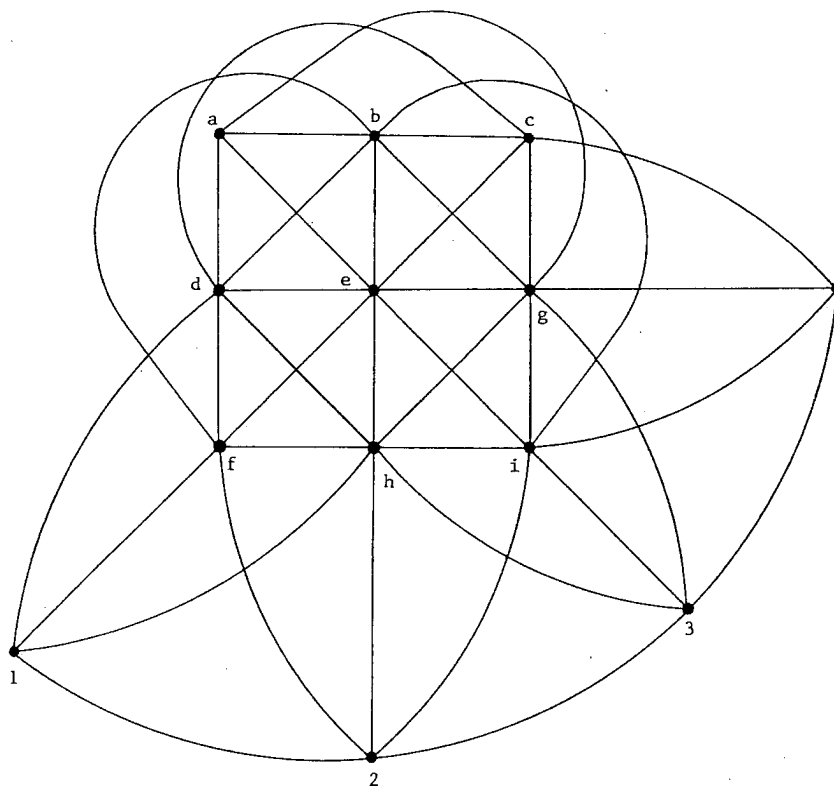


FIGURE 8

follows that N is a restriction of $AG(2, 3)$. Thus N is one of \mathcal{W}^3 , P_7 , $AG(2, 3) - p$, or $AG(2, 3)$, where we note that each of the matroids obtained by adding one of g , h , and i to $PG(2, 3) \setminus \{a, b, c, d, e, f\}$ is isomorphic to P_7 . To show that none of \mathcal{W}^3 , P_7 , $AG(2, 3) - p$, or $AG(2, 3)$ has an $M(K_4)$ -minor, it suffices to observe that $AG(2, 3)$ has no $M(K_4)$ -restriction. The latter is true because $M(K_4)$ is not affine over $GF(3)$, since K_4 is not a 3-colourable graph. ■

The next result follows immediately from the proof of the preceding lemma. We label the ground set of \mathcal{W}^3 as in Fig. 9 and view \mathcal{W}^3 as a restriction of $PG(2, 3)$. The closure of a set X in $PG(2, 3)$ will be denoted by $\sigma(X)$.

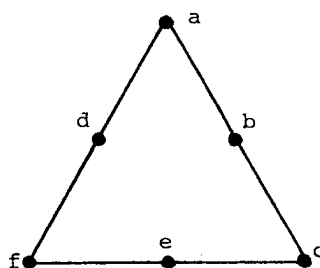


FIGURE 9

(2.5) COROLLARY. *There are exactly three points of $PG(2, 3)$ that can be added to \mathcal{W}^3 to give a member of Ω . These points are $\sigma\{a, e\} \cap \sigma\{b, d\}$, $\sigma\{c, d\} \cap \sigma\{b, e\}$, and $\sigma\{b, f\} \cap \sigma\{d, e\}$. Adding any one of these points to \mathcal{W}^3 gives a matroid isomorphic to P_7 .*

Another consequence of the last proof is

(2.6) COROLLARY. *$S(5, 6, 12)$ has no $M(K_4)$ -minor.*

Proof. If $S(5, 6, 12)$ does have an $M(K_4)$ -minor, then, as $S(5, 6, 12)$ has rank 6 and $M(K_4)$ has rank 3, there is an independent 3-element subset U of $E(S(5, 6, 12))$ such that $M(K_4)$ is a restriction of $S(5, 6, 12)/U$. But $S(5, 6, 12)/U \cong AG(2, 3)$ and so we get the contradiction that $M(K_4)$ is a restriction of $AG(2, 3)$. ■

The next lemma completes the proof that if M has a \mathcal{W}^4 -minor, it must be a whirl.

(2.7) LEMMA. *Every non-trivial ternary extension of \mathcal{W}^4 has an $M(K_4)$ -minor.*

Proof. Let N be a non-trivial ternary extension of \mathcal{W}^4 . By Theorem 1.5, we lose no generality in assuming that N is represented by the matrix

$$N = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ & & & & | & 1 & 0 & 0 & 1 & x_1 \\ & & & & | & -1 & 1 & 0 & 0 & x_2 \\ & & & & | & 0 & -1 & 1 & 0 & x_3 \\ & & & & | & 0 & 0 & -1 & 1 & x_4 \end{bmatrix},$$

where each of x_1, x_2, x_3 , and x_4 is in $\{0, 1, -1\}$. By Lemma 2.2 and the symmetry of \mathcal{W}^4 , we can assume that $N/5 \setminus 1$ is a non-trivial extension of \mathcal{W}^3 . Thus, by Corollary 2.5, $N/5 \setminus 1 \cong P_7$. It follows that $(x_2 + x_1, x_3, x_4)$ is one of $(1, 1, 1)$, $(1, -1, -1)$, or $(1, 1, -1)$. Therefore, there are nine possibilities for (x_1, x_2, x_3, x_4) : $(1, 0, 1, 1)$, $(1, 0, -1, -1)$, $(1, 0, 1, -1)$, $(0, 1, 1, 1)$, $(0, 1, -1, -1)$, $(0, 1, 1, -1)$, $(-1, -1, 1, 1)$, $(-1, -1, -1, -1)$, and $(-1, -1, 1, -1)$.

On subtracting row 4 from row 1 in N and then deleting row 1 and column 8, we get the following matrix representing $N/8$:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 9 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & x_2 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & x_3 \\ -1 & 0 & 0 & 1 & -1 & 0 & -1 & x_4 - x_1 \end{bmatrix}.$$

If (x_1, x_2, x_3, x_4) is $(1, 0, 1, -1)$ or $(-1, -1, -1, -1)$, then $N/8$ is a 7-point matroid having a 4-point line. As $N/8$ is not the parallel connection of two 4-point lines, Lemma 2.3 implies that it has an $M(K_4)$ -minor. Hence we can assume that (x_1, x_2, x_3, x_4) is neither $(1, 0, 1, -1)$ nor $(-1, -1, -1, -1)$. Moreover, we can also suppose that (x_1, x_2, x_3, x_4) is not $(-1, -1, 1, 1)$, otherwise $N/8 \setminus \{7, 1\} \cong M(K_4)$.

Next we observe that if row 3 is added to row 4 in N and then row 3 and column 7 are deleted, we get the following matrix representing $N/7$:

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 8 & 9 \\ \left[\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 1 & 0 & 1 & x_1 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & x_2 \\ 0 & 0 & 1 & 1 & 0 & -1 & 1 & x_4 + x_3 \end{array} \right]. \end{array}$$

If (x_1, x_2, x_3, x_4) is any of $(1, 0, 1, 1)$, $(0, 1, -1, -1)$, or $(-1, -1, 1, -1)$, then $N/7$ is a 7-point matroid having a 4-point line and it follows using Lemma 2.3 that $N/7$ has an $M(K_4)$ -minor.

There remain just three possibilities for (x_1, x_2, x_3, x_4) , namely $(0, 1, 1, -1)$, $(0, 1, 1, 1)$, and $(1, 0, -1, -1)$. It is straightforward to check that, in the first case, $N/6$ is a 7-point matroid having a 4-point line and hence, by Lemma 2.3, having an $M(K_4)$ -minor; in the second case, $N/3 \setminus \{6, 7\} \cong M(K_4)$, while, in the third case, $N/4 \setminus \{7, 8\} \cong M(K_4)$. ■

For the remainder of the proof of Theorem 2.1, we shall assume that M is not a whirl and focus attention on the sequence $M_0, M_1, M_2, \dots, M_n$ of 3-connected matroids that begins at a whirl \mathcal{W} of rank at least 3 and ends at M . It follows, on combining Lemmas 2.2 and 2.7 with Theorem 1.4 that $M_0 \cong \mathcal{W}^3$. The sequence $M_0, M_1, M_2, \dots, M_n$ will be used to construct all the members of Ω that are not whirls and have at least six elements. Evidently the class Ω is closed under duality, so whenever a new member of Ω is determined, we know immediately that its dual is also in Ω .

Since \mathcal{W}^3 is self-dual, we may assume that M_1 is a lift of M_0 . But, by Corollary 2.5, P_7 is the unique 3-connected ternary extension of \mathcal{W}^3 having no $M(K_4)$ -minor. Thus P_7^* is the unique lift of \mathcal{W}^3 in Ω and so $M_1 \cong P_7^*$.

Since P_7 can be represented by the matrix

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \left[\begin{array}{cccccc|cccc} & & & & 0 & 1 & 1 & 1 \\ & & & & 1 & 0 & 1 & -1 \\ & & & & 1 & 1 & 0 & 1 \end{array} \right], \end{array}$$

P_7^* can be represented by

$$A = \begin{bmatrix} & 4 & 5 & 6 & 7 & & 1 & 2 & 3 \\ & & & & & I_4 & \begin{array}{c} 0 \\ 1 \\ 1 \\ 1 \end{array} & \begin{array}{c} 1 \\ 0 \\ 1 \\ -1 \end{array} & \begin{array}{c} 1 \\ 1 \\ 0 \\ 1 \end{array} \end{bmatrix},$$

or, by the matrix we get by subtracting rows 1 and 3 from row 4, namely

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ -1 & 0 & -1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Figure 10 is a Euclidean representation for P_7^* . Note that the elements $(1, 0, 0, 0)^T$ and $(0, 0, 1, 0)^T$, which are not in P_7^* , have been marked on the same diagram.

If M_2 is a lift of P_7^* , then by Lemma 2.4, $M_2 \cong (AG(2, 3) - p)^*$. Furthermore, if M_3 is a lift of this last matroid, then $M_3 \cong (AG(2, 3))^*$. We shall

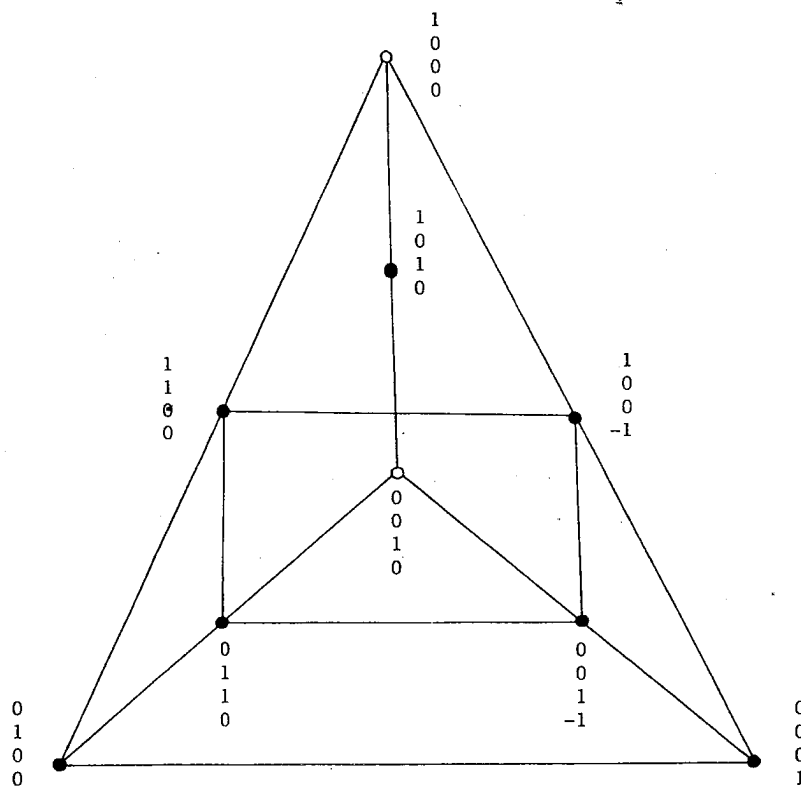


FIGURE 10

delay till Lemma 2.16 consideration of the extensions of $(AG(2, 3) - p)^*$ and $(AG(2, 3))^*$ that are in Ω .

We now assume that M_2 is an extension of M_1 . To determine the different possibilities for M_2 , we consider the columns that can be added to A to give a matrix representing a member of Ω . Since adding the negative of a column gives an isomorphic matroid to that obtained by adding the column itself, we shall not distinguish a column from its negative here. A matrix will be said to be in Ω if it represents a member of Ω .

(2.8) LEMMA. Suppose that $(x_1, x_2, x_3, x_4)^T$ is a column that is added to the matrix A to give a member of Ω . Then $(x_1, x_2, x_3, x_4)^T$ is one of $e_0 = (0, 1, 0, 1)^T$, $e_1 = (1, -1, 1, 0)^T$, $e_2 = (1, -1, -1, 1)^T$, and $e_3 = (1, 1, -1, -1)^T$.

Proof. M_2 is represented by the matrix

$$C = \begin{array}{ccccccccc} & 4 & 5 & 6 & 7 & 1 & 2 & 3 & 8 \\ \left[\begin{array}{cccc|cccc} & & & & 0 & 1 & 1 & x_1 \\ & & & & 1 & 0 & 1 & x_2 \\ & & & & 1 & 1 & 0 & x_3 \\ & & & & 1 & -1 & 1 & x_4 \end{array} \right] \end{array}$$

where each of x_1, x_2, x_3 , and x_4 is in $\{0, 1, -1\}$. Now suppose that $j \in \{4, 5, 6, 7\}$. Then it is routine to check that $P_7^*/j \cong \mathcal{W}^3$. Therefore M_2/j is either a 3-connected extension or a parallel extension of \mathcal{W}^3 . Since M_2/j is represented by the matrix $C_{j-3, j-3}$ obtained by deleting row $j-3$ and column $j-3$ from C , if M_2/j is a parallel extension of \mathcal{W}^3 , the last column of $C_{j-3, j-3}$ must be a nonzero scalar multiple of one of the first six columns. If M_2/j is a 3-connected extension of \mathcal{W}^3 , then $M_2/j \cong P_7$ and, by Corollary 2.5, there are exactly three possibilities for the last column of $C_{j-3, j-3}$.

To complete our determination of the possibilities for the element 8 of M_2 , we shall use Table I. In this table, it is assumed that $M_2/j \in \Omega$ and the possibilities for the last column of $C_{j-3, j-3}$ and hence for $(x_1, x_2, x_3, x_4)^T$ are listed. The former were determined using Corollary 2.5.

Now suppose that $8 = (1, 1, -1, x_4)^T$. Then, since the last column of $C_{1,1}$ must be a nonzero scalar multiple of another column of $C_{1,1}$ or of one of the three possibilities tabulated below, we are forced to get that 8 is $(1, 1, -1, -1)^T$ or $(1, 1, -1, 0)^T$. But, in addition, the last column of $C_{2,2}$ must be a nonzero scalar multiple of another column of $C_{2,2}$ or of one of the three possibilities tabulated. From this it follows that $8 \neq (1, 1, -1, 0)^T$. Hence, if $8 = (1, 1, -1, x_4)^T$, then $x_4 = -1$.

TABLE I

j	Possible last columns of $C_{j-3,j-3}$	Possibilities for $(x_1, x_2, x_3, x_4)^T$
4	$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$	$\begin{pmatrix} x_1 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} x_1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$
5	$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ x_2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ x_2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ x_2 \\ -1 \\ -1 \end{pmatrix}$
6	$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ -1 \\ x_3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ x_3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ x_3 \\ -1 \end{pmatrix}$
7	$\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ -1 \\ x_4 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ x_4 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ x_4 \end{pmatrix}$

By a similar argument, if $8 = (1, -1, 1, x_4)^T$ then $x_4 = 0$, and if $8 = (1, -1, -1, x_4)^T$ then $x_4 = 1$.

If $M_2/7$ is not 3-connected, then $(x_1, x_2, x_3)^T$ is a scalar multiple of some other column of $C_{4,4}$. Now, arguing as above, it is routine to check that 8 cannot be any of the columns $(1, 0, 0, x_4)^T$, $(0, 0, 1, x_4)^T$, $(0, 1, 1, x_4)^T$, $(1, 1, 0, x_4)^T$, $(1, 0, 1, 1)^T$, and $(0, 1, 0, -1)^T$. However, 8 may equal $(1, 0, 1, 0)^T$ or $(0, 1, 0, 1)^T = e_0$. But, in the representation for M_2 that is depicted in Fig. 10 and is obtained from the matrix **B**, the element corresponding to $(1, 0, 1, 0)^T$ is $(1, 0, 1, 1)^T$. (Recall that rows 1 and 3 were subtracted from row 4 to get **B** from **A**.) By symmetry, the matroid obtained by adding $(1, 0, 1, 1)^T$ to **B** is isomorphic to that obtained by adding $(1, -1, 1, 0)^T$ to **B** (see Fig. 10). But, in the representation for M_2 determined by **A**, the column corresponding to the element $(1, -1, 1, 0)^T$ from the representation determined by **B** is $(1, -1, 1, -1)^T$ and we have already eliminated this as a possibility for 8. Therefore $8 \neq (1, 0, 1, 0)^T$ and the proof of Lemma 2.8 is complete. ■

By the last lemma, P_7^* has up to four different extensions that are in Ω . We now examine these extensions more closely. For i in $\{0, 1, 2, 3\}$, let $P_7^* + e_i$ be the dependence matroid of the matrix obtained by letting 8 equal e_i in **C**. We first observe from Fig. 11 that $P_7^* + e_0 \cong J$ where we note

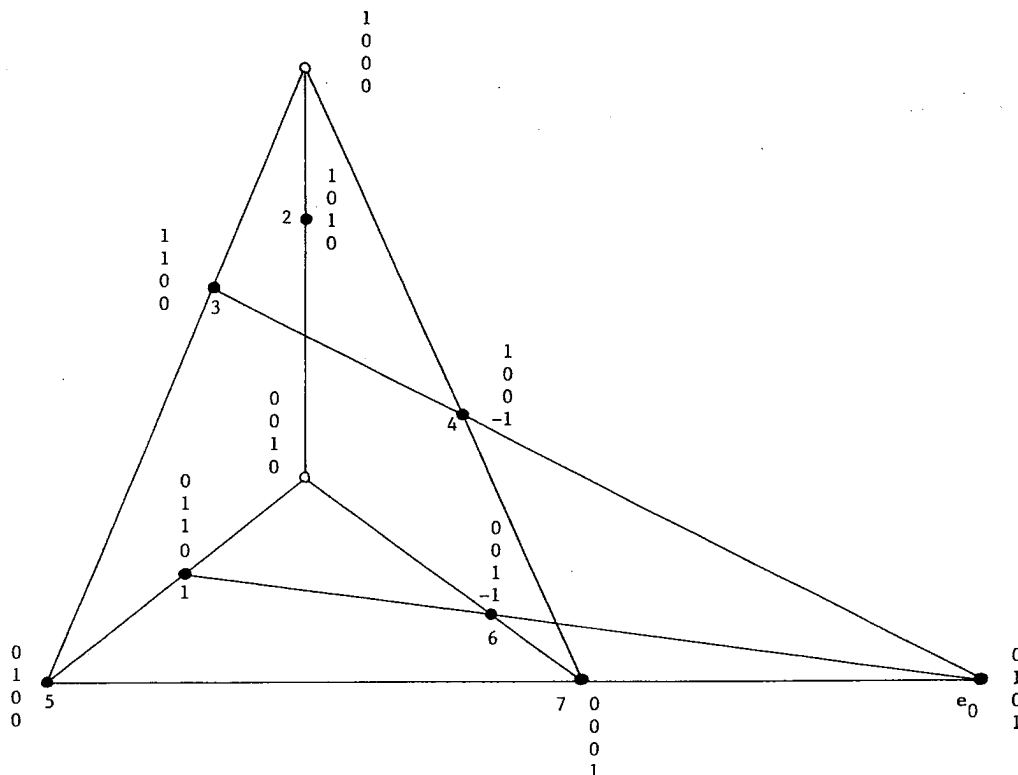


FIGURE 11

that the transformation that takes the matrix A to the matrix B leaves e_0 fixed. Moreover,

(2.9) LEMMA. J is a self-dual member of Ω having no extension or lift in Ω .

Proof. In this proof we shall identify J with $P_7^* + e_0$. Evidently J is 3-connected and ternary. Now suppose that J has an $M(K_4)$ -minor. Then, as $\text{rk } J > \text{rk } M(K_4)$, for some element f of J , J/f has an $M(K_4)$ -minor. By the construction of J , $f \notin \{4, 5, 6, 7\}$. Moreover, from the symmetry of Fig. 11, we see that $f \notin \{1, 3\}$. Again using Fig. 11, we get that J/e_0 is a parallel extension of $U_{3,4}$ and $J/2 \cong P_7$. Thus $f \notin \{e_0, 2\}$. We conclude that $f \notin E(J)$. This contradiction establishes that J has no $M(K_4)$ -minor, that is, $J \in \Omega$.

To establish that J is self-dual we use the fact noted above that $J/2 \cong P_7$. Therefore $J^* \setminus 2 \cong P_7^*$, that is, J^* is a member of Ω that is an extension of P_7^* . Thus, by Lemma 2.8, J^* is isomorphic to one of J , $P_7^* + e_1$, $P_7^* + e_2$, or $P_7^* + e_3$. As none of the last three matroids has a 3-circuit, whereas J^* does, it follows that $J^* \cong J$.

To show that J is a maximal member of Ω , it suffices to establish that every ternary 3-connected extension of J has an $M(K_4)$ -minor. Since the only elements that can be added to P_7^* without producing an $M(K_4)$ -minor

are e_0, e_1, e_2 , and e_3 , the only elements that we need consider adding to J are e_1, e_2 , and e_3 . But on adding any one of these elements to J and then contracting 2 we get an 8-point rank-3 matroid containing a 4-point line. By Lemma 2.3 we get a contradiction. ■

We shall show next that the matroids $P_7^* + e_1, P_7^* + e_2$, and $P_7^* + e_3$, are all isomorphic and each is self-dual.

(2.10) LEMMA. $P_7^* + e_1 \cong P_7^* + e_2 \cong P_7^* + e_3$ and $P_7^* + e_1$ is a self-dual member of Ω .

Proof. We first note that if we take $8 = e_1$ in \mathbf{C} , then the matrix we get representing $P_7^* + e_1$ is

$$\mathbf{D} = \begin{bmatrix} 4 & 5 & 6 & 7 & 1 & 2 & 3 & e_1 \\ & & & & 0 & 1 & 1 & 1 \\ & & & & 1 & 0 & 1 & -1 \\ & & I_4 & & 1 & 1 & 0 & 1 \\ & & & & 1 & -1 & 1 & 0 \end{bmatrix}.$$

Since this matrix has the form $[I|A]$ where A is symmetric, $P_7^* + e_1$ is self-dual.

To show that $P_7^* + e_1, P_7^* + e_2$, and $P_7^* + e_3$ are isomorphic, we begin by deleting rows 5 and 6 and columns 5 and 6 from \mathbf{X} . The resulting matrix \mathbf{X}' represents $S(5, 6, 12)/s_5, s_6$. If we now delete columns 9 and 10 from \mathbf{X}' , we get the matrix \mathbf{D} that represents $P_7^* + e_1$. If, instead, we delete columns 8 and 10 from \mathbf{X}' , we get the matrix \mathbf{C} with $8 = e_2$, a representation for $P_7^* + e_2$. Finally, if we delete columns 8 and 9 from \mathbf{X}' , we get the matrix \mathbf{C} with $8 = e_3$, a representation for $P_7^* + e_3$. Thus, each of $P_7^* + e_1, P_7^* + e_2$, and $P_7^* + e_3$ can be obtained from $S(5, 6, 12)$ by contracting two elements and then deleting two elements. Since the automorphism group of $S(5, 6, 12)$ is 5-transitive, it follows that $P_7^* + e_1, P_7^* + e_2$, and $P_7^* + e_3$ are isomorphic. As an alternative to using the 5-transitivity of the automorphism group of $S(5, 6, 12)$ to obtain these isomorphisms, we can use the following more elementary argument. Consider the representations for $P_7^* + e_1, P_7^* + e_2$, and $P_7^* + e_3$ obtained from \mathbf{C} by letting 8 equal e_1, e_2 , and e_3 , respectively. Then one can check using a sequence of projective operations and column interchanges on these matrices that the map which takes 4, 5, 6, 7, 1, 2, 3, and e_1 to 4, 1, 7, 6, 5, 2, 3, and e_2 , respectively, is an isomorphism between $P_7^* + e_1$ and $P_7^* + e_2$. Similarly, the map which takes 4, 5, 6, 7, 1, 2, 3, and e_1 to 5, 4, 1, 3, 6, 2, 7, and e_3 , respectively, is an isomorphism between $P_7^* + e_1$ and $P_7^* + e_3$.

As $P_7^* + e_1$ is a minor of $S(5, 6, 12)$, Corollary 2.6 implies that it has no $M(K_4)$ -minor. We conclude that $P_7^* + e_1 \in \Omega$. ■

We now suppose that M_2 is $P_7^* + e_1$. Since this matroid is self-dual, we may assume that M_3 is an extension of M_2 . Thus M_3 is one of the matroids $P_7^* + e_1 + e_2$ or $P_7^* + e_1 + e_3$. But each of these matroids is obtained from $S(5, 6, 12)$ by contracting two elements and deleting one. Hence, each is isomorphic to $S(3, 4, 10) - p$, the unique matroid obtained from $S(3, 4, 10)$ by deleting a single element. Thus $M_3 \cong P_7^* + e_1 + e_2 \cong S(3, 4, 10) - p$.

If M_4 is an extension of M_3 , then clearly $M_4 \cong P_7^* + e_1 + e_2 + e_3 \cong S(3, 4, 10)$. By Lemmas 2.8 and 2.9, $P_7^* + e_1 + e_2 + e_3$ has no extension in Ω . We delay until Lemma 2.16 consideration of the possible lifts of $P_7^* + e_1 + e_2 + e_3$ that are in Ω . Suppose now that M_4 is a lift of M_3 , where we recall that the latter is $P_7^* + e_1 + e_2$. Instead of determining the lifts of $P_7^* + e_1 + e_2$ that are in Ω , we shall solve the dual problem of finding all extensions of $(P_7^* + e_1 + e_2)^*$ that are in Ω . Now $P_7^* + e_1 + e_2$ is the dependence matroid of the matrix

$$E = \begin{bmatrix} & 4 & 5 & 6 & 7 & & 1 & 2 & 3 & e_1 & e_2 \\ & & & & & | & 0 & 1 & 1 & 1 & 1 \\ & & & & & | & 1 & 0 & 1 & -1 & -1 \\ & & & & & | & 1 & 1 & 0 & 1 & -1 \\ & & & & & | & 1 & -1 & 1 & 0 & 1 \end{bmatrix}$$

Thus $(P_7^* + e_1 + e_2)^*$ is represented by the matrix

$$E^* = \begin{bmatrix} & 1 & 2 & 3 & e_1 & e_2 & & 4 & 5 & 6 & 7 \\ & & & & & & | & 0 & 1 & 1 & 1 \\ & & & & & & | & 1 & 0 & 1 & -1 \\ & & & & & & | & 1 & 1 & 0 & 1 \\ & & & & & & | & 1 & -1 & 1 & 0 \\ & & & & & & | & 1 & -1 & -1 & 1 \end{bmatrix}$$

(2.11) LEMMA. *There are exactly two columns that can be added to E^* to give a member of Ω . These columns are*

$$f_1 = (1, -1, -1, 1, 0)^T \text{ and } f_2 = (1, 1, -1, -1, 1)^T.$$

The proof of this lemma is similar to the proof of Lemma 2.8. Indeed, the rest of the proof of Theorem 2.1 will use this technique. On combining the information from two minors of $(P_7^* + e_1 + e_2)^*$, each of which we recognize as having only a small number of extensions in Ω , we are able to show that $(P_7^* + e_1 + e_2)^*$ itself has the same property. We now give the details of this argument.

Proof of Lemma 2.11. The matrix $E_{5,5}^*$ that is obtained by deleting the fifth row and the fifth column from E^* equals the matrix D . Since the latter represents $P_7^* + e_1$, we know that the only extensions of it in Ω involve adding one of the columns $(1, -1, -1, 1)^T$ or $(1, 1, -1, -1)^T$. Thus the only columns that can be added to E^* to give a member of Ω must have their first four rows equal to a nonzero scalar multiple of a column of $E_{5,5}^*$ or of $(1, -1, -1, 1)^T$ or $(1, 1, -1, -1)^T$. This still leaves quite a large number of possibilities. To further reduce these, consider $(P_7^* + e_1 + e_2)^*/e_1$. This is isomorphic to the self-dual matroid $(P_7^* + e_2)^*$. By deleting row 4 and column 4 from E^* we get the following matrix representing $(P_7^* + e_2)^*$ and hence $P_7^* + e_2$:

$$F = \left[\begin{array}{c|cccc} & 0 & 1 & 1 & 1 \\ & 1 & 0 & 1 & -1 \\ & 1 & 1 & 0 & 1 \\ & 1 & -1 & -1 & 1 \\ \hline I_4 & & & & \end{array} \right].$$

Now $P_7^* + e_2 \cong P_7^* + e_1$ and there are exactly two columns that can be added non-trivially to the latter to give a member of Ω . In each case the resulting extension of $P_7^* + e_1$ is isomorphic to $S(3, 4, 10) - p$. It follows from Theorem 1.5 that there are exactly two columns that can be added non-trivially to $P_7^* + e_2$ to give a member of Ω and, for each of these, the resulting extension of $P_7^* + e_2$ is isomorphic to $S(3, 4, 10) - p$. To find these two columns, consider the matrix X'' that is obtained from X by deleting rows 4 and 6 and columns 4 and 6. Since this matrix represents $S(3, 4, 10)$ and F can be obtained from it by deleting its last two columns, these two columns, namely $(1, -1, -1, 0)^T$ and $(1, 1, -1, 1)^T$, are the two columns that can be added non-trivially to $P_7^* + e_2$ to give a member of Ω .

We can now show relatively quickly that if $(x_1, x_2, x_3, x_4, x_5)^T$ is a column that is added to E^* to give a member of Ω , then $(x_1, x_2, x_3, x_4, x_5)^T$ is f_1 or f_2 . First, suppose that $(x_1, x_2, x_3, x_4) = (1, -1, -1, 1)$. Then, as $(x_1, x_2, x_3, x_5)^T$ must be a nonzero scalar multiple of $(1, -1, -1, 0)^T$, $(1, 1, -1, 1)^T$, or some column of F , it follows easily that $(x_1, x_2, x_3, x_4, x_5)^T = f_1$. Similarly, if $(x_1, x_2, x_3, x_4) = (1, 1, -1, -1)$, then it is straightforward to show that $(x_1, x_2, x_3, x_4, x_5)^T = f_2$.

Finally, if $(x_1, x_2, x_3, x_4)^T$ is a nonzero scalar multiple of a column of $E_{5,5}^*$ and $(x_1, x_2, x_3, x_5)^T$ is a nonzero scalar multiple of $(1, -1, -1, 0)^T$, $(1, 1, -1, 1)^T$, or a column of F , then $(x_1, x_2, x_3, x_4, x_5)^T$ is $(0, 0, 0, 1, 1)^T$ or $(0, 0, 0, 1, -1)^T$. If we now adjoin the first of these columns to E^* , then, on deleting rows 1 and 2 and columns 1 and 2, we get a rank-3 8-point matroid having a 4-point line and hence an $M(K_4)$ -minor. Similarly, if we adjoin $(0, 0, 0, 1, -1)^T$ to E^* , then the deletion of rows 1 and 3 and

columns 1 and 3 again gives a rank-3 8-point matroid with a 4-point line and hence an $M(K_4)$ -minor. ■

In the next lemma, the matroid $(P_7^* + e_1 + e_2)^* + f_1$ is the dependence matroid of the following matrix G that is obtained by adjoining f_1 to E^* :

$$G = \begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & e_1 & e_2 & 4 & 5 & 6 & 7 & f_1 \end{array} \\ \left[\begin{array}{ccccccccc|ccccc} & & & & & & & & & & 0 & 1 & 1 & 1 & 1 \\ & & & & & & & & & & 1 & 0 & 1 & -1 & -1 \\ & & & & & & & & & & 1 & 1 & 0 & 1 & -1 \\ & & & & & & & & & & 1 & -1 & 1 & 0 & 1 \\ & & & & & & & & & & 1 & -1 & -1 & 1 & 0 \end{array} \right] \end{array}$$

Likewise, $(P_7^* + e_1 + e_2)^* + f_2$ is the dependence matroid of the matrix that is obtained by adjoining f_2 to E^* . Although the notation for members of Ω is becoming rather cumbersome, we shall retain it for its illustrative value.

(2.12) LEMMA. *The matroid $(P_7^* + e_1 + e_2)^* + f_1$ is self-dual. Moreover, $(P_7^* + e_1 + e_2)^* + f_1 \cong (P_7^* + e_1 + e_2)^* + f_2$.*

Proof. Since the matrix G representing $(P_7^* + e_1 + e_2)^* + f_1$ has the form $[I|A]$ where A is symmetric, $(P_7^* + e_1 + e_2)^* + f_1$ is self-dual.

To establish the isomorphism of $(P_7^* + e_1 + e_2)^* + f_1$ and $(P_7^* + e_1 + e_2)^* + f_2$, we begin by deleting row 6 and column 6 from X . The resulting matrix X''' represents $S(4, 5, 11)$. If we now delete column 11 from X''' , we get the matrix G that represents $(P_7^* + e_1 + e_2)^* + f_1$. If, instead, we delete column 10 from X''' , the resulting matrix is the same as the matrix obtained by adjoining f_2 to E^* and therefore represents $(P_7^* + e_1 + e_2)^* + f_2$. Since the automorphism group of $S(4, 5, 11)$ is transitive, it follows that $(P_7^* + e_1 + e_2)^* + f_1 \cong (P_7^* + e_1 + e_2)^* + f_2$.

As in the proof of Lemma 2.10, one can obtain a more elementary proof of the isomorphism between $(P_7^* + e_1 + e_2)^* + f_1$ and $(P_7^* + e_1 + e_2)^* + f_2$ as follows. Using the representations considered above for these two matroids, it is straightforward to check, by a sequence of projective operations and column interchanges, that the map which takes 1, 2, 3, e_1 , e_2 , 4, 5, 6, 7, and f_1 to 1, 4, 2, 3, e_2 , e_1 , 7, 6, f_2 , and 5, respectively, is an isomorphism. ■

Next we consider the extensions and lifts of $(P_7^* + e_1 + e_2)^* + f_1$ that are in Ω . As $(P_7^* + e_1 + e_2)^* + f_1$ is self-dual, we need only look at the extensions of it that are in Ω . By Lemma 2.11, the only such extension is $(P_7^* + e_1 + e_2)^* + f_1 + f_2$, this matroid being isomorphic to $S(4, 5, 11)$. Furthermore, the last matroid has no extensions in Ω . The next lemma

shows that it has exactly one lift in Ω . The matroid $((P_7^* + e_1 + e_2)^* + f_1 + f_2)^*$ is represented by the matrix

$$H = \left[\begin{array}{c|ccccc} & 0 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & -1 & -1 \\ & 1 & 1 & 0 & 1 & -1 \\ & 1 & -1 & 1 & 0 & 1 \\ & 1 & -1 & -1 & 1 & 0 \\ & 1 & 1 & -1 & -1 & 1 \end{array} \right].$$

(2.13) LEMMA. *There is only one column that can be added to the matrix H to give a member of Ω . This column is $g_1 = (1, 1, -1, -1, 1, 0)^T$.*

Proof. On deleting row 6 and column 6 from H we get the matrix G . The only column that can be added to this matrix to give a member of Ω is $(1, 1, -1, -1, 1)^T$. Similarly, it is not difficult to show that $(1, 1, -1, -1, 0)^T$ is the only column that gives a member of Ω when added to the matrix obtained from H by deleting row 5 and column 5. Now, arguing as in Lemma 2.11, it is straightforward to check that g_1 is the only column that can be added to H to give a member of Ω . ■

(2.14) LEMMA. *$((P_7^* + e_1 + e_2)^* + f_1 + f_2)^* + g_1$ is isomorphic to $S(5, 6, 12)$ and every non-trivial extension or lift of it has an $M(K_4)$ -minor.*

Proof. Adding g_1 to H gives the matrix X that represents $S(5, 6, 12)$. Thus $((P_7^* + e_1 + e_2)^* + f_1 + f_2)^* + g_1 \cong S(5, 6, 12)$. Since no column other than g_1 can be added to H to give a member of Ω and $S(5, 6, 12)$ is self-dual, $S(5, 6, 12)$ has no non-trivial extensions or lifts without an $M(K_4)$ -minor. ■

To complete the proof of Theorem 2.1, there are still some details left to check. The following table lists all the members of Ω with 4 or more elements that we have found so far together with their duals. The fact that this list is complete will follow from combining Lemma 2.16 with our earlier results. By the 5-transitivity of the automorphism group of $S(5, 6, 12)$, each of the non-whirl matroids in this list except $P_7^* + e_0$ can be expressed in terms of $S(5, 6, 12)$ or its derived systems. We have done this for all listed matroids with 9 or more elements.

To establish the completeness of Table II, we shall use the following result.

(2.15) LEMMA [27, Theorem 4.7]. *Let M be a minimally 3-connected matroid of rank r where $3 \leq r \leq 6$. Then $|E(M)| \leq 2r$.*

TABLE II
Members of Ω with at least 4 Elements

Number of elements	Rank	Matroid	Remarks
$2r$	r	\mathcal{W}^r	$r \geq 2$
7	3	P_7	Unique extension of \mathcal{W}^3 in Ω
	4	P_7^*	Unique lift of \mathcal{W}^3 in Ω
8	3	$AG(2, 3) - p$	Unique extension of P_7 in Ω
	4	$P_7^* + e_0$	Self-dual; isomorphic to J
			Has no extension or lift in Ω
	4	$P_7^* + e_1$	Self-dual
9	5	$(AG(2, 3) - p)^*$	
	3	$AG(2, 3) \cong S(2, 3, 9)$	Unique extension of $AG(2, 3) - p$ in Ω . Has no extension in Ω
	4	$P_7^* + e_1 + e_2$	Unique extension of $P_7^* + e_1$ in Ω ; isomorphic to $S(3, 4, 10) - p$
	5	$(P_7^* + e_1 + e_2)^*$	Unique lift of $P_7^* + e_1$ in Ω ; isomorphic to $(S(3, 4, 10) - p)^*$
10	6	$(AG(2, 3))^*$	Isomorphic to $(S(2, 3, 9))^*$
	4	$P_7^* + e_1 + e_2 + e_3$ $\cong S(3, 4, 10)$	Unique extension of $P_7^* + e_1 + e_2$ in Ω . Has no extension in Ω
	5	$(P_7^* + e_1 + e_2)^* + f_1$	Self-dual; isomorphic to $S(4, 5, 11) - p$
	6	$(P_7^* + e_1 + e_2 + e_3)^*$	Isomorphic to $(S(3, 4, 10))^*$
11	5	$(P_7^* + e_1 + e_2)^* + f_1 + f_2$ $\cong S(4, 5, 11)$	Unique extension of $(P_7^* + e_1 + e_2)^* + f_1$ in Ω
	6	$((P_7^* + e_1 + e_2)^* + f_1 + f_2)^*$	Has no extension in Ω Isomorphic to $(S(4, 5, 11))^*$ and $S(5, 6, 12) - p$
12	6	$((P_7^* + e_1 + e_2)^* + f_1 + f_2)^* + g_1$ $\cong S(5, 6, 12)$	Self-dual. Has no extension or lift in Ω

(2.16) LEMMA. *The only lifts of $AG(2, 3) - p$, $AG(2, 3)$, and $S(3, 4, 10)$ that are in Ω are $S(3, 4, 10) - p$, $S(3, 4, 10)$, and $S(4, 5, 11)$, respectively.*

Proof. Let N be a lift of $AG(2, 3) - p$ that is in Ω . Then N is 3-connected having rank 4 and corank 5. By Lemma 2.15, N is not minimally 3-connected. Thus, for some element x of N , $N \setminus x$ is 3-connected and hence is in Ω . But $N \setminus x$ has rank 4 and corank 4 and is therefore isomorphic to $P_7^* + e_1$. Since the only extension of $P_7^* + e_1$ that is in Ω is $P_7^* + e_1 + e_2$, and it is isomorphic to $S(3, 4, 10) - p$, we conclude that N is isomorphic to $S(3, 4, 10) - p$. Similar arguments complete the proof in the other two cases. ■

The preceding lemma finishes the proof of Theorem 2.1. An immediate consequence of Theorem 1.2 is that one can construct all ternary matroids with no $M(K_4)$ -minor by beginning with the members of Ω and repeatedly using the operations of 2-sum and direct sum.

3. THE CRITICAL PROBLEM

In this section we shall verify Brylawski's conjecture that a loopless ternary matroid with no $M(K_4)$ -minor has critical exponent at most two. The proof will use the following result.

(3.1) LEMMA [26, Corollary 3.6]. *Let M be a matroid representable over $GF(q)$. Suppose that $E(M)$ can be covered by cocircuits each with fewer than q^k elements. Then $c(M; q) \leq k$.*

(3.2) THEOREM. *Let M be a loopless ternary 3-connected matroid having no $M(K_4)$ -minor. Then M is affine unless M is isomorphic to J , a whirl of even rank, $S(3, 4, 10)$, $S(4, 5, 11)$, or $S(4, 5, 11) - p$. In the exceptional cases, M has critical exponent two.*

Proof. All the members of Ω that are not whirls have rank and corank less than 7. Each such matroid can be covered by cocircuits with at most 7 elements and hence, by Lemma 3.1, each such matroid has critical exponent at most two.

It is routine to verify by induction on r that for all $r \geq 2$,

$$P(\mathcal{W}^r; \lambda) = (\lambda - 1) \sum_{j=0}^{r-1} (-1)^j (\lambda - 2)^{r-1-j}.$$

From this, we get that $P(\mathcal{W}^r; 3)$ is 0 for r even and is 2 for r odd. It follows that $c(\mathcal{W}^r; 3)$ is 1 for r odd and is 2 for r even.

To show that none of J , $S(3, 4, 10)$, or $S(4, 5, 11) - p$ is affine, one needs only to check that no hyperplane of the ternary projective space of the appropriate rank avoids the ground set of the specified matroid. This is routine and we omit the details. Since $S(4, 5, 11)$ is an extension of $S(4, 5, 11) - p$ and the latter is non-affine, the same is true of the former.

To complete the proof we shall show that each of the remaining members of Ω is affine. Evidently all loopless members of Ω with at most three elements are affine. Moreover, $AG(2, 3)$ is affine and therefore so are $AG(2, 3) - p$ and P_7 . The matroid $P_7^* + e_1 + e_2$ is represented by the matrix E . No column of this matrix is in the hyperplane $x_1 - x_2 - x_3 + x_4 = 0$ of $PG(3, 3)$ and therefore $P_7^* + e_1 + e_2$ is affine. It follows that $P_7^* + e_1$ and P_7^* are also affine.

The matroid $S(5, 6, 12)$ is represented by the matrix X . No column of this matrix is in the hyperplane $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 0$ of $PG(5, 3)$, hence $S(5, 6, 12)$ is affine. Now, each matroid in the following sequence is isomorphic to a restriction of its predecessor: $S(5, 6, 12)$, $S(5, 6, 12) - p$, $(P_7^* + e_1 + e_2 + e_3)^*$, $(AG(2, 3))^*$. Since $S(5, 6, 12)$ is affine, the other three matroids listed are also affine.

The matroid $(P_7^* + e_1 + e_2)^*$ is represented by the matrix E^* . No column of this matrix is in the hyperplane $x_1 + x_2 - x_3 - x_4 - x_5 = 0$ of $PG(4, 3)$. Therefore $(P_7^* + e_1 + e_2)^*$ and its restriction $(AG(2, 3) - p)^*$ are affine, and the theorem is proved. ■

We can now verify Brylawski's conjecture [6, p. 159].

(3.3) COROLLARY. *Let M be a ternary loopless matroid having no $M(K_4)$ -minor. Then $c(M; 3) \leq 2$.*

Proof. We argue by induction on $|E(M)|$. If M is 3-connected, then the result follows immediately from the last theorem. If M is not connected, the result follows by the induction assumption. Finally, if M is connected but not 3-connected, then by Theorem 1.2, $M = S((M_1, p), (M_2, p))/p$ for some minors M_1 and M_2 of M , where each of M_1 and M_2 has at least three elements and p is not a loop or a coloop of M_1 or M_2 . Walton and Welsh [37, p. 5] showed that, under these circumstances,

$$P(M; \lambda) = (\lambda - 1)^{-1} P(M_1; \lambda) P(M_2; \lambda) + P(M_1/p; \lambda) P(M_2/p; \lambda).$$

By the induction assumption, neither $c(M_1; 3)$ nor $c(M_2; 3)$ exceeds 2. Thus both $P(M_1; 9)$ and $P(M_2; 9)$ are positive. Since both $P(M_1/p; 9)$ and $P(M_2/p; 9)$ are nonnegative, we conclude that $P(M; 9) > 0$. Hence $c(M; 3) \leq 2$ and the corollary is proved. ■

4. TERNARY GAMMOIDS AND RELATED CLASSES

It was shown by Brylawski [5, 6] and Ingleton [17] that a binary matroid is a gammoid if and only if it has no minor isomorphic to $M(K_4)$. Ingleton [19, p. 127] raised the question of finding an excluded-minor description for the class $\Gamma(3)$ of ternary gammoids noting that the task of finding a nicely determined complete list of excluded minors for the class of all gammoids is "probably futile." Brylawski also considered the class $\Gamma(3)$ and conjectured [6, p. 157] that it consists of precisely those gammoids which contain neither $U_{2,5}$ nor $U_{3,5}$ as a minor. In this section we shall use our main theorem to answer Ingleton's question and prove Brylawski's conjecture. We also consider when a ternary matroid M is base-orderable and when it is strongly base-orderable. Using our main theorem, it is straightforward to deduce that the former occurs if and only if M has no

minor isomorphic to $M(K_4)$ or J . The corresponding result for binary matroids, which was proved by de Sousa and Welsh [15], is that a binary matroid is base-orderable if and only if it has no $M(K_4)$ -minor. On combining this with the characterization of binary gammoids noted above, we get that for binary matroids the properties of being a gammoid, being strongly base-orderable, being base-orderable, and having no $M(K_4)$ -minor are equivalent. The results of this section show that, for ternary matroids, these four properties are all different. We shall use a number of elementary properties of base-orderable and strongly base-orderable matroids. A summary of these can be found in Welsh [38, Sect. 14.1].

(4.1) THEOREM. *A matroid is a ternary gammoid if and only if it has no minor isomorphic to any of the matroids $U_{2,5}$, $U_{3,5}$, $M(K_4)$, P_7 , or P_7^* .*

Proof. The class $\Gamma(3)$ of ternary gammoids is closed under minors and duality. Suppose that M is in this class. Then, as $M(K_4)$ is not a gammoid, M has no minor isomorphic to $M(K_4)$. Moreover, by [19, p. 128], neither P_7 nor P_7^* is a gammoid, so M has no minor isomorphic to P_7 or P_7^* . Finally, as M is ternary, it has no minor isomorphic to $U_{2,5}$ or $U_{3,5}$.

Now assume that M is a matroid having no minor isomorphic to any of $U_{2,5}$, $U_{3,5}$, $M(K_4)$, P_7 , or P_7^* . We shall show that $M \in \Gamma(3)$. Evidently, since M has no $M(K_4)$ -minor, it has no minor isomorphic to the Fano matroid, F_7 , or its dual. Furthermore, M has no minor isomorphic to $U_{2,5}$ or $U_{3,5}$ and therefore, by Theorem 1.3, M is ternary.

To show that M is a gammoid, we shall argue by induction on $|E(M)|$. This is trivially true for $|E(M)| = 1$. Assume it is true for $|E(M)| < n$ and let $|E(M)| = n$. If M is not connected, then since a direct sum of gammoids is also a gammoid, it follows by the induction assumption that M is a gammoid. Thus we may suppose that M is connected. If, in addition, M is 3-connected, then, as M has no $M(K_4)$ -minor and no minor isomorphic to P_7 or P_7^* , Table II implies that $|E(M)| \leq 3$ or M is a whirl. But all matroids on three or fewer elements are transversal as are all whirls (see, for example, [38, p. 241]). Hence M is a gammoid.

We may now suppose that M is connected but not 3-connected. Then, by Theorem 1.2, $M = S(M_1, M_2)/p$, where both M_1 and M_2 have fewer elements than M and are isomorphic to minors of M . By the induction assumption, both M_1 and M_2 are gammoids. Hence $S(M_1, M_2)$ is a gammoid. This last fact can be deduced, for example, from the fact that the class of gammoids is closed under the operation of matroid union [38, p. 224]. As $S(M_1, M_2)$ is a gammoid, $S(M_1, M_2)/p$ is also a gammoid, that is, M is a gammoid. ■

The next two results are consequences of the last proof. The first of these verifies a conjecture of Brylawski [6, p. 157].

(4.2) COROLLARY. *A gammoid is ternary if and only if it has no minor isomorphic to $U_{2,5}$ or $U_{3,5}$.*

Proof. As noted in the preceding proof, none of $M(K_4)$, P_7 , or P_7^* is a gammoid. Using this, the corollary follows from the theorem. ■

(4.3) COROLLARY. *Let M be a 3-connected ternary gammoid having at least 4 elements. Then $M \cong \mathcal{W}^r$ for some $r \geq 2$.*

By contrast with the last result, we note that there are no 3-connected binary gammoids having 4 or more elements.

The next theorem is an excluded-minor description of the class of ternary strongly base-orderable matroids. To prove it, we shall use Theorem 2.1 together with the following three lemmas. The first of these was proved by Ingleton [18] and Davies [13].

(4.4) LEMMA. *The following statements are equivalent for a rank-3 matroid M .*

- (i) *M is strongly base-orderable.*
- (ii) *M is base-orderable.*
- (iii) *M has no restriction isomorphic to $M(K_4)$.*

(4.5) LEMMA. *J is not base-orderable and hence is not strongly base-orderable.*

Proof. Since $J \cong P_7^* + e_0$, it suffices to show that the latter is not base-orderable. A Euclidean representation for $P_7^* + e_0$ is shown in Fig. 11. Now let $B_1 = \{2, e_0, 4, 6\}$ and $B_2 = \{1, 3, 5, 7\}$. Then B_1 and B_2 are bases of $P_7^* + e_0$. Suppose $\psi: B_1 \rightarrow B_2$ is a bijection with the property that, for all elements x of B_1 , both $(B_1 - x) \cup \psi(x)$ and $(B_2 - \psi(x)) \cup x$ are bases of $P_7^* + e_0$. Then, since $\{e_0, 5, 7\}$ is a circuit, $\psi(e_0) \notin \{1, 3\}$. Moreover, as $\{1, 6, e_0\}$ and $\{3, 4, e_0\}$ are circuits, $\psi(2) \neq 1$ and $\psi(2) \neq 3$. Thus $\psi(\{e_0, 2\}) = \{5, 7\}$, so either $\psi(e_0) = 7$ or $\psi(2) = 7$. The former cannot occur because $\{2, 7, 4, 6\}$ is a circuit, and the latter cannot occur because $\{1, 3, 5, 2\}$ is a circuit. We conclude that ψ does not exist and therefore J is not base-orderable. ■

(4.6) LEMMA. *$P_7^* + e_1$ is not strongly base-orderable.*

Proof. $P_7^* + e_1$ is represented by the matrix D . It has no 3-circuits and its 4-circuits include $\{2, 4, 6, 7\}$, $\{e_1, 4, 5, 6\}$, $\{1, 3, 4, 6\}$, $\{1, 3, e_1, 7\}$, and $\{1, 2, 3, 5\}$. Now let $B_1 = \{1, 2, 3, e_1\}$ and $B_2 = \{4, 5, 6, 7\}$. Then B_1 and B_2 are bases of $P_7^* + e_1$. Suppose that $\psi: B_1 \rightarrow B_2$ is a bijection with the property that, for all subsets X of B_1 , both $(B_1 - X) \cup \psi(X)$ and

$(B_2 - \psi(X)) \cup X$ are bases. As $\{2, 4, 6, 7\}$ and $\{1, 3, e_1, 7\}$ are circuits, $\psi(2)$ is not 5 or 7. Moreover, as both $\{1, 2, 3, 5\}$ and $\{e_1, 4, 5, 6\}$ are circuits, $\psi(e_1)$ is not 5 or 7 either. Thus $\psi(\{2, e_1\}) = \{4, 6\}$. But $(B_1 - \{2, e_1\}) \cup \psi(\{2, e_1\}) = \{1, 3, 4, 6\}$, which is a circuit; a contradiction. We conclude that $P_7^* + e_1$ is not strongly base-orderable. ■

(4.7) THEOREM. *A ternary matroid is strongly base-orderable if and only if it has no minor isomorphic to $M(K_4)$, J , or $P_7^* + e_1$.*

Before proving this theorem, we note that, on combining it with Theorem 1.3, we get

(4.8) COROLLARY. *A matroid is ternary and strongly base-orderable if and only if it has no minor isomorphic to any of $U_{2,5}$, $U_{3,5}$, $M(K_4)$, J , or $P_7^* + e_1$.*

Proof of Theorem 4.7. By Lemmas 4.4, 4.5, and 4.6, $M(K_4)$, J , and $P_7^* + e_1$ are not strongly base-orderable. Thus, as the class of strongly base-orderable matroids is closed under minors, if a ternary matroid is strongly base-orderable, it has no minor isomorphic to $M(K_4)$, J , or $P_7^* + e_1$. Conversely, suppose that a ternary matroid M has no minor isomorphic to $M(K_4)$, J , or $P_7^* + e_1$. Then, as the class of strongly base-orderable matroids is closed under the operations of direct sum, 2-sum, and series-parallel extension, we can assume that M is 3-connected having at least 4 elements. Thus $M \in \Omega$ and, from Table II, M is a whirl or one of P_7 , $AG(2, 3) - p$, or $AG(2, 3)$, or their duals. As all whirls are gammoids, they are strongly base-orderable. Moreover, by Lemma 4.4, each of P_7 , $AG(2, 3) - p$, and $AG(2, 3)$ is strongly base-orderable. Since the dual of a strongly base-orderable matroid is strongly base-orderable, the theorem follows. ■

The following is an immediate consequence of the last proof.

(4.9) COROLLARY. *Let M be a ternary 3-connected matroid having at least four elements. Then M is strongly base-orderable if and only if M is isomorphic to one of P_7 , P_7^* , $AG(2, 3) - p$, $(AG(2, 3) - p)^*$, $AG(2, 3)$, $(AG(2, 3))^*$, or \mathcal{W}^r for some $r \geq 2$.*

Next we give an excluded-minor description of the class of ternary base-orderable matroids.

(4.10) THEOREM. *A ternary matroid is base-orderable if and only if it has no minor isomorphic to $M(K_4)$ or J .*

Before proving this theorem, we note the following corollary that comes from combining the theorem with Theorem 1.3.

(4.11) COROLLARY. *A matroid is ternary and base-orderable if and only if it has no minor isomorphic to $U_{2,5}$, $U_{3,5}$, $M(K_4)$, or J .*

The key step in the proof of Theorem 4.10 is contained in the next lemma.

(4.12) LEMMA. *$S(5, 6, 12)$ is base-orderable.*

Proof. Suppose that B_1 and B_2 are bases of $S(5, 6, 12)$, where $B_1 = \{x_1, x_2, \dots, x_6\}$ and $B_2 = \{y_1, y_2, \dots, y_6\}$. We want to show that there is a bijection ψ from B_1 to B_2 so that, for all x in B_1 , both $(B_1 - x) \cup \psi(x)$ and $(B_2 - \psi(x)) \cup x$ are bases of $S(5, 6, 12)$. Now, all the circuits of $S(5, 6, 12)$ have exactly 6 elements and no two distinct circuits have 5 common elements. Thus, for each i in $\{1, 2, \dots, 6\}$, there is at most one element $\sigma(i)$ so that $(B_1 - x_i) \cup y_{\sigma(i)}$ is not a base, and there is at most one element $\tau(i)$ so that $(B_2 - y_i) \cup x_{\tau(i)}$ is not a base. Moreover, if $i_1 \neq i_2$, then $\sigma(i_1) \neq \sigma(i_2)$, and $\tau(i_1) \neq \tau(i_2)$. Now, for each i in $\{1, 2, \dots, 6\}$, let Y_i be obtained from B_2 by deleting the elements $y_{\sigma(i)}$ and $y_{\tau^{-1}(i)}$, where we observe that these elements need not exist. The existence of the required bijection ψ corresponds to the existence of a transversal for (Y_1, Y_2, \dots, Y_6) . But, it follows easily by Hall's marriage theorem (see, for example, [38, p. 98]) that the latter exists. We conclude that $S(5, 6, 12)$ is base-orderable. ■

The rest of the proof of Theorem 4.10 is just like the proof of Theorem 4.7 and we omit the details. The final result of this section comes from combining Theorems 2.1 and 4.10.

(4.13) COROLLARY. *A ternary 3-connected matroid having at least four elements is base-orderable if and only if it is isomorphic to one of the matroids, other than J , listed in Table II.*

5. A BOUND ON THE NUMBER OF ELEMENTS

Dirac [16] proved that, for all $n \geq 3$, a simple n -vertex graph with no subgraph homeomorphic from K_4 has at most $2n - 3$ edges. Since every binary matroid having no $M(K_4)$ -minor is a series-parallel network [5, Theorem 7.6] and hence is graphic, Dirac's result gives that, for all $r \geq 2$, a simple binary matroid of rank r having no $M(K_4)$ -minor has at most $2r - 1$ elements. In this section we use the main theorem to determine the maximum number of elements in a simple ternary matroid having no $M(K_4)$ -minor.

(5.1) THEOREM. *Let M be a simple ternary matroid of rank r having no $M(K_4)$ -minor. Then*

$$|E(M)| \leq \begin{cases} 4r-3 & \text{if } r \text{ is odd;} \\ 4r-4 & \text{if } r \text{ is even.} \end{cases}$$

Moreover, the following list contains all the matroids that attain this bound:

- (i) $r=1$, $U_{1,1}$;
- (ii) $r=2$, $U_{2,4}$;
- (iii) $r=3$, $AG(2, 3)$;
- (iv) $r=2t+1$ for $t \geq 2$, all matroids that can be formed from t copies of $AG(2, 3)$ using $t-1$ parallel connections;
- (v) $r=2t$ for $t \geq 2$, all matroids that can be formed from $t-1$ copies of $AG(2, 3)$ and one copy of $U_{2,4}$ using $t-1$ parallel connections.

Proof. Evidently each of the matroids listed attains equality in the bound. Now let M_0 be a simple ternary matroid of rank r with no $M(K_4)$ -minor that, among all such matroids, has the greatest number of elements. Then, from the matroids listed in (i)–(v), we know that $|E(M_0)|$ is at least $4r-3$ if r is odd and at least $4r-4$ if r is even. We shall argue by induction on r to show that M_0 is one of the matroids listed. If $r=1$, then clearly $M_0 \cong U_{1,1}$, as required. Now assume the proposition is true for $r < n$ and let $r=n \geq 2$. If M_0 is not connected, it follows easily from the induction assumption that $|E(M_0)| < 4r-4$, a contradiction. Thus M_0 is connected. If M_0 is also 3-connected, then from Table II, either $r=2$ and $M_0 \cong U_{2,4}$, or $r=3$ and $M_0 \cong AG(2, 3)$.

We may now suppose that M_0 is not 3-connected. Then, by Theorem 1.2, $M_0 = P((M_1, p), (M_2, p)) \setminus p$ for some minors M_1 and M_2 of M_0 each having at least three elements. Evidently both M_1 and M_2 are ternary having no $M(K_4)$ -minor. Moreover, either both M_1 and M_2 are simple, or exactly one of them, say M_1 , has a single element q in parallel with the basepoint p . In the former case, we may add an element s in parallel with p , thereby increasing the number of elements in the 2-sum without destroying the property of simplicity or creating an $M(K_4)$ -minor. This contradicts the choice of M_0 . In the latter case, M_0 is isomorphic to the parallel connection of $M_1 \setminus q$ and M_2 , and both these matroids are simple. We conclude that we may assume that M_0 is the parallel connection of two simple ternary matroids each having no $M(K_4)$ -minor. It is now routine to complete the proof of the theorem using the induction assumption and we omit the details. ■

ADDENDUM: $GF(4)$ -REPRESENTABLE MATROIDS

In this section, we apply the same method that was used to prove Theorem 2.1 to characterize the class of $GF(4)$ -representable matroids having no minor isomorphic to either $M(K_4)$ or \mathcal{W}^3 . The large number of cases makes it difficult to apply this method to determine the $GF(4)$ -representable matroids when $M(K_4)$, the rank-3 wheel, is the only excluded minor. However, excluding both $M(K_4)$ and \mathcal{W}^3 , the rank-3 whirl, makes the task manageable. Moreover, in view of the fundamental building-block role that wheels and whirls play within the class of matroids (see Theorem 1.4 or [36]), \mathcal{W}^3 is a natural addition to the set of excluded minors. As noted earlier, Brylawski [5] characterized series-parallel networks as being those matroids with no minor isomorphic to the rank-2 whirl or the rank-3 wheel. As a natural extension of this, the author [33] has recently characterized the class of matroids with no minor isomorphic to the rank-2 whirl or the rank-4 wheel.

The details given in this section will be brief. The proof of the main theorem will rely on the following result of Kahn.

(A.1) THEOREM [21, Theorem 1]. *Let A and A' be $r \times n$ matrices over $GF(4)$ such that the map which, for all i in $\{1, 2, \dots, n\}$, takes the i th column of A to the i th column of A' is an isomorphism from $D(A)$ to $D(A')$. If $D(A)$ is 3-connected, then A and A' are projectively equivalent.*

For every integer r exceeding 1, let K_r be the following $r \times (2r + 1)$ matrix over $GF(4)$. Here, and throughout this section, we shall identify $GF(4)$ with $GF(2)(\omega)$, where $\omega^2 = \omega + 1$,

$$K_r = \begin{bmatrix} x_1 & x_2 & \cdots & x_r & y_1 & y_2 & y_3 & y_4 & \cdots & y_r & z_r \\ & & & & \omega & \omega + 1 & 1 & 1 & \cdots & 1 & 1 \\ & & & & \omega + 1 & \omega & 1 & 1 & \cdots & 1 & 1 \\ & & & & 1 & 1 & & & & & 0 \\ & & & & 1 & 1 & & & & & 0 \\ & & & & \vdots & \vdots & & & I_{r-2} & & \vdots \\ & & & & 1 & 1 & & & & & 0 \end{bmatrix}$$

Let X_r be the dependence matroid of K_r . One can obtain X_r geometrically by the following straightforward construction. Let $P(rU_{2,3})$ be the parallel connection of r 3-point lines [5], that is, $P(rU_{2,3})$ is the rank- $(r+1)$ matroid consisting of r 3-point lines all passing through a common point p . Now add a new point q freely to $P(rU_{2,3})$ and then contract out q . The resulting matroid is the truncation $T(P(rU_{2,3}))$ of $P(rU_{2,3})$.

(A.2) LEMMA. $X_r \cong T(P(rU_{2,3}))$.

To prove this lemma it is useful to exploit the symmetry of X_r as expressed in the following result.

(A.3) LEMMA. *The automorphism group of X_r is transitive on $\{x_1, x_2, \dots, x_r, y_1, y_2, \dots, y_r\}$. Moreover, for $r \geq 3$, every automorphism of X_r fixes z_r .*

We omit the routine proof of this lemma as well as the remaining straightforward details of the proof of Lemma A.2.

The next result, the main theorem of this section, lists all 3-connected $GF(4)$ -representable matroids having no minor isomorphic to $M(K_4)$ or \mathcal{W}^3 . We shall denote by Ψ the class of all such matroids. By Theorem 1.2, one can construct all $GF(4)$ -representable matroids having no $M(K_4)$ - or \mathcal{W}^3 -minor by beginning with the members of Ψ and repeatedly using the operations of direct sum and 2-sum.

(A.4) THEOREM. *A matroid M is 3-connected and $GF(4)$ -representable having no minor isomorphic to $M(K_4)$ or \mathcal{W}^3 if and only if*

(i) $M \cong X_r, X_r^*, X_r \setminus z_r$, or $X_r \setminus y_r$ for some $r \geq 2$; or

(ii) $M \cong U_{0,1}, U_{1,1}, U_{1,2}, U_{1,3}$, or $U_{2,3}$.

The only pairs of matroids in the above list that have the same rank and corank are $X_r \setminus z_r$ and $X_r \setminus y_r$. When $r = 2$, these matroids are isomorphic. When $r \geq 3$, the matroids are non-isomorphic since the latter has a 3-element circuit while the former does not.

A consequence of Theorem A.4 is the following partial result, in the case when $q = 4$, towards Brylawski's conjecture that a loopless $GF(q)$ -representable matroid with no $M(K_4)$ -minor has critical exponent at most two [6].

(A.5) COROLLARY. *Suppose that M is a loopless $GF(4)$ -representable matroid having no minor isomorphic to $M(K_4)$ or \mathcal{W}^3 . Then $c(M; 4) \leq 2$. Moreover, if M is 3-connected, then $c(M; 4) = 1$ unless M is isomorphic to X_r for some even integer r .*

Proof of Theorem A.4. All the matroids listed in (ii) have fewer than four elements and so cannot have an $M(K_4)$ - or \mathcal{W}^3 -minor. The fact that none of the matroids listed in (i) has such a minor follows from the next lemma, the proof of which uses Lemma A.3 and resembles the proof of Lemma 2.3 of [33]. We omit the details.

(A.6) LEMMA. X_r has no minor isomorphic to $M(K_4)$ or \mathcal{W}^3 .

Now suppose that M is a 3-connected $GF(4)$ -representable matroid having no minor isomorphic to $M(K_4)$ or \mathcal{W}^3 . If $|E(M)| < 6$, it is straightforward to check that M is listed under (i) or (ii). Now suppose $|E(M)| \geq 6$. Then, as neither $U_{2,6}$ nor $U_{4,6}$ is representable over $G(4)$, both

the rank and corank of M exceed two. Therefore, by [29, Theorem 2.5], M has a minor isomorphic to one of $M(K_4)$, \mathcal{W}^3 , P_6 , Q_6 , or $U_{3,6}$. By assumption, M has no $M(K_4)$ - or \mathcal{W}^3 -minor. Moreover, P_6 is not representable over $GF(4)$ (see, for example, [31]). Therefore, M has a Q_6 - or $U_{3,6}$ -minor where we note that $Q_6 \cong X_3 \setminus y_3$, while $U_{3,6} \cong X_3 \setminus z_3$.

The next result, a straightforward consequence of Theorem 1.4, shows that if $|E(M)| \geq 7$ then M has a Q_6 -minor.

(A.7) LEMMA. *Let N be a 3-connected $GF(4)$ -representable matroid and suppose that N has no minor isomorphic to Q_6 . If N has a $U_{3,6}$ -minor, then $N \cong U_{3,6}$.*

Proof. By Theorem 1.4, if $N \not\cong U_{3,6}$, then N has a non-trivial extension or lift of $U_{3,6}$ as a minor. It is straightforward to check that every such extension has a $U_{2,6}$ - or a Q_6 -minor. Hence, by duality, every non-trivial lift has a $U_{4,6}$ - or a Q_6 -minor. Since $U_{2,6}$ and $U_{4,6}$ are not representable over $GF(4)$, the lemma follows. ■

We shall now suppose that $M \not\cong U_{3,6}$. Then M has a Q_6 -minor. Therefore, by Theorem 1.4 again, there is a sequence N_0, N_1, \dots, N_n of 3-connected matroids such that $N_0 \cong Q_6$, $N_n = M$, and, for all i in $\{1, 2, \dots, n\}$, N_i is an extension or lift of N_{i-1} . This sequence will be used to construct all the members of $\Psi - \{U_{3,6}\}$ having rank and corank exceeding two. In this construction, we shall make frequent use of the fact that Ψ is closed under duality. Since Q_6 is self-dual, we can assume that N_1 is a lift of N_0 .

Now it is straightforward to show that $T(P(3U_{2,3}))$ is the only extension of Q_6 that is in ψ . Moreover, $T(P(3U_{2,3}))$ has no extensions in Ψ . Since, by Lemma A.2, $T(P(3U_{2,3})) \cong X_3$, it follows that $N_1 \cong X_3^*$ and that N_2 is an extension of N_1 . As X_3^* is 3-connected, Theorem A.1 implies that we lose no generality by arguing in terms of a particular representation K_3^* for it, where, for all $r \geq 3$, K_r^* is the following $(r+1) \times (2r+1)$ matrix that represents X_r^* :

$$K_r^* = \begin{bmatrix} y_1 & y_2 & \cdots & y_r & z_r & x_1 & x_2 & x_3 & x_4 & \cdots & x_r \\ \hline & & & & & \omega & \omega+1 & 1 & 1 & \cdots & 1 \\ & & & & & \omega+1 & \omega & 1 & 1 & \cdots & 1 \\ \hline & & & & & 1 & 1 & & & & \\ & & & & & 1 & 1 & & & & \\ & & & & & \vdots & \vdots & & & & \\ & & & & & 1 & 1 & & & & \\ \hline & & & & & 1 & 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

We note from this representation that $X_r^* \cong X_{r+1} \setminus y_{r+1}, z_{r+1}$.

As in Section 2, we shall not distinguish here between a column and its non-zero scalar multiples. The next lemma essentially finishes the proof of Theorem A.4.

(A.8) LEMMA. Suppose that $r \geq 3$ and that $v = (v_1, v_2, \dots, v_{r+1})^T$ is a column that is adjoined to K_r^* to give a matrix L over $GF(4)$ representing a member N of Ψ . Then v is $(1, 1, 0, 0, \dots, 0, 0)^T$ or $(1, 1, 0, 0, \dots, 0, 1)^T$.

Proof. We argue by induction on r . First suppose that $r = 3$. The proof in this case resembles the proof of Lemma 2.8. On deleting the first row and column from L , we get the following matrix representing the connected matroid N/y_1 :

$$L_{1,1} = \left[\begin{array}{c|ccc} & \omega+1 & \omega & 1 & v_2 \\ & 1 & 1 & 1 & v_3 \\ & 1 & 1 & 0 & v_4 \\ \hline I_3 & & & & \end{array} \right].$$

The first six columns of this matrix represent Q_6 . Since N/y_1 is an extension of Q_6 , if N/y_1 is in Ψ , it is isomorphic to $T(P(3U_{2,3}))$ and it is straightforward to check that (v_2, v_3, v_4) is $(1, 0, 1)$. If N/y_1 is not in Ψ , it is a parallel extension of Q_6 , so $(v_2, v_3, v_4)^T$ is a nonzero scalar multiple of some other column of $L_{1,1}$. It also follows immediately from this, using the symmetry of the matrix L , that $(v_1, v_3, v_4)^T$ is $(1, 0, 1)^T$ or a nonzero scalar multiple of some column of $L_{1,1}$.

On deleting the third row and column from L , we get the following matrix representing N/y_3 :

$$L_{3,3} = \left[\begin{array}{c|ccc} & \omega & \omega+1 & 1 & v_1 \\ & \omega+1 & \omega & 1 & v_2 \\ & 1 & 1 & 0 & v_4 \\ \hline I_3 & & & & \end{array} \right].$$

By a similar argument to the above, it follows that $(v_1, v_2, v_4)^T$ is $(1, 1, 1)^T$ or a nonzero scalar multiple of another column of $L_{3,3}$. In the first case, using the constraints on (v_2, v_3, v_4) and (v_1, v_3, v_4) obtained above, we find that (v_1, v_2, v_3, v_4) is $(1, 1, 0, 1)$. Using these constraints again in the case that $(v_1, v_2, v_4)^T$ is a nonzero scalar multiple of another column of $L_{3,3}$, we get that (v_1, v_2, v_3, v_4) is one of $(1, 1, 0, 0)$, $(1, 0, 1, 0)$, or $(0, 1, 1, 0)$. To eliminate the last two possibilities, assume that (v_1, v_2, v_3, v_4) is one of them. By symmetry, we can suppose it is $(1, 0, 1, 0)$. In that case, it is straightforward to show that N/x_1 has a \mathcal{W}^3 -minor, a contradiction. We conclude that the lemma holds for $r = 3$.

Now assume the lemma is true for all integers less than r and consider K_r^* , where $r \geq 4$. On applying the induction assumption to each of the

matrices $L_{r+1,r+1}$, $L_{r,r}$, and $L_{r-1,r-1}$ and combining this information as before, it is not difficult to complete the proof of the lemma. ■

From considering the matrix K_r , we see that, for all r , both $X_r \setminus y_r$ and $X_r \setminus z_r$ are self-dual. On combining this information with the last lemma and using the chain of matroids $N_0, N_1, N_2, \dots, N_n$ guaranteed by Theorem 1.4, we get that if $M \in \Psi$ and $|E(M)| \geq 4$, then M is listed in (i).

We have not explicitly checked here that each of the matroids listed in (i) is 3-connected, but this follows inductively from the way in which these matroids can be built up from $U_{2,4}$. This completes the proof of Theorem A.4. ■

Proof of Corollary A.5. Suppose that M is a loopless $GF(4)$ -representable matroid having no minor isomorphic to $M(K_4)$ or \mathcal{W}^3 . We shall prove

If M is 3-connected, then $c(M; 4) = 1$ unless M is isomorphic to X_r for some even integer r ; in the exceptional case, $c(M; 4) = 2$. (A.9)

The corollary follows from this by a straightforward induction argument similar to the proof of Corollary 3.3. If $|E(M)| \leq 3$, then M is listed under (i) and it is easy to check that M is affine. We show next that

$$c(T(P(rU_{2,3})); 4) = \begin{cases} 1 & \text{if } r \text{ is odd,} \\ 2 & \text{if } r \text{ is even.} \end{cases} \quad (\text{A.10})$$

To verify this, we first observe that, by [5, Theorem 6.16], the chromatic polynomial of $P(rU_{2,3})$ is

$$(\lambda - 1)(\lambda - 2)^r = (\lambda - 1) \sum_{i=0}^r \binom{r}{i} (-2)^i \lambda^{r-i}.$$

Hence, by [7, Theorem 7.3], $T(P(rU_{2,3}))$ has chromatic polynomial

$$(\lambda - 1) \sum_{i=0}^{r-1} \binom{r}{i} (-2)^i \lambda^{r-i-1} = \lambda^{-1}(\lambda - 1)[(\lambda - 2)^r - (-2)^r].$$

On evaluating this expression at $\lambda = 4$ and at $\lambda = 4^2$, we immediately obtain (A.10).

By Lemma A.2, $X_r \cong T(P(rU_{2,3}))$. Moreover, $X_r^* \cong X_{r+1} \setminus y_{r+1}, z_{r+1}$. It therefore follows, using (A.10), that to complete the proof of (A.9) we need only show that, for all positive integers m , both $X_{2m} \setminus z_{2m}$ and $X_{2m} \setminus y_{2m}$ are affine. The first of these follows because every column of K_{2m} except z_{2m} is orthogonal to $(1, 1, \dots, 1)^T$. To get the second, note that, by Lemma A.3, $X_{2m} \setminus y_{2m} \cong X_{2m} \setminus y_2$; the latter is affine since every column of K_{2m} except y_2 is orthogonal to $(\omega + 1, 1, 1, 1, \dots, 1)^T$. ■

We conclude this Addendum with one further consequence of Theorem

2.1, another partial result towards the characterization of $GF(4)$ -representable matroids having no $M(K_4)$ -minor.

(A.11) THEOREM. *Let M be a matroid having at least four elements. Then M is 3-connected, $GF(4)$ -representable and has no minor isomorphic to $M(K_4)$ or Q_6 if and only if*

- (i) $M \cong \mathcal{W}^r$ for some $r \geq 2$;
- (ii) $M \cong U_{2,5}$, $U_{3,5}$, or $U_{3,6}$; or
- (iii) M is isomorphic to J or to one of P_7 , $AG(2, 3) - p$ or $AG(2, 3)$, or their duals.

The proof of this theorem will require the following two lemmas.

(A.12) LEMMA. *Let M be a 3-connected $GF(4)$ -representable matroid having no minor isomorphic to $M(K_4)$ or Q_6 . If both the rank and corank of M exceed two, then either M is ternary or $M \cong U_{3,6}$.*

Proof. Assume that $M \not\cong U_{3,6}$. Then, by Lemma A.7, M has no $U_{3,6}$ -minor. Now suppose that M is not ternary. Then, by Theorem 1.3 and the fact that M has no $M(K_4)$ -minor, it follows that M has a minor isomorphic to $U_{2,5}$ or $U_{3,5}$. By duality, we can assume that the latter occurs. Then, using Theorem 1.4, it is not difficult to show that M has one of $U_{4,6}$, $U_{3,6}$, P_6 , or Q_6 as a minor. Since each of these possibilities leads to a contradiction, the lemma holds. ■

(A.13) LEMMA [31]. $P_7^* + e_1$ is not representable over $GF(4)$.

Proof of Theorem A.11. Since Q_6 is non-ternary, it follows easily that none of the matroids in (i)–(iii) has a Q_6 -minor. Moreover, by Theorem 2.1, all the matroids in (i) and (iii) are 3-connected and have no $M(K_4)$ -minor. Evidently the same is true for the matroids in (ii). The matroids in (i) and (ii) are well known to be $GF(4)$ -representable. To establish $GF(4)$ -representability for the matroids in (iii), it suffices to observe that this holds for both $AG(2, 3)$ (see, for example, [22, p. 14]) and J . To verify the latter, one can easily check that the following matrix is a $GF(4)$ -representation for $P_7^* + e_0$ and hence for J :

$$\begin{array}{ccccccccc} 4 & 5 & 6 & 7 & 1 & 2 & 3 & e_0 \\ \left[\begin{array}{cccc|cc} & & & & 0 & 1 & 1 & 0 \\ & & & & 1 & 0 & 1 & 1 \\ & & & & 1 & \omega & 0 & 0 \\ & & & & 1 & \omega + 1 & 1 & 1 \end{array} \right] \end{array}$$

To prove the converse, let M be 3-connected and $GF(4)$ -representable having no $M(K_4)$ - or Q_6 -minor. We shall assume that both the rank and corank of M exceed two since the result is easily checked otherwise. Then, by Lemma A.12, $M \cong U_{3,6}$ or M is ternary. In the former case, the result holds. In the latter case, M is 3-connected and ternary having no $M(K_4)$ -minor, so M is listed in Table II. Using that table and Lemma A.13, we deduce that, in this case, $M \cong \mathcal{W}^r$ for some $r \geq 3$ or M is listed under (iii). ■

It follows easily from Theorem A.11 and Lemma 3.1 that a loopless $GF(4)$ -representable matroid M having no minor isomorphic to $M(K_4)$ or Q_6 has critical exponent $c(M; 4)$ at most two.

REFERENCES

1. R. E. BIXBY, On Reid's characterization of the ternary matroids, *J. Combin. Theory Ser. B* **26** (1979), 174–204.
2. J. A. BONDY, Transversal matroids, base-orderable matroids and graphs, *Quart. J. Math. Oxford* (2) **23** (1972), 81–89.
3. R. A. BRUALDI, Comments on bases in dependence structures, *Bull. Austral. Math. Soc.* **7** (1969), 161–167.
4. R. A. BRUALDI, Induced matroids, *Proc. Amer. Math. Soc.* **29** (1971), 221–231.
5. T. H. BRYLAWSKI, A combinatorial model for series-parallel networks, *Trans. Amer. Math. Soc.* **154** (1971), 1–22.
6. T. H. BRYLAWSKI, An affine representation for transversal geometries, *Studies in Appl. Math.* **54** (1975), 143–160.
7. T. H. BRYLAWSKI, Modular constructions for combinatorial geometries, *Trans. Amer. Math. Soc.* **203** (1975), 1–44.
8. T. H. BRYLAWSKI AND D. LUCAS, Uniquely representable combinatorial geometries, in "Teorie Combinatorie, Proc. 1973 Internat. Colloq.," Accademia Nazionale dei Lincei, Roma, 1976, pp. 83–104.
9. P. J. CAMERON, Extremal results and configuration theorems for Steiner systems, in "Topics on Steiner Systems," Ann. Discrete Math. Vol. 7, pp. 43–63, North-Holland, Amsterdam/New York, 1980.
10. P. J. CAMERON AND J. H. VAN LINT, "Graph Theory, Coding Theory and Block Designs," London Math. Soc. Lecture Notes Ser. Vol. 19, Cambridge Univ. Press, Cambridge, 1975.
11. H. S. M. COXETER, Twelve points in $PG(5, 3)$ with 95040 self-transformations, *Proc. Roy. Soc. London Ser. A* **247** (1958), 279–293.
12. H. H. CRAPO AND G.-C. ROTA, "On the Foundations of Combinatorial Theory: Combinatorial Geometries," preliminary ed., M.I.T. Press, Cambridge, Mass., 1970.
13. J. DAVIES, "Some Problems in Matroid Theory," D. Phil. thesis, Univ. of Oxford, 1975.
14. P. DEMBOWSKI, "Finite Geometries," Springer-Verlag, Berlin, 1968.
15. J. DE SOUSA AND D. J. A. WELSH, A characterisation of binary transversal structures, *J. Math. Anal. Appl.* **40** (1972), 55–59.
16. G. A. DIRAC, In abstrakten Graphen vorhande vollständige 4-Graphen und ihre Unterteilungen, *Math. Nachr.* **22** (1960), 61–85.
17. A. W. INGLETON, A geometrical characterization of transversal independence structures, *Bull. London Math. Soc.* **3** (1971), 47–51.

18. A. W. INGLETON, Non-base-orderable matroids, in "Proc. Fifth British Comb. Conf." (C. St. J. A. Nash-Williams and J. Sheehan, Eds.), pp. 355-359, Utilitas Math., Winnipeg, 1976.
19. A. W. INGLETON, Transversal matroids and related structures, in "Higher Combinatorics" (M. Aigner, Ed.), pp. 117-131, Reidel, Dordrecht, 1977.
20. A. W. INGLETON AND M. J. PIFF, Gammoids and transversal matroids, *J. Combin. Theory Ser. B* **15** (1973), 51-68.
21. J. KAHN, On the uniqueness of matroid representations over $GF(4)$, submitted.
22. W. M. KANTOR, Envelopes of geometric lattices, *J. Combin. Theory Ser. A* **18** (1975), 12-26.
23. D. G. KELLY AND G.-C. ROTA, Some problems in combinatorial geometry, in "A Survey of Combinatorial Theory" (J. N. Srivastava *et al.*, Eds.), pp. 309-312, North-Holland, Amsterdam, 1973.
24. J. H. MASON, On a class of matroids arising from paths in graphs, *Proc. London Math. Soc.* (3) **25** (1972), 55-74.
25. E. MATHIEU, Mémoire sur l'étude des fonctions de plusieurs quantités, *J. Math. Pures Appl. (Liouville)* **6** (1861), 241-323.
26. J. G. OXLEY, Colouring, packing and the critical problem, *Quart. J. Math. Oxford* (2) **29** (1978), 11-22.
27. J. G. OXLEY, On matroid connectivity, *Quart. J. Math. Oxford* (2) **32** (1981), 193-208.
28. J. G. OXLEY, On 3-connected matroids, *Canad. J. Math.* **33** (1981), 20-27.
29. J. G. OXLEY, On the intersections of circuits and cocircuits in matroids, *Combinatorica* **4** (1984), 187-195.
30. J. G. OXLEY, On non-binary 3-connected matroids, *Trans. Amer. Math. Soc.*, in press.
31. J. G. OXLEY, On the matroids representable over $GF(4)$, *J. Combin. Theory Ser. B* **41** (1986), 250-252.
32. J. G. OXLEY, On ternary transversal matroids, *Discrete Math.* **62** (1986), 71-83.
33. J. G. OXLEY, The binary matroids with no 4-wheel minor, *Trans. Amer. Math. Soc.*, in press.
34. P. D. SEYMOUR, Matroid representation over $GF(3)$, *J. Combin. Theory Ser. B* **26** (1979), 159-173.
35. P. D. SEYMOUR, Decomposition of regular matroids, *J. Combin. Theory Ser. B* **28** (1980), 305-359.
36. W. T. TUTTE, Connectivity in matroids, *Canad. J. Math.* **18** (1966), 1301-1324.
37. P. N. WALTON AND D. J. A. WELSH, On the chromatic number of binary matroids, *Mathematika* **27** (1980), 1-9.
38. D. J. A. WELSH, "Matroid Theory," Academic Press, London, 1976.
39. E. WITT, Die 5-fach transitiven Gruppen von Mathieu, *Abh. Math. Sem. Univ. Hamburg* **12** (1938), 256-264.
40. E. WITT, Über Steinersche Systeme, *Abh. Math. Sem. Univ. Hamburg* **12** (1938), 265-275.