

THE CONTINUOUS WAVELET TRANSFORM AND SYMMETRIC SPACES

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ABSTRACT. The continuous wavelet transform has become a widely used tool in applied science during the last decade. In this article we discuss some generalizations coming from actions of closed subgroups H of $GL(n, \mathbb{R})$ acting on \mathbb{R}^n . If \mathbb{R}^n has finitely many open orbits under the transposed action of H such that the union has full measure, then $L^2(\mathbb{R}^n)$ decomposes into finitely many irreducible representations, $L^2(\mathbb{R}^n) \simeq V_1 \oplus \dots \oplus V_k$ under the action of the semidirect product $H \times_s \mathbb{R}^n$. It is well known, that the space V_j contains an admissible vector if and only if the stabilizer in H^t of every point in V_j is compact. In this article we discuss the case where the stabilizer of a generic point in \mathbb{R}^n is not compact, but a symmetric subgroup, a case that has not previously been discussed in the literature. In particular we show, that the wavelet transform can always be inverted in this case.

INTRODUCTION

The wavelet transform, and more generally time frequency analysis, has become a widely used and studied tool in mathematics, physics, engineering, and applied science during the last decade. One of the interesting aspect is the role played abstract harmonic analysis and representation theory of locally compact groups. In wavelet theory one studies square integrable representations of semidirect products $H \times_s \mathbb{R}^n$, and in time frequency analysis representations of the Heisenberg group are used to understand Gabor frames build from a lattice $\Gamma \subset \mathbb{R}^{2d}$.

The best known example for the continuous wavelet transform deals with wavelets on the real line. In the language of representation theory the *matrix coefficients*, i.e., inner products with translations and dilations of the wavelet, defines a wavelet transform which can be used to reconstruct the function from the dilations and translations of the wavelet. This process helps compensate for the local–nonlocal behavior of the Fourier transform.

Translations and dilations form the so-called $ax + b$ -group of transformations of the real line. These act in a natural manner as unitary operators on $L^2(\mathbb{R})$. In that way, wavelet transforms are simply a part of the representation theory of the $ax + b$ -group. This observation is the basis for the generalization of the continuous wavelet transform to higher dimensions and more general settings, and was already made by A. Grossmann, J. Morlet, and T. Paul in 1985, see [19, 20]. In [19] the connection to square integrable representations and the relation to the fundamental paper of M. Dufflo and C. C. Moore

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[9] was already pointed out. Since then several people have worked on wavelets related to actions of topological groups acting on \mathbb{R}^n . Without trying to be complete we would like to name the work of S.T. Ali, J.-P. Antoine, and J.-P. Gazeau, [2, 3], D. Bernier and Taylor [7], H. Für and Führ and Mayer [13, 14, 15, 16], and finally Laugesen, Weaver, Weiss, and Wilson [24]. In most of these cases the group generalizing the $ax + b$ -group is a semidirect product $H \times_s \mathbb{R}^n$ where H is a closed subgroup of $\mathrm{GL}(n, \mathbb{R})$ and the group H is not abelian. But the abelian groups have turned up as important tools for the *discrete* wavelet transform and in generalizations of Gabor analysis, [1, 6, 5, 11, 17, 21, 29], and for Gabor analysis on \mathbb{R}^n the Heisenberg group and its representations show up in a natural way [10, 18]. We would finally like to point out to the reader the nice overview articles [22, 30] and the special volume, *Journal of Mathematical Physics*, **39**, (8), August 1998, where several aspects of the theory are discussed. For further directions see [3].

In this article we review some of the basic ideas for the general wavelet transform for groups acting on \mathbb{R}^n . We start by reviewing the classical one dimensional wavelet transform as motivation. In this simple setting, the usual definition of a wavelet is equivalent to the corresponding matrix coefficients being square integrable on the $ax + b$ -group. A natural generalization of this to an arbitrary representation (π, \mathbb{H}) of a topological group G is to say, that a vector $u \in \mathbb{H} \setminus \{0\}$ is a wavelet, or an admissible vector, if $a \mapsto (v | \pi(a)u)$ is square integrable for all $v \in \mathbb{H}$. The generalized wavelet transform is then $W_u(v)(a) := (v | \pi(a)u)$. The basic facts for this transform and, in particular, the inversion formula are presented in section 2.

In section 3 these notions are applied to topological groups H acting on \mathbb{R}^n by a representation π . Let $G = H \times_\pi \mathbb{R}^n$ be the semi-direct product of H and \mathbb{R}^n . Then G acts on \mathbb{R}^n and $L^2(\mathbb{R}^n)$. When the action of H on \mathbb{R}^n is sufficiently well behaved, the decomposition of $L^2(\mathbb{R}^n)$ under G can be described in terms of the orbits in \mathbb{R}^n under the transpose (contragredient) representation π' where $\pi'(a) = \pi(a^{-1})^t$. In general, one may not even have irreducible subrepresentations inside $L^2(\mathbb{R}^n)$, but in the case when there are open orbits, one has sufficiently many irreducible subspaces to decompose all functions in $L^2(\mathbb{R}^n)$. In this situation, those irreducible subspaces corresponding to orbits which have compact stabilizers will be precisely those subspaces for which wavelets exist. Some typical examples come from identifying the nilradical of a parabolic subalgebra $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ in a semisimple Lie algebra \mathfrak{g} with \mathbb{R}^n . Several such examples can be found in the literature [8].

Thus an interesting situation occurs when \mathbb{R}^n decomposes up to set of measure zero into finitely many open orbits $\{\mathcal{O}_i\}_{i=1}^k$ under $\pi(H)^t$. In this case, $L^2(\mathbb{R}^n) = \bigoplus_{i=1}^k L^2_{\mathcal{O}_i}(\mathbb{R}^n)$ where $L^2_{\mathcal{O}_i}(\mathbb{R}^n) = \left\{ f \in L^2(\mathbb{R}^n) \mid \hat{f}|_{\mathbb{R}^n \setminus \mathcal{O}_i} = 0 \right\}$ gives a decomposition of $L^2(\mathbb{R}^n)$ into irreducible subspaces. As mentioned earlier and as will be seen in section 3, the subspace $L^2_{\mathcal{O}_i}(\mathbb{R}^n)$ contains a wavelet vector in the above sense if and only if the stabilizers of points in \mathcal{O}_i are compact. In each of these cases, there will be wavelet transforms. The question then becomes what can be done in the general case where the stabilizer of a point in the orbit \mathcal{O}_i is non-compact. This situation occurs already in the simple example of the group

$\mathbb{R}^+ \text{SO}_o(1, n)$ acting on \mathbb{R}^n ; this we discuss in detail in section 5. It falls into the category where one has symmetric orbits, a case we show how to handle.

Let $x \in \mathcal{O}_i$ and $L = \{a \in H \mid \pi(a)^t x = x\}$. Assume that the group H is reductive and that there exists an involution $\tau : H \rightarrow H$ such that $H_o^\tau \subset L \subset H^\tau = \{a \in H \mid \tau(a) = a\}$. Then using a classical result of Matsuki [27, 28], one can show that there exist a closed group $R = R_i \subset H$ and a finite set $x_1, \dots, x_r \in \mathcal{O}_i$ such that each $\pi(R)^t x_q$ is open, $\pi(R)^t x_q \cap \pi(R)^t x_{q'} = \emptyset$ ($q \neq q'$), $\pi(R)^t x_1 \cup \dots \cup \pi(R)^t x_r$ is dense in \mathcal{O}_i , and the stabilizer of each x_q in R is compact (cf. [28]). These ideas are used in section 4 to decompose each of the G irreducible spaces $L_{\mathcal{O}_i}^2(\mathbb{R}^n)$ into irreducible subspaces under $G_i = R_i \times_\pi \mathbb{R}^n$ having G_i -wavelet vectors. These vectors and the results from section 3 can then be used to construct a wavelet transform and an inversion formula for the wavelet transform. We give simple examples of this in the final section.

Lastly, we hope that presenting a relatively accessible account of many of the fundamental concepts in continuous wavelet theory will be useful to the nonspecialist and those using wavelet analysis in applications.

1. WAVELETS AND THE $ax + b$ -GROUP

The $ax + b$ group is the group of affine transformations $T_{a,b}$ acting on \mathbb{R} by dilation by $a > 0$ followed by translation by b . Thus

$$T_{a,b}x = ax + b.$$

Note

$$T_{a,b}T_{a',b'}x = aa'x + ab' + b = T_{aa',b+ab'}$$

and

$$T_{a,b}^{-1} = T_{\frac{1}{a}, -\frac{b}{a}}.$$

We identify (a, b) where $a > 0$ and $b \in \mathbb{R}$ to the transformation $T_{a,b}$ and take G to be collection of all such pairs. The action of G on \mathbb{R} “induces” a natural action ρ of G on the space of L^2 functions on \mathbb{R} . This action is called the left *quasi-regular representation* of G on \mathbb{R} . It is defined by

$$\rho(a, b)f(x) = J(T_{a,b}^{-1})^{\frac{1}{2}}f(T_{a,b}^{-1}x) = a^{-\frac{1}{2}}f\left(\frac{x-b}{a}\right)$$

where $J(\cdot)$ denotes the Jacobian of the transformation. If we also allow a to be negative the action has to be replaced by $|a|^{-1/2} f\left(\frac{x-b}{a}\right)$. The presence of J is to force the natural action to be unitary. Because of the composition laws in G , $\rho(a, b)\rho(a', b') = \rho((a, b)(a', b'))$. The space $L^2(\mathbb{R})$ under the action ρ is known to contain precisely two closed proper invariant subspaces. Thus to decompose a function $f \in L^2(\mathbb{R})$ into “wavelets”, one needs to know how one can project f into each of these subspaces. The wavelet transform and the square integrability of ρ provides the key ingredient. For completeness, we describe the decomposition of $L^2(\mathbb{R})$ into its two closed proper invariant subspaces. The argument we

give is elementary; it decomposes $L^2(\mathbb{R})$ into a the sum $H_+^2 \oplus H_-^2$, the classical Hardy spaces of functions f satisfying $\hat{f}(\pm y) = 0$ for $y < 0$.

Theorem 1.1. *The two subspaces H_+^2 and H_-^2 are the only proper closed invariant subspaces under ρ and $L^2(\mathbb{R}) = H_+^2 \oplus H_-^2$.*

Proof. The Fourier transform $f \mapsto \hat{f}$ given formally by

$$\hat{f}(y) = \frac{1}{\sqrt{2\pi}} \int f(x) e^{-ixy} dx$$

is a linear isometry from $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$. If one defines $\hat{\rho}(a, b)$ by

$$\hat{\rho}(a, b)\hat{f} = \widehat{\rho(a, b)f},$$

then

$$\begin{aligned} \hat{\rho}(a, b)\hat{f}(y) &= \frac{1}{\sqrt{2\pi}} \int a^{-1/2} f(a^{-1}(x-b)) e^{-ixy} dx \\ &= \frac{1}{\sqrt{2\pi}} \int a^{-\frac{1}{2}} f(a^{-1}x) e^{-i(x+b)y} dx \\ (1) \quad &= \frac{1}{\sqrt{2\pi}} e^{-iby} \int a^{-\frac{1}{2}} f(x) e^{-iaxy} d(ax) \\ &= \frac{1}{\sqrt{2\pi}} a^{\frac{1}{2}} e^{-iby} \int f(x) e^{-iaxy} dx \\ &= \sqrt{a} e^{-iby} \hat{f}(ay). \end{aligned}$$

Since $a > 0$, this implies that the subspaces \hat{H}_+^2 and \hat{H}_-^2 are invariant under $\hat{\rho}$ and $\hat{H}_+^2 \oplus \hat{H}_-^2 = \hat{L}_2(\mathbb{R})$.

Let S be a closed invariant subspace of $\hat{L}_2(\mathbb{R})$. Note if $h \in L^1(\mathbb{R})$ and $\hat{f} \in S$, then the vector

$$M_{\hat{h}}\hat{f}(y) \equiv \frac{1}{\sqrt{2\pi}} \int h(b)\rho(1, b)\hat{f}(y) db = \frac{1}{\sqrt{2\pi}} \int h(b)e^{-iby}\hat{f}(y) db = \hat{h}(y)\hat{f}(y)$$

is a limit in $\hat{L}_2(\mathbb{R})$ of linear combinations of vectors of form $\rho(1, b_k)f$ and thus is in S . Hence $\hat{h}\hat{f} \in S$ for any $h \in L^1(\mathbb{R})$. Since $|\hat{h}|_\infty \leq |h|_1$, one can argue that given any $\hat{f} \in S$ and any continuous bounded function h , one has $h\hat{f} \in S$. Consequently $h\hat{f} \in S$ for any bounded measurable function h . In particular, if P is an orthogonal projection commuting with all the operators $\hat{\rho}(1, b)$, P commutes with all multiplication operators M_h and thus must have form M_{χ_E} where E is some measurable set; i.e., $Pf = \chi_E f$ for

all $f \in \hat{L}_2(\mathbb{R})$. If P is the orthogonal projection onto S , we see since

$$\begin{aligned} \sqrt{a}\chi_E(ay)f(ay) &= \sqrt{a}Pf(ay) \\ &= \hat{\rho}(a, 1)P\hat{f}(y) \\ &= P\hat{\rho}(a, 1)\hat{f}(y) \\ &= \chi_E(y)\sqrt{a}\hat{f}(ay), \end{aligned}$$

that $\chi_{a^{-1}E} = \chi_E$ for all $a > 0$. Thus E is essentially the empty set, the set $(-\infty, 0]$, the set $[0, \infty)$, or \mathbb{R} . This shows $L^2(\mathbb{R})$ has precisely four closed invariant subspaces, the only proper ones being H_+^2 and H_-^2 . \square

The following discussion on wavelets can be extended to all of $L^2(\mathbb{R})$, but we will concentrate on the Hardy space H_+^2 , for the analysis is the same in the other case. We normalize the Haar measure on G to be

$$dg = \frac{dad b}{2\pi a^2}.$$

Definition 1.2. Let ψ be a nonzero vector in H_+^2 . Then ψ is called a **wavelet** if $\hat{\psi} \in L^2(\mathbb{R}, \frac{dt}{|t|})$.

Let ψ be a wavelet. By (1), $\rho(a, b)\psi$ is a wavelet for each (a, b) . If $f \in L^2(\mathbb{R})$, then the *wavelet transform* of f is defined by

$$W_\psi f(a, b) := (f | \rho(a, b)\psi) = a^{-1/2} \int_{-\infty}^{\infty} f(t)\psi\left(\frac{t-b}{a}\right) dt.$$

The function $G \ni (a, b) \mapsto Wf(a, b)$ is continuous on G .

Note if ψ is any nonzero element in H_+^2 , then the vectors $\rho(a, b)\psi$ span a dense subspace of H_+^2 . It is well known that any function f can be recovered from the wavelet transform $W_\psi f$. The key is the square integrability of the wavelet transform. As the Fourier transform is an unitary isomorphism we get the following lemma, cf. Lemma 3.1

Lemma 1.3. *Let the notation be as above. Then*

$$W_\psi f(a, b) = a^{1/2} \int \hat{f}(\omega)\overline{\hat{\psi}(a\omega)} e^{i\omega b} d\omega.$$

This implies the following theorem, cf. 3.4:

Theorem 1.4. *Let ψ be a wavelet. Set $C_\psi^2 = \int_0^\infty \frac{|\hat{\psi}(\omega)|^2}{\omega} d\omega > 0$. Then for any $f \in H_+^2$, $W_\psi f$ is in $L^2(G)$. Moreover, for $f, h \in H_+^2$,*

$$(W_\psi f, W_\psi g) = C_\psi^2(f, g).$$

Recall that if (π, \mathbb{H}) and (ρ, \mathbb{K}) are two representations of a group G , then a continuous map $T : \mathbb{H} \rightarrow \mathbb{K}$ is called an *intertwining operator* if for all $x \in G$ we have $T\pi(x) = \rho(x)T$. We can then describe the wavelet transform W_ψ as an intertwining operator $H_+^2 \rightarrow L^2(G)$:

Lemma 1.5. *Let $\psi \in H_+^2$ be a wavelet. Then*

$$H_+^2 \ni f \mapsto W_\psi f \in L^2(G)$$

is continuous with norm $C_\psi > 0$. Furthermore, W_ψ intertwines the representation ρ and the left regular representation L on G ; i.e., $W_\psi \rho(a, b) = L(a, b)W_\psi$ for all $(a, b) \in G$.

This lemma is the basis for understanding and generalizing wavelet transforms to other groups. The reproducing property and inversion formula for the wavelet transform are simple consequences of intertwining the representation with the regular representation and the irreducibility of the representation on H_+^2 . We discuss this general framework in the next section.

2. WAVELETS AND SQUARE INTEGRABLE REPRESENTATIONS

In this section G denotes a locally compact Hausdorff topological group. We fix a left Haar measure $d\mu = dx$ for G . We define a *wavelet vector*, or *admissible vector* with respect to an unitary representation of G in the following way:

Definition 2.1. Suppose (ρ, \mathbb{H}) is an irreducible unitary representation of G . Then a **wavelet** ψ for ρ is a nonzero vector such that $\rho_{v, \psi}(g) := (v | \rho(g)\psi)$ is in $L^2(G)$ for all $v \in \mathbb{H}$.

Lemma 2.2. *Assume ψ is a wavelet for the irreducible representation ρ on the Hilbert space V_ρ . Then the map $W_\psi : V_\rho \rightarrow L^2(G)$ sending v to $\rho_{v, \psi}$ is a continuous intertwining operator.*

Proof. This linear mapping has closed graph. Hence it is continuous. We show that W_ψ is intertwining. Indeed,

$$W_\psi(\rho(x)u)(y) = (\rho(x)u | \rho(y)\psi) = (u | \rho(x^{-1}y)\psi) = L_x W_\psi u(y).$$

□

Remark. Duflo and Moore [9] showed for any ψ , $\rho_{u, \psi}$ is square integrable for all u if it is square integrable for one nonzero u and defined an irreducible representation to be square integrable if it has a nonzero square integrable matrix coefficient. Hence the wavelets for ρ are the ψ which give square integrable matrix coefficients. Moreover, since $\|\rho_{u, \rho(g)\psi}\|_2^2 = \Delta(g)^{-1} \|\rho_{u, \psi}\|_2^2$, the wavelets form an invariant and hence dense linear subspace of V_ρ .

The special case where $G = P = HN$ is a *parabolic subgroup* in a semisimple Lie group is discussed in detail in [25, 26, 31]. Here H is the Levi-factor of P and N is the nilradical. In particular, H is reductive and acts on the Lie algebra of N , which in many situations is isomorphic to \mathbb{R}^n . These special cases are important for our discussions on the generalization of the wavelet transform.

Definition 2.3. Let X be a set and let \mathbb{H} be a Hilbert space of functions $f : X \rightarrow \mathbb{V}$ where \mathbb{V} is a Hilbert space. The space \mathbb{H} is called a **reproducing Hilbert space** if the evaluation maps $\text{ev}_x : \mathbb{H} \rightarrow \mathbb{V}$, $f \mapsto f(x)$, are continuous for all $x \in X$.

Lemma 2.4. *Let \mathbb{H} be a reproducing Hilbert space. Define $K(x, y) := \text{ev}_x \text{ev}_y^* : \mathbb{V} \rightarrow \mathbb{V}$. Let $K_y(x) := K(x, y)$. Then the following hold:*

- (a) $K(x, y) : \mathbb{V} \rightarrow \mathbb{V}$ is continuous and linear; i.e., $K(x, y) \in \text{Hom}(\mathbb{V}, \mathbb{V})$
- (b) $K_y(\cdot)v \in \mathbb{H}$ for all $v \in \mathbb{V}$, and $(f | K_y v) = (f(y) | v)$ for all $f \in \mathbb{H}$ and all $v \in \mathbb{V}$.
- (c) The set of finite linear combinations $\sum c_j K_{y_j}(\cdot)v_j$ is dense in \mathbb{H} .
- (d) $K(x, y)^* = K(y, x)$

Proof. As $\text{ev}_x : \mathbb{H} \rightarrow \mathbb{V}$ is continuous and linear for all $x \in X$, it follows that $\text{ev}_x^* : \mathbb{V} \rightarrow \mathbb{H}$ is continuous and linear. Hence $K(x, y) : \mathbb{V} \rightarrow \mathbb{V}$ is continuous and linear. Let $v \in \mathbb{V}$ and $y \in X$. As $K(\cdot, y)v = \text{ev}_y^* v \in \mathbb{H}$ for all $y \in \mathbb{V}$, we see:

$$\begin{aligned} (f(y) | v) &= (\text{ev}_y f | v) \\ &= (f | \text{ev}_y^* v) \\ &= (f | K(\cdot, y)v). \end{aligned}$$

Hence (b) follows. Assume that $f \in \mathbb{H}$ is perpendicular to all the linear combinations $\sum c_j K_{y_j}(\cdot)v_j$. Then in particular

$$(f(y) | v) = (f | K(\cdot, y)v) = 0$$

for all $y \in X$ and all $v \in \mathbb{V}$. Hence $f(y) = 0$ for each y and thus $f = 0$. So (c) follows. Finally (d) follows directly from the definition as

$$K(x, y)^* = (\text{ev}_x \text{ev}_y^*)^* = \text{ev}_y \text{ev}_x^* = K(y, x).$$

□

Definition 2.5. The map $K : X \times X \rightarrow \text{Hom}(V, V)$ is called the **reproducing kernel** of \mathbb{H} .

Let (π, \mathbb{H}) be a unitary representation of G and let f be a measurable function on G such that the integral $\int_G f(x)(\pi(x)u | v) dx$ converges for all $u, v \in \mathbb{H}$ and for which there is a constant C with $|\int_G f(x)(\pi(x)u | v) dx| \leq C\|u\|\|v\|$. Then we can define a continuous linear map $\pi(f) : \mathbb{H} \rightarrow \mathbb{H}$ by

$$(\pi(f)u | v) = \int_G f(x)(\pi(x)u | v) dx.$$

Theorem 2.6. *Let (ρ, \mathbb{H}) and (τ, \mathbb{K}) be square integrable representations with wavelet vectors ψ and η , respectively. Then the following hold:*

- (a) If ρ and τ are not equivalent, then $(W_\psi u | W_\eta v) = 0$ for all $u \in V_\rho$ and all $v \in V_\tau$. Thus the images of W_ψ and W_η are orthogonal subspaces of $L^2(G)$.
- (b) Let $f \in L^2(G)$. Then $\rho(f)\psi$ is defined and $W_\psi^* f = \rho(f)\psi$.
- (c) If $\rho = \tau$, then there exists a constant $C_{\psi, \eta}$ such that

$$(W_\psi u | W_\eta v) = C_{\psi, \eta}(u | v)$$

for all $u, v \in V_\rho$.

- (d) $\text{Im}(W_\psi)$ is a closed reproducing Hilbert subspace of $L^2(G)$ contained in $L^2(G) \cap C(G)$.
- (e) We have $C_{\psi,\psi} > 0$. Let $c_\psi = 1/\sqrt{C_{\psi,\psi}} > 0$. Define $U_\psi : \mathbb{H} \rightarrow L^2(G)$ by $u \mapsto c_\psi W_\psi u = W_{c_\psi \psi} u$. Then U_ψ is an unitary isomorphism $\mathbb{H} \simeq \text{Im}(W_\psi)$.
- (f) Assume that $\rho = \tau$ and that $C_{\psi,\eta} \neq 0$. Then

$$u = \frac{1}{C_{\psi,\eta}} \rho(W_\psi u) \eta = \frac{1}{C_{\psi,\eta}} \int_G (u \mid \rho(x)\psi) \rho(x)\eta \, dx,$$

for all $u \in V_\rho$. In particular,

$$u = \frac{1}{C_{\psi,\psi}} \rho(W_\psi u) \psi = \frac{1}{C_{\psi,\psi}} \int_G (u \mid \rho(x)\psi) \rho(x)\psi \, dx.$$

- (g) The reproducing kernel for $\text{Im}(W_\psi)$ is given by $K(x, y) = C_{\psi,\psi}^{-1} W_\psi \psi(y^{-1}x)$
- (h) If $f \in \text{Im}(W_\psi)$, then $f = C_{\psi,\psi}^{-1} f * W_\psi(\psi)$.

Proof. Define $\beta : \mathbb{H} \times \mathbb{K} \rightarrow \mathbb{C}$ by

$$\beta(u, v) := (W_\psi u \mid W_\eta v) = \int W_\psi u(x) \overline{W_\eta v(x)} \, dx.$$

As the wavelet transform is continuous, it follows that

$$|\beta(u, v)| = |(W_\psi u \mid W_\eta v)| \leq \|W_\psi u\| \|W_\eta v\| \leq \|W_\psi\| \|W_\eta\| \|u\| \|v\|.$$

Consequently, there is a continuous linear map $T : \mathbb{H} \rightarrow \mathbb{K}$ such that

$$\beta(u, v) = (Tu \mid v).$$

Let $a \in G$. Then, using that the wavelet transform is intertwining,

$$\begin{aligned} (T\rho(a)u \mid v) &= \beta(\rho(a)u, v) \\ &= \int [W_\psi \rho(a)u](x) \overline{W_\eta v(x)} \, dx \\ &= \int W_\psi u(a^{-1}x) \overline{W_\eta v(x)} \, dx \\ &= \int W_\psi u(x) \overline{W_\eta v(ax)} \, dx \\ &= \int W_\psi u(x) \overline{[W_\eta \tau(a^{-1})v](x)} \, dx \\ &= \beta(u, \tau(a^{-1})v) \\ &= (Tu \mid \tau(a)^*v) \\ &= (\tau(a)Tu \mid v). \end{aligned}$$

As this holds for all u and v , it follows that $T\rho(a) = \tau(a)T$. Thus T is an intertwining operator. It follows by Schur's Lemma that T is zero when ρ is not equivalent to τ , and $T = C_{\psi,\eta}I$ for some scalar when $\rho = \tau$. Moreover, if $\psi = \eta$, then $T > 0$ and thus $C_{\psi,\psi} > 0$.

Consequently, $\frac{1}{\sqrt{C_{\psi,\psi}}}W_\psi$ is an isometry of V_ρ onto its image in $L^2(G)$. Hence (a), (c), and (e) hold.

Let $f \in L^2(G)$ and $u \in V_\rho$. Then

$$\begin{aligned} (W_\psi^* f | u) &= (f | W_\psi u) \\ &= \int f(x) \overline{(u | \rho(x)\psi)} dx \\ &= \int f(x) (\rho(x)\psi | u) dx \\ &= (\rho(f)\psi | u) \end{aligned}$$

Hence $\rho(f)\psi$ is weakly defined and (b) holds.

(d) The functions $f = W_\psi u \in \text{Im}(W_\psi)$ are continuous functions. In particular, $\text{ev}_a(f) = f(a)$ is well defined on $\text{Im}(W_\psi)$. Since U_ψ is an isometry,

$$\begin{aligned} |\text{ev}_a(f)| &= |W_\psi u(a)| \\ &= |(u, \rho(a)\psi)| \\ &\leq \|u\| \|\psi\| \\ &\leq \|U_\psi u\| \|\psi\| \\ &= c_\psi \|W_\psi u\| \|\psi\| \\ &= c_\psi \|\psi\| \|f\|_2. \end{aligned}$$

Hence point evaluations are continuous linear mappings.

Note (f) follows from (b) and (c) for:

$$\begin{aligned} \frac{1}{C_{\psi,\eta}}(\rho(W_\psi u)\eta | v) &= \frac{1}{C_{\psi,\eta}}(W_\eta^*(W_\psi u)\eta | v) \\ &= \frac{1}{C_{\psi,\eta}}(W_\psi u | W_\eta v) \\ &= (u | v). \end{aligned}$$

Note if $f = W_\psi u$, then

$$\begin{aligned}
 (2.1) \quad f(y) &= W_\psi u(y) \\
 &= (u \mid \rho(y)\psi) \\
 &= \frac{1}{C_{\psi,\psi}} (\rho(W_\psi u)\psi \mid \rho(y)\psi) \\
 &= \frac{1}{C_{\psi,\psi}} \int W_\psi u(x) (\rho(x)\psi \mid \rho(y)\psi) dx \\
 &= \frac{1}{C_{\psi,\psi}} \int f(x) (\rho(y^{-1}x)\psi \mid \psi) dx \\
 &= \frac{1}{C_{\psi,\psi}} \int f(x) \overline{W_\psi \psi(y^{-1}x)} dx.
 \end{aligned}$$

Hence $K(x, y) = W_\psi \psi(y^{-1}x)$ is the reproducing kernel for the image of W_ψ .

Moreover, 2.1 shows $f(y) = C_{\psi,\psi}^{-1} \int f(x) (\psi, \rho(x^{-1}y)\psi) dx = C_{\psi,\psi}^{-1} f * W_\psi(\psi)(y)$. Thus (g) and (h) follow. \square

3. GENERALIZATIONS OF THE WAVELET TRANSFORM

Let H be a locally compact Hausdorff topological group, and let $\pi : H \rightarrow \text{GL}(n, \mathbb{R})$ be a continuous homomorphism. Then we can define the semi-direct product $G = H \times_\pi \mathbb{R}^n$ where the product is given by

$$(a, x)(b, y) = (ab, x + \pi(a)y).$$

The group G is locally compact; and if dh is a left Haar measure on H and dx is the standard Lebesgue measure on \mathbb{R}^n , then a left Haar measure on G is given by

$$d\mu_G(h, x) = \frac{\Delta_\pi(h)}{(2\pi)^n} dh dx$$

where

$$\Delta_\pi(h) = |\det(\pi(h))|^{-1}.$$

Indeed, for continuous functions f with compact support in G ,

$$\begin{aligned}
\int f((k, y)(h, x)) d\mu_G(h, x) &= (2\pi)^{-n} \int \int f(kh, y + \pi(k)x) |\det(\pi(k^{-1}kh))|^{-1} dh dx \\
&= (2\pi)^{-n} \int \int f(h, y + \pi(k)x) |\det(\pi(k^{-1}h))|^{-1} dh dx \\
&= (2\pi)^{-n} \iint f(h, y + x) |\det(\pi(h))^{-1} \det(\pi(k))| d(\pi(k)^{-1}x) dh \\
&= (2\pi)^{-n} \iint f(h, y + x) |\det(\pi(h))|^{-1} dx dh \\
&= (2\pi)^{-n} \iint f(h, x) |\det(\pi(h))|^{-1} dx dh \\
&= \int f(h, x) d\mu_G(h, x).
\end{aligned}$$

As before, the action of G on \mathbb{R}^n given by

$$T_{h,x}(y) = \pi(h)y + x$$

induces a unitary representation ρ of G on $L^2(\mathbb{R}^n)$. Namely,

$$\rho(h, x)f(y) = J(T_{h,x}^{-1})^{\frac{1}{2}}f(T_{h,x}^{-1}y) = \Delta_\pi(h)^{1/2}f(\pi(h)^{-1}(y - x)).$$

If $x \cdot y = \sum_{i=1}^n x_i y_i$ is the usual inner product on \mathbb{R}^n , then the Fourier transform defined formally by

$$\hat{f}(y) = \frac{1}{(2\pi)^{n/2}} \int f(x) e^{-ix \cdot y} dx$$

is a unitary mapping from $L^2(\mathbb{R}^n)$ onto itself.

To analyze the representation ρ , we again look at its Fourier transform $\hat{\rho}$. It is given by

$$\hat{\rho}(h, x)\hat{f} = \widehat{\rho(h, x)f}.$$

Lemma 3.1. *Let $f \in L^2(\mathbb{R}^n)$. Denote by $\pi(h)^t$ the transpose of the matrix $\pi(h)$ in $\text{GL}(n, \mathbb{R})$. Then*

$$\hat{\rho}(h, x)\hat{f} = \widehat{\rho(h, x)f} = \Delta_\pi(h)^{-1/2} e^{-ix \cdot y} \hat{f}(\pi(h)^t y).$$

Proof. This is a simple calculation:

$$\begin{aligned}
\hat{\rho}(h, x)\hat{f}(y) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \rho(h, x)f(u)e^{-iu \cdot y} du \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \Delta_\pi(h)^{1/2} f(\pi(h^{-1})(u-x))e^{-iu \cdot y} du \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \Delta_\pi(h)^{1/2} f(\pi(h^{-1})u)e^{-i(u+x) \cdot y} du \\
&= \frac{1}{(2\pi)^{n/2}} e^{-ix \cdot y} \int_{\mathbb{R}^n} \Delta_\pi(h)^{1/2} f(\pi(h^{-1})u)e^{-iu \cdot y} du \\
&= \frac{1}{(2\pi)^{n/2}} e^{-ix \cdot y} \int_{\mathbb{R}^n} \Delta_\pi(h)^{1/2} f(u)e^{-i\pi(h)u \cdot y} d(\pi(h)u) \\
&= \frac{1}{(2\pi)^{n/2}} e^{-ix \cdot y} \int_{\mathbb{R}^n} \Delta_\pi(h)^{1/2} |\det(\pi(h))| f(u)e^{-iu \cdot \pi(h)^t y} du \\
&= \Delta_\pi(h)^{-1/2} e^{-ix \cdot y} \hat{f}(\pi(h)^t y).
\end{aligned}$$

□

Let A be a measurable subset of \mathbb{R}^n with positive measure. Let $L_A^2(\mathbb{R}^n)$ be the closed linear subspace of $L^2(\mathbb{R}^n)$ consisting of those L^2 functions with $\hat{f}(y) = 0$ for $y \notin A$. The subspaces $L_A^2(\mathbb{R}^n)$ are precisely the closed subspaces of $L^2(\mathbb{R}^n)$ invariant under translation.

Definition 3.2. A $\pi(H)^t$ invariant measurable subset A of \mathbb{R}^n is ergodic (under Lebesgue measure) if any invariant measurable subset of A has measure 0 or has complement in A with measure 0.

In particular, all homogeneous spaces obtained from the action of $\pi(H)^t$ are ergodic.

The action of $\pi(H)^t$ on \mathbb{R}^n is said to have a **measurable cross section** if there is a Borel set E in \mathbb{R}^n meeting each set $\pi^t(H)y$ in precisely one point.

For completeness we present the following well known lemma.

Lemma 3.3. *Suppose there is a Borel cross section E for the action of $\pi(H)^t$ on \mathbb{R}^n and A is an ergodic invariant Borel measurable set. Then A contains a homogeneous space whose complement in A has measure 0.*

Proof. Denote by L the subgroup $\pi(H)^t$ of $\text{GL}(n, \mathbb{R})$. There is nothing to do if A has measure 0. Suppose A has positive measure. Let μ be a probability measure equivalent to Lebesgue measure on A . Set $F = E \cap A$ where E is a Borel set meeting each homogeneous space Ly in exactly one point. Define $p : A \rightarrow F$ by $\{p(y)\} = Ly \cap F$. Then p is a Borel function from A onto F . Let \mathcal{C} be a countable algebra of Borel subsets of F which separate points in F . Set $\mathcal{C}_0 = \{W \in \mathcal{C} \mid \mu(p^{-1}(W)) > 0\}$. Since $p^{-1}(W)$ are invariant sets with positive measure, their complements in A have measure 0. Hence $\cap p^{-1}(W)$ has complement in A with measure 0. Thus $\cap p^{-1}(W)$ has positive measure. Suppose it contains two disjoint L -orbits Ly and Ly' . This implies $\cap W$ contains distinct points $p(y)$

and $p(y')$. Since \mathcal{C} is a separating family, there is a $U \in \mathcal{C}$ with $p(y) \in U$ and $p(y') \notin U$. Hence U or $F - U$ belongs to \mathcal{C}_0 . If $U \in \mathcal{C}_0$, then $p(y') \notin \cap W$ and if $F - U \in \mathcal{C}_0$, then $p(y) \notin \cap W$. Thus $\cap p^{-1}(W)$ is a homogeneous space whose complement in A has measure 0. \square

Theorem 3.4. *Let $\pi : H \rightarrow \text{GL}(n, \mathbb{R})$ be a continuous representation as before. Then the following hold:*

- (a) *A nonzero closed subspace M of $L^2(\mathbb{R}^n)$ is invariant under the representation ρ if and only if $M = L^2_A(\mathbb{R}^n)$ for some measurable $\pi(H)^t$ invariant subset A of \mathbb{R}^n having positive measure. Moreover, this subspace is irreducible if and only if A is ergodic.*
- (b) *If the action of $\pi(H)^t$ on \mathbb{R}^n has a Borel cross section, then every Borel measurable invariant ergodic subset A of \mathbb{R}^n contains a $\pi(H)^t$ homogeneous subset whose complement in A has measure 0. In particular, if H is a Lie group, the irreducible subspaces of ρ correspond to the open $\pi(H)^t$ homogeneous subspaces of \mathbb{R}^n .*
- (c) *Let A be ergodic under $\pi(H)^t$ with positive measure. Assume ψ is a nonzero vector in $L^2_A(\mathbb{R}^n)$. Then ψ is a wavelet vector if and only if $h \mapsto \hat{\psi}(\pi(h)^t y)$ is in $L^2(H)$ for almost all $y \in A$. In particular, the stabilizer H^y is compact for almost all $y \in A$.*
- (d) *Assume A is open and homogeneous under $\pi(H)^t$ and the stabilizer H^y is compact for one (and hence all) $y \in A$. Then any nonzero ψ with $\hat{\psi} \in C_c(A)$ is a wavelet vector and the linear space of wavelet vectors is dense in $L^2_A(\mathbb{R}^n)$.*

Proof. Let M be a closed nonzero invariant subset of $L^2(\mathbb{R}^n)$. Set $S = \hat{M}$. By Lemma 3.1, S is invariant under the unitary operators $\hat{\rho}(e, x)\hat{f}(y) = e^{-ix \cdot y}\hat{f}(y)$. Now every multiplication operator $M_h f = hf$ for $f \in L^2(\mathbb{R}^n)$ and $h \in L^\infty(\mathbb{R}^n)$ can be weakly approximated by a finite linear combination of the operators $\hat{\rho}(e, x)$. This implies that S is invariant under M_h for all $h \in L^\infty(\mathbb{R}^n)$. In particular, if P is the orthogonal projection onto S , then $PM_h = M_h P$ for all h . Hence $P = M_{\chi_A}$ for some Borel measurable subset A of \mathbb{R}^n . Thus $M = L^2_A(\mathbb{R}^n)$.

Since S is invariant under ρ , we have $\hat{\rho}(h, 0)(\chi_A f) = \chi_A \hat{\rho}(h, 0)f$ for all $f \in L^2(\mathbb{R}^n)$. Thus

$$\begin{aligned} \Delta_\pi(h)^{-\frac{1}{2}} \chi_A(\pi(h)^t y) f(\pi(h)^t(y)) &= \hat{\rho}(h, 0)(\chi_A f)(y) \\ &= \chi_A(y) \hat{\rho}(h, 0) f(y) \\ &= \Delta_\pi(h)^{-\frac{1}{2}} \chi_A(y) f(\pi(h)^t y) \end{aligned}$$

a.e. y for each $h \in H$. Thus $\chi_A(\pi(h)^t y) = \chi_A(y)$ a.e. y for each h . Hence the sets $\pi(h^{-1})^t A$ and A are essentially equal for all h .

Set $A' = \{y \mid \pi(h)^t y \in A \text{ for a.e. } h\}$. By the left invariance of the measure dh , A' is invariant under $\pi^t(H)$. Moreover, by Fubini's Theorem, A' equals A up to a set of measure 0. Replacing A by A' , we see if M is a nonzero closed ρ invariant subspace of

$L^2(\mathbb{R}^n)$, then $M = L^2_A(\mathbb{R}^n)$ for some invariant Borel measurable $\pi(H)^t$ invariant subset of \mathbb{R}^n with positive measure. The converse follows directly from Lemma 3.1.

Next note the action of $\pi(H)^t$ on A is not ergodic if and only if there exists an invariant subset A_0 of A so that both A_0 and $A - A_0$ have positive measure. This occurs if and only if M has a proper invariant subspace of form $L^2_{A_0}(\mathbb{R}^n)$ which is equivalent to M being reducible. We thus have (a).

The first statement in (b) follows from Lemma 3.3. To see the second, suppose $\pi(H)^t y$ is a homogeneous space with positive Lebesgue measure. Note $h \mapsto \pi(h)^t y$ will have range with positive measure in \mathbb{R}^n if and only if the rank of this transformation is n . But this occurs only if $\pi(H)^t y$ is an open subset of \mathbb{R}^n .

To see (c) we set $W_\psi f = (f \mid \rho(h, x)\psi)$. By Lemma 3.1

$$W_\psi f(h, x) = (\hat{f} \mid \hat{\rho}(h, x)\hat{\psi}) = \int_A \hat{f}(\omega) \Delta_\pi(h)^{-1/2} e^{ix \cdot \omega} \overline{\hat{\psi}(\pi(h)^t \omega)} d\omega.$$

Define $F(\omega) = \int_H |\hat{\psi}(\pi(a)^t \omega)|^2 da$ for $\omega \in A$. Note

$$\begin{aligned} F(\pi(h)^t \omega) &= \int_H |\hat{\psi}(\pi(a)^t \pi(h)^t \omega)|^2 da \\ &= \int_H |\hat{\psi}(\pi(ha)^t \omega)|^2 da \\ &= \int_H |\hat{\psi}(\pi(a)^t \omega)|^2 da \\ &= F(\omega). \end{aligned}$$

The invariance of F and the ergodicity of A imply $F(\omega)$ is essentially constant on A ; i.e., there is a constant $C_\psi^2 \leq \infty$ with

$$\int_H |\hat{\psi}(\pi(a)^t \omega)|^2 da = C_\psi^2 \text{ for a.e. } \omega \in A.$$

Assume f is in $L^2_A(\mathbb{R}^n)$. Then

$$\begin{aligned} \int |W_\psi f(h, x)|^2 dx &= \int \int \int \hat{f}(\omega) \overline{\hat{f}(\omega')} \Delta_\pi(h)^{-1} \overline{\hat{\psi}(\pi(a)^t \omega)} \hat{\psi}(\pi(h)^t \omega') e^{ix \cdot \omega} e^{-ix \cdot \omega'} d\omega d\omega' dx \\ &= \Delta_\pi(h)^{-1} \int \int \int \hat{f}(\omega) \overline{\hat{\psi}(\pi(a)^t \omega)} e^{ix \cdot \omega} \overline{\hat{f}(\omega')} \hat{\psi}(\pi(h)^t \omega') e^{-ix \cdot \omega'} d\omega d\omega' dx. \end{aligned}$$

Set $F(\omega) = \overline{\hat{f}(\omega)}\hat{\psi}(\pi(h)^t\omega)$. Then

$$\begin{aligned} \int |W_\psi f(h, x)|^2 dx &= \Delta_\pi(h)^{-1} \int \int \int \hat{f}(\omega)\overline{\hat{\psi}(\pi(a)^t\omega)} e^{ix\cdot\omega} \overline{\hat{f}(\omega')} \hat{\psi}(\pi(h)^t\omega') e^{-ix\cdot\omega'} d\omega d\omega' dx \\ &= (2\pi)^n \Delta_\pi(h)^{-1} \int |\hat{F}(x)|^2 dx \\ &= (2\pi)^n \Delta_\pi(h)^{-1} \int_A |F(\omega)|^2 d\omega \\ &= (2\pi)^n \Delta_\pi(h)^{-1} \int_A |\hat{f}(\omega)|^2 |\hat{\psi}(\pi(h)^t\omega)|^2 d\omega. \end{aligned}$$

Using Fubini's Theorem and

$$\int_H |\hat{\psi}(\pi(h)^t\omega)|^2 dh = C_\psi^2$$

for a.e. ω in A , we see

$$\begin{aligned} \int_{H \times_\pi \mathbb{R}^n} |W_\psi f(h, x)|^2 d(h, x) &= \frac{1}{(2\pi)^n} \int_H \Delta_\pi(h) \int |W_\psi f(h, x)|^2 dx dh \\ &= \int_H \int_A |\hat{f}(\omega)|^2 |\hat{\psi}(\pi(h)^t\omega)|^2 d\omega dh \\ (3.1) \qquad &= \int_A |\hat{f}(\omega)|^2 \int_H |\hat{\psi}(\pi(h)^t\omega)|^2 dh d\omega \\ &= C_\psi^2 \int_A |\hat{f}(\omega)|^2 d\omega \\ &= C_\psi^2(f, f). \end{aligned}$$

Hence ψ is a wavelet vector if and only if $0 < C_\psi^2 < \infty$.

Now suppose $0 < \int_H |\hat{\psi}(\pi(h)^t\omega)|^2 dh < \infty$. Set $F(h) = |\hat{\psi}(\pi(h)^t\omega)|^2$ and $K = H^\omega = \{a : \pi(a)^t\omega = \omega\}$. Note $F(ah) = F(a)$ for $a \in K$. Let $p : H \rightarrow K \setminus H$ be the natural projection and suppose μ is a regular right quasi-invariant measure on $K \setminus H$. Let $\int_{K \setminus H} m_{Kh} d\mu(Kh)$ be the disintegration of left Haar measure m over the fibers of p .

Note

$$\begin{aligned} m(E) &= m(aE) = \int_{K \setminus H} m_{Kh}(aE) d\mu(Kh) \\ &= \int_{K \setminus H} m_{Kh}(E) d\mu(Kh). \end{aligned}$$

Uniqueness of disintegrations yields $m_{Kh}(E) = m_{Kh}(aE)$ for all E for each a a.e. Kh . Hence if $\sigma : K \setminus H \rightarrow H$ is a Borel cross section, then since left Haar measures on K are proportional, we have

$$dm_{Kh}(a\sigma(Kh)) = c(Kh) da$$

where da is a left Haar measure on K and $c(Kh) \geq 0$. Consequently, since $0 < \int_H F(h) dh < \infty$ and

$$\int F(h) dh = \int_{K \setminus H} c(Kh) \int_K F(a\sigma(Kh)) da d\mu(Kh) = \int_K da \int c(Kh) F(\sigma(Kh)) d\mu(Kh),$$

we see Haar measure on K is finite and thus K is compact. We thus conclude H^y is compact for almost all y .

Note (d) is consequence of A is homeomorphic to H/H^y . \square

4. WAVELETS AND SYMMETRIC SPACES

There are, as we will see in the next section, natural examples where \mathbb{R}^n contains finitely many open orbits some of which do not have compact stabilizers. The obvious question is therefore, what do we do in these cases? One possible solution is given in [2] where the authors propose to parameterize the wavelets by *homogeneous spaces* G/L . This idea was used in [4] to define a wavelet transform for the n -dimensional sphere S^n using so called principal series representations of the group $G = \mathrm{SO}(1, n + 1)$ and a homogeneous space G/N where N is a particular maximal unipotent subgroup of G . In this section we will discuss another solution to this problem in the case where the group H is reductive.

Let us recall that the compactness of the stabilizer was used at one point in the proof of Theorem 3.4 to allow us to use Fubini's Theorem to interchange integration over A and H , see (3.1):

$$\int_H \int_A |\hat{f}(\omega)|^2 |\hat{\psi}(\pi(h)^t \omega)|^2 d\omega dh = \int_A |\hat{f}(\omega)|^2 \int_H |\hat{\psi}(\pi(h)^t \omega)|^2 dh d\omega.$$

We notice that the right hand side makes perfect sense if $h \mapsto |\hat{\psi}(\pi(h)^t \omega)|^2$ is integrable over $H^\omega \setminus H$, where $H^\omega = \{h \in H \mid \pi(h)^t \omega = \omega\}$. The problem is, that the stabilizer group H^ω depends on the point ω . We will show how to overcome this difficulty in case where H^ω is a *symmetric* subgroup of H . We will show that in this case we can replace H by a closed subgroup R such that the open orbit essentially decomposes into finitely many smaller open orbits which are homogeneous under R and have compact stabilizers. The drawback is that in most cases the group R depends on the orbit, but at least this allows us to reconstruct the function from its wavelet transform. Several interesting examples can be found in [8]. The tools that we use are the structure theory of reductive symmetric spaces corresponding to an involution $\tau : H \rightarrow H$. In particular we will need a *Cartan involution* $\theta : H \rightarrow H$ commuting with τ and the structure of minimal $\theta\tau$ -stable parabolic subgroups. To avoid technical details we will only discuss this for a linear Lie group invariant under transposition. Then $h \mapsto (h^{-1})^t$ is a Cartan involution on H . We refer to [28] for the discussion of the general case. We thus will be dealing with closed subgroups H of $\mathrm{GL}(n, \mathbb{R})$ invariant under transposition. A conjugate being invariant under transposition is one of several alternative definitions for reductive linear groups.

Definition 4.1. A closed subgroup $H \subset \mathrm{GL}(n, \mathbb{R})$ is called **reductive** if there exists a $x \in \mathrm{GL}(n, \mathbb{R})$ such that xHx^{-1} is invariant under transposition, $a \mapsto a^t$.

Example. Let $S \in \mathrm{GL}(n, \mathbb{R})$ be symmetric and regular. Define a non-degenerate bilinear form $\beta_S : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$\beta_S(u, v) = (Su \mid v)$$

and let

$$\mathrm{O}(\beta_S) = \{a \in \mathrm{GL}(n, \mathbb{R}) \mid \forall u, v \in \mathbb{R}^n : \beta_S(au, av) = \beta_S(u, v)\}.$$

Then $\mathrm{O}(\beta_S)$ can also be described by

$$\mathrm{O}(\beta_S) = \{a \in \mathrm{GL}(n, \mathbb{R}) \mid a^t S a = S\}$$

and obviously $\mathrm{O}(\beta_S)$ is invariant under transposition. Notice that for $a \in \mathrm{O}(\beta_S)$ we have $\det(S) = \det(a)2\det(S)$. As S is regular it follows that $\det(a) = \pm 1$. We let

$$\mathrm{SO}(\beta_S) = \{a \in \mathrm{O}(\beta_S) \mid \det(a) = 1\}.$$

The Lie algebra of $\mathrm{O}(\beta_S)$ is given by

$$\begin{aligned} \mathfrak{o}(\beta_S) &= \{X \in M_n(\mathbb{R}) \mid \forall t \in \mathbb{R} : e^{tX} \in \mathrm{O}(\beta_S)\} \\ &= \{X \in M_n(\mathbb{R}) \mid X^t S + S X = 0\} \end{aligned}$$

If $S = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$, $p + q = n$, then $\beta_S(u, v) = u_1 v_1 + \dots + u_p v_p - u_{p+1} v_{p+1} - \dots - u_n v_n$ and $\mathrm{O}(\beta_S) = \mathrm{O}(p, q)$.

We will now assume that H is reductive. For simplicity we can then assume that $H^t = H$. Let π be the natural action of H on \mathbb{R}^n given by $\pi(h)v = hv$. If ρ is the natural representation of $H \times_{\pi} \mathbb{R}^n$ on $L^2(\mathbb{R}^n)$, we will be interested in the irreducible subrepresentations of ρ corresponding to open orbits in \mathbb{R}^n under the contragredient action $h \cdot v = (h^{-1})^t v$. We recall if \mathcal{O} is such an orbit, then the subspace $L^2_{\mathcal{O}}(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) \mid \hat{f} \text{ vanishes off } \mathcal{O}\}$ is an irreducible subspace. In order to handle cases where $\mathcal{O} = H \cdot u$ do not have compact stabilizers, we will assume that there exists an involution $\tau : H \rightarrow H$ such that

$$H^{\tau}_o \subset H^u = \{h \in H \mid h^t \cdot u = u\} \subset H^{\tau}$$

where

$$H^{\tau} = \{h \in H \mid \tau(h) = h\}$$

and the subscript $_o$ indicates the connected component containing the identity element. Let $L = H^u$. Then $\mathcal{O} = L \backslash H$ is a *semisimple symmetric space*. We can assume that L is also invariant under transposition. Then $\theta : h \mapsto (h^t)^{-1}$ and τ commute. Let $K = \mathrm{O}(n) \cap H = \{k \in H \mid k^t = k^{-1}\}$. Then K is a maximal compact subgroup of H and $L \cap K$ is a maximal compact subgroup of L . Denote by \mathfrak{h} the Lie algebra of H . One has

$$\mathfrak{h} = \{X \in M_n(\mathbb{R}) \mid \forall t \in \mathbb{R} : e^{tX} \in H\}.$$

Then \mathfrak{h} decomposes as

$$\begin{aligned}\mathfrak{h} &= \mathfrak{k} \oplus \mathfrak{s} \\ &= \mathfrak{l} \oplus \mathfrak{q} \\ &= \mathfrak{k} \cap \mathfrak{l} \oplus \mathfrak{k} \cap \mathfrak{q} \oplus \mathfrak{s} \cap \mathfrak{l} \oplus \mathfrak{s} \cap \mathfrak{q}\end{aligned}$$

where

$$\mathfrak{s} = \{X \in \mathfrak{h} \mid X^t = X\}$$

is the subspace of symmetric matrices, and

$$\mathfrak{q} = \{X \in \mathfrak{h} \mid \tau(X) = -X\} .$$

Notice that

$$[\mathfrak{l}, \mathfrak{q}] \subset \mathfrak{q} \quad \text{and} \quad [\mathfrak{k}, \mathfrak{s}] \subset \mathfrak{s} .$$

Recall the linear maps $\text{Ad}(a), \text{ad}(X) : \mathfrak{h} \rightarrow \mathfrak{h}$, $a \in H$, $X \in \mathfrak{h}$, are given by $\text{ad}(X)Y = XY - YX$ and $\text{Ad}(a)Y = aYa^{-1}$. Let \mathfrak{a} be a maximal commutative subspace of $\mathfrak{s} \cap \mathfrak{q}$; thus $XY - YX = 0$ for all $X, Y \in \mathfrak{a}$. Then the algebra $\text{ad}(\mathfrak{a})$ is also commutative. Define an inner product $(\cdot \mid \cdot)$ on \mathfrak{h} by $(X \mid Y) := \text{Tr}(XY^t)$. Then

$$\begin{aligned}(\text{ad}(X)Y \mid Z) &= \text{Tr}(XY - YX)Z^t \\ &= -\text{Tr}(Y(ZX^t - X^tZ)^t) \\ &= (Y \mid \text{ad}(X^t)Z) .\end{aligned}$$

Thus $\text{ad}(X)^t = \text{ad}(X^t)$. In particular, if $X \in \mathfrak{s}$ then $\text{ad}(X)$ is symmetric. It follows that we can diagonalize the action of \mathfrak{a} on \mathfrak{h} . Specifically, for $\alpha \in \mathfrak{h}^*$ set

$$\mathfrak{h}_\alpha = \{Y \in \mathfrak{h} \mid \forall X \in \mathfrak{a} : \text{ad}(X)Y = \alpha(X)Y\} .$$

Let $\Delta = \{\alpha \in \mathfrak{h}^* \mid \mathfrak{h}_\alpha \neq \{0\}\} \setminus \{0\}$. Notice that the set Δ is finite. Hence there is a $X_r \in \mathfrak{a}$ such that $\alpha(X_r) \neq 0$ for all $\alpha \in \Delta$. Let $\Delta^+ = \{\alpha \mid \alpha(X_r) > 0\}$ and $\Delta^- = \{\alpha \mid \alpha(X_r) < 0\} = -\Delta^+$. Let

$$\mathfrak{n} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{h}_\alpha \quad \text{and} \quad \bar{\mathfrak{n}} = \bigoplus_{\alpha \in \Delta^-} \mathfrak{h}_\alpha .$$

Let $\mathfrak{m}_1 = \{X \in \mathfrak{h} \mid [\mathfrak{a}, X] = \{0\}\}$, and $\mathfrak{m} = \{X \in \mathfrak{m}_1 \mid \forall Y \in \mathfrak{a} : (X \mid Y) = 0\}$. Then $\mathfrak{m}_1 = \mathfrak{m} \oplus \mathfrak{a}$. Furthermore $\bar{\mathfrak{n}}$, \mathfrak{m} , \mathfrak{n} , and $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ are subalgebras of \mathfrak{h} and

$$\begin{aligned}\mathfrak{h} &= \bar{\mathfrak{n}} \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n} \\ &= \mathfrak{k} + \mathfrak{p} .\end{aligned}$$

Notice that the last sums are not direct because $\mathfrak{k} \cap \mathfrak{p} = \mathfrak{k} \cap \mathfrak{m}$ and $\mathfrak{l} \cap \mathfrak{p} = \mathfrak{l} \cap \mathfrak{m}$. Furthermore $P \cap L$ is not necessarily compact. To deal with this, we let \mathfrak{m}_2 be the smallest subalgebra containing $\mathfrak{m} \cap \mathfrak{s}$. Thus $\mathfrak{m}_2 = [\mathfrak{m} \cap \mathfrak{s}, \mathfrak{m} \cap \mathfrak{s}] \oplus \mathfrak{m} \cap \mathfrak{s}$.

Lemma 4.2. \mathfrak{m}_2 is an ideal in \mathfrak{m} and contained in $\mathfrak{m} \cap \mathfrak{l}$.

Proof. That $\mathfrak{m} \cap \mathfrak{s} \subset \mathfrak{l}$ follows from the fact that \mathfrak{a} is maximal abelian in $\mathfrak{q} \cap \mathfrak{s}$. Hence $\mathfrak{m}_2 \subset \mathfrak{l}$ follows from the fact that \mathfrak{l} is an algebra. By definition we have $\mathfrak{m} = \mathfrak{m} \cap \mathfrak{k} + \mathfrak{m}_2$. We have by definition $[\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{m}_2$. Therefore we only have to show that $[\mathfrak{m} \cap \mathfrak{k}, \mathfrak{m}_2] \subset \mathfrak{m}_2$. But $[\mathfrak{k}, \mathfrak{s}] \subset \mathfrak{s}$ and hence $[\mathfrak{m} \cap \mathfrak{k}, \mathfrak{m} \cap \mathfrak{s}] \subset \mathfrak{m} \cap \mathfrak{s}$. The claim follows now by the Jacobi identity and the fact that \mathfrak{m}_2 is generated by $\mathfrak{m} \cap \mathfrak{s}$. \square

Let

$$\mathfrak{b} := \{X \in \mathfrak{m} \mid \forall Y \in \mathfrak{m}_2 : (X, Y) = 0\} = \mathfrak{m}_2^\perp.$$

Then \mathfrak{b} is an ideal in \mathfrak{m} and $\mathfrak{m} = \mathfrak{b} \oplus \mathfrak{m}_2$. Let

$$N_K(\mathfrak{a}) = \{k \in K \mid \forall X \in \mathfrak{a} : \text{Ad}(a)X \in \mathfrak{a}\}$$

and

$$N_{L \cap K}(\mathfrak{a}) = N_K(\mathfrak{a}) \cap L.$$

Finally let

$$M_K = Z_K(\mathfrak{a}) = \{k \in K \mid \forall X \in \mathfrak{a} : aXa^{-1} = X\}$$

and

$$M_H = Z_H(\mathfrak{a}).$$

Then

$$(4.1) \quad W = N_K(\mathfrak{a})/M_K$$

is a finite group. Let

$$(4.2) \quad W_0 = N_{K \cap L}(\mathfrak{a})/M_K \cap L.$$

Then W_0 is a subgroup of W . Choose $w_0 = e, w_1, \dots, w_r \in W$ such that

$$(4.3) \quad W = \bigcup w_j W_0 \quad (\text{disjoint union}).$$

Let $s_j \in N_K(\mathfrak{a})$ be such that $s_j M_K = w_j$, and $s_0 = e$. Let $P = \{a \in H \mid \text{Ad}(a)\mathfrak{p} = \mathfrak{p}\}$. Then P is a closed subgroup of H . Let $A = \{e^X \mid X \in \mathfrak{a}\}$ and $N = \{e^X \mid X \in \mathfrak{n}\}$. Then A , and N are closed subgroups of P . Let M_2 be the group generated by $\exp(\mathfrak{m}_2)$, and $B_o = \exp(\mathfrak{b})$. Then $F = \exp(i\mathfrak{a}) \cap K \subset M_K$ is finite and such that $B = FB_o$ is a group. Furthermore

$$B \times M_2 \times A \ni (b, m, a) \mapsto bma \in Z_H(A)$$

is a diffeomorphism. Notice that by definition F is central in $Z_H(A)$ and $mFm^{-1} = F$ for all $m \in N_K(\mathfrak{a})$. Let $M = BM_2$. Then $P = MAN$. Furthermore each element $p \in P$ has a unique expression $p = man$ with $m \in M, a \in A, n \in N$. The group P is called a *minimal $\theta\tau$ stable parabolic subgroup of H* . This is still not the correct group for us to work with because $P \cap s_j L s_j^{-1}$ may not be compact. We therefore set

$$R = BAN.$$

Then R is a closed subgroup of H with Lie algebra $\mathfrak{r} = \mathfrak{b} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Notice that $\mathfrak{h} = \mathfrak{l} + (\mathfrak{b} \oplus \mathfrak{a} \oplus \mathfrak{n})$ and $\mathfrak{l} \cap (\mathfrak{b} \oplus \mathfrak{a} \oplus \mathfrak{n}) = \mathfrak{l} \cap \mathfrak{b}$.

Lemma 4.3. *Let the notation be as above. Then the following hold:*

- (a) If $s \in N_K(\mathfrak{a})$, then $sBs^{-1} = B$ and $sM_2s^{-1} = M_2$.
- (b) $P \cap s_j Ls_j^{-1} = M \cap s_j Ls_j^{-1}$.
- (c) We have $R \cap s_j Ls_j^{-1} \subset K$ is compact.

Proof.

- (a) First we notice that if $X \in \mathfrak{m}$, $Y \in \mathfrak{a}$ and $s \in N_K(\mathfrak{a})$ then

$$[\text{Ad}(s)X, Y] = \text{Ad}(s)[X, \text{Ad}(s)^{-1}Y] = 0.$$

Hence $\text{Ad}(s)\mathfrak{m} = \mathfrak{m}$. Furthermore $\text{Ad}(s)\mathfrak{s} = \mathfrak{s}$ as $s \in K$. It follows that $\text{Ad}(s)\mathfrak{m}_2 = \mathfrak{m}_2$. As $\text{Ad}(s)$ is an orthogonal transformation it follows that $\text{Ad}(s)\mathfrak{b} = \mathfrak{b}$. Thus $sB_0s^{-1} = B$ and $sM_2s^{-1} = M_2$. As $sFs^{-1} = F$ it follows that $sBs^{-1} = B$.

- (b) See [28].

- (c) Let $p \in R \cap s_j Ls_j^{-1}$. Then again by using [28] one has $p \in M \cap L \cap B \subset M \cap K$. \square

We are now able to state the following classical result of Matsuki [27].

Theorem 4.4 (Matsuki). *The sets Ls_0P, \dots, Ls_rP are disjoint and open. Furthermore $\bigcup_{j=0}^r Ls_jP$ is open and dense in H .*

As a consequence of this we get (cf. [28], p. 611):

Corollary 4.5. *The sets Ls_0R, \dots, Ls_rR are disjoint and open. Furthermore $\bigcup_{j=0}^r Ls_jR$ is open and dense in H .*

Proof. This follows from the fact that $Ls_jM = Ls_jB$. \square

One can actually state more than this, see [28].

The groups H and L being reductive are unimodular. This fact is important for the integral formulas in the next theorem.

Theorem 4.6. *Let the notation be as above. Then the following hold:*

- (a) *There exists a real analytic function $p : H \rightarrow \mathbb{R}_0^+ = \{t \in \mathbb{R} \mid t \geq 0\}$ such that*

$$\bigcup_{j=0}^r Ls_jR = \{x \in H \mid p(x) > 0\}.$$

In particular it follows that $H \setminus \bigcup_{j=0}^r Ls_jR$ has measure zero in H .

- (b) *We can normalize the Haar measures on the groups R and L so that*

$$\int_H f(x) dh = \int_R \int_L f(ls_jr) dld_r r$$

for all $f \in L^1(H)$ with $\text{Supp}(f) \subset Ls_jR$, where $d_r r$ is the right invariant measure on R .

From this we now get the following result:

Theorem 4.7. *Let H be a reductive group acting on \mathbb{R}^n . Assume that $\mathcal{O} = H^t \cdot u$ is an open orbit and H^u is noncompact and symmetric. Then there exists a closed subgroup R of H and elements $y_0 = y, \dots, y_r \in \mathcal{O}$ such that the following hold:*

- (a) *The orbits $\mathcal{U}_j = R^t \cdot y_j \subset \mathcal{O}$ are open and disjoint;*
- (b) *If $x \in \mathcal{U}_j$, then $R^x = \{r \in R \mid r^t \cdot x = x\}$ is compact;*
- (c) *$\bigcup \mathcal{U}_j \subset \mathcal{O}$ is open and dense, and the complement has measure zero;*
- (d) *The space $L^2_{\mathcal{O}}(\mathbb{R}^n)$ decomposes orthogonally as a representation of $R \times_{\pi} \mathbb{R}^n$ into irreducible parts*

$$L^2_{\mathcal{O}}(\mathbb{R}^n) = \bigoplus_{j=0}^r L^2_{\mathcal{U}_j}(\mathbb{R}^n).$$

- (e) *The space of $R \times_{\pi} \mathbb{R}^n$ -wavelets is dense in each of the spaces $L^2_{\mathcal{U}_j}(\mathbb{R}^n)$.*

Thus ρ on $L^2_{\mathcal{O}}(\mathbb{R}^n)$ restricted to $G_0 = R \times_{\pi} \mathbb{R}^n$ decomposes into a finite sum irreducible representations, one for each \mathcal{U}_j .

Let P_j be the orthogonal projection of $L^2(\mathbb{R}^n)$ onto $L^2_{\mathcal{U}_j}(\mathbb{R}^n)$. Then $\rho_j = \rho|_{G_0}$ on $L^2_{\mathcal{U}_j}(\mathbb{R}^n)$ is irreducible and any nonzero ψ_j with $\hat{\psi}_j \in C_c(\mathcal{U}_j)$ is a G_0 wavelet for ρ_j .

Thus the mapping

$$W_j : L^2_{\mathcal{U}_j}(\mathbb{R}^n) \rightarrow L^2(G_0)$$

defined by

$$W_j f(g) = (f \mid \rho(g)\psi_j)$$

is an intertwining operator of ρ_j into $L^2(G_0)$. Moreover, there is a scalar C_j such that

$$(W_j f \mid W_j g) = C_j^2 (f \mid g)$$

for $f, g \in L^2_{\mathcal{U}_j}(\mathbb{R}^n)$.

Also for $f \in L^2(\mathbb{R}^n)$,

$$P_j f = \frac{1}{C_j^2} \rho_0(W_j f) = \frac{1}{C_j^2} \int_{G_0} (f \mid \rho_0(g)\psi_j) \rho_0(g)\psi_j dg.$$

Hence if $f \in L^2_{\mathcal{O}}(\mathbb{R}^n)$, we have

$$f = \sum P_j f = \sum_{j=0}^r \frac{1}{C_j^2} \int_{G_0} (f \mid \rho(g)\psi_j) \rho(g)\psi_j dg.$$

By normalizing the ψ_j , we may assume $C_j = 1$ for all j . Then if $f \in L^2(\mathbb{R}^n)$ and $P = \sum P_j$, then

$$P f = \int_{G_0} \sum_{j=0}^r (f \mid \rho(g)\psi_j) \rho(g)\psi_j dg.$$

We thus can reconstruct $P f$ weakly from the finite collection of wavelet transforms $W_j(f)$ which we call a wavelet package. This remains possible even if there are no genuine wavelets for G on the orbit \mathcal{O} .

Remark. The authors thought it feasible to describe which functions are wavelets in the symmetric case by determining if their Fourier transforms were square integrable over the homogeneous orbits $H^u \backslash H$. However, the finiteness of the integrals $\int_R |\hat{\psi}(\pi(r)^t y_j)|^2 dr$ in Theorem 3.4 involve left Haar measure and the measure needed on $H^u \backslash H$ involves the right invariant measure on R . The nonunimodularity of R prevents these from being matched up.

Remark. One can say that the drawback of this method is that the group R , which we use instead of the original group H , depends on the orbit. But first of all this method allows us to invert the continuous wavelet transform. It would also be interesting to see if this can be used to generalize the ideas from [7], where in the case of free actions, one constructs *frames* for \mathbb{R}^n .

5. EXAMPLES

In this section we present two often cited examples. In these examples we will use the natural action for both the action and the contragredient action. This simplifies the presentation. To obtain this situation, one would either have to use the contragredient of the natural action for the action and the natural action for the contragredient action or replace the subgroup P by its transpose.

In the first example, we discuss in detail the group $H = \mathbb{R}^+ \text{SO}_o(1, n)$ acting on \mathbb{R}^{n+1} . It is a case where one has one open symmetric orbit with noncompact stabilizers. For a clear and detailed discussion of other cases where our assumptions are satisfied, i.e., the pre-homogeneous vector spaces, parabolic case, and the explicit construction of involutions corresponding to the stabilizer of points in an open orbit, we refer to the preprint by N. Bopp and H. Rubenthaler [8]. The variety of these examples shows that the symmetric case gives a class of wavelets and wavelet transforms worth studying in more detail.

In the second example, we take H to be the group $\text{GL}(n, \mathbb{R})$ and let it act on the Euclidean space $\text{Sym}(n, \mathbb{R})$ of symmetric $n \times n$ -matrices by $a \cdot X := aXa^t$. As indicated above, we assume the transposed action is also given by $(a^{-1})^t \cdot X := aXa^t$. With this action one has different symmetric spaces as open orbits and the groups used to obtain wavelets vary from orbit to orbit.

We mention there are other further examples where H is an automorphism group of a symmetric open convex cone; e.g., take H to be $\mathbb{R}^+ \text{SL}(n, \mathbb{C})$ acting on the space $\text{Sym}(n, \mathbb{C})$ of symmetric $n \times n$ -matrices by $a \cdot X = aXa^*$, or by taking H to be $\mathbb{R}^+ \text{SU}^*(2n)$ or $\mathbb{R}^+ E_{6(-26)}$ with similarly defined actions.

Example ($\mathbb{R}^+ \text{SO}_o(1, n), \mathbb{R}^{1+n}$):

Let $H = \mathbb{R}^+ \text{SO}(1, n)$ where \mathbb{R}^+ stands for the group of matrices $\{\lambda \mathbf{I}_{1+n} : \lambda > 0\}$ and \mathbf{I}_k is the $k \times k$ identity matrix. Notice that \mathbb{R}^+ is central in H . The group H acts on \mathbb{R}^{1+n} by matrix multiplication on \mathbb{R}^{1+n} . If $\lambda \in \mathbb{R}^+$ and $h \in \text{SO}(1, n)$, then

$$\Delta(\lambda h) = \det(\lambda h)^{-1} = \lambda^{-(1+n)}.$$

We write elements in \mathbb{R}^{1+n} as (t, x) with $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$. If one defines a bilinear form β on \mathbb{R}^{1+n} by

$$\beta((t, x), (s, y)) = ts - (x | y)$$

where $(\cdot | \cdot)$ is the usual inner product on \mathbb{R}^n , then $\text{SO}_o(1, n)$ is the connected component of the identity of the group

$$\text{O}(\beta) := \{g \in \text{GL}(1+n, \mathbb{R}) \mid \forall u, v \in \mathbb{R}^{1+n} : \beta(gu, gv) = \beta(u, v)\}.$$

The Lie algebra of $\text{SO}_o(1, n)$ consists of all $n+1$ by $n+1$ matrices X satisfying $X^t B + BX = 0$ where $B = \begin{pmatrix} 1 & 0 \\ 0 & -I_n \end{pmatrix}$.

We will when convenient write elements in $\text{SO}_o(1, n)$ in block form

$$g = \begin{pmatrix} a & x \\ y^t & A \end{pmatrix}, \quad a \in \mathbb{R}, x, y \in \mathbb{R}^n, A \in M_n(\mathbb{R})$$

Notice that actually $a > 0$ and that $H^t = H$.

Denote the matrix $(\delta_{\nu,i}\delta_{\mu,j})_{i,j=1}^{n+1}$ by $E_{\nu,\mu}$ and let $X = E_{1,n+1} + E_{n+1,1} \in \mathfrak{g}$. Let $\mathfrak{a} = \mathbb{R}X + \mathbb{R}I_{n+1}$, where the second factor corresponds to the dilations. Then \mathfrak{a} is a maximal abelian subalgebra of \mathfrak{g} . We will write

$$a_t = e^{tX} = \begin{pmatrix} \cosh(t) & 0 & \sinh(t) \\ 0 & I_{n-1} & 0 \\ \sinh(t) & 0 & \cosh(t) \end{pmatrix} \in A.$$

Imbed $\text{SO}_o(n)$ into $\text{GL}(1+n, \mathbb{R})$ by

$$k \mapsto \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}.$$

We will also imbed $\text{SO}_o(1, n-1)$ into $\text{SO}_o(1, n)$ by

$$g' \mapsto g = \begin{pmatrix} g' & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $\text{SO}_o(n) \subset \text{SO}_o(1, n)$ and $K = \text{SO}_o(n)$ is a maximal compact subgroup of H . Let

$$\mathcal{O}_1 = \{u = (t, x) \in \mathbb{R}^{1+n} \mid \beta(u, u) > 0, t > 0\}$$

$$\mathcal{O}_2 = \{u = (t, x) \in \mathbb{R}^{1+n} \mid \beta(u, u) > 0, t < 0\}$$

$$\mathcal{O}_3 = \{u = (t, x) \in \mathbb{R}^{1+n} \mid \beta(u, u) < 0\}$$

$$\mathcal{C} = \{u = (t, x) \in \mathbb{R}^{1+n} \mid \beta(u, u) = 0\}$$

Then \mathcal{O}_1 , \mathcal{O}_2 , and \mathcal{O}_3 are open, and the complementary set \mathcal{C} is the zero set of β and is a union of orbits of smaller dimension. We have the decomposition:

$$\mathbb{R}^{1+n} = \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{C} \quad (\text{disjoint union}).$$

Lemma 5.1. *The sets \mathcal{O}_1 , \mathcal{O}_2 , and \mathcal{O}_3 are homogeneous under H . Furthermore the following hold:*

- (a) The sets \mathcal{O}_1 and \mathcal{O}_2 are homogeneous self dual convex cones.
- (b) If $u \in \mathcal{O}_1 \cup \mathcal{O}_2$, then $H^u \simeq \text{SO}_o(n)$ is compact.
- (c) If $u \in \mathcal{O}_3$, then $H^u \simeq \text{SO}_o(1, n-1)$ is noncompact.
- (d) If we replace H by the non-connected group $H^* = \mathbb{R}^* \text{SO}_o(1, n-1)$, then $\mathcal{U}_2 = \mathcal{O}_1 \cup \mathcal{O}_2$ is homogeneous.

Proof. All of this is well known so that we only prove that \mathcal{O}_3 is homogeneous and that $H^u \simeq \text{SO}_o(1, n-1)$ for $u \in \mathcal{O}_3$. Considering vectors in \mathbb{R}^k as column vectors, we let $u = e_{n+1} = (0, \dots, 0, 1)^t$. In the following we will also view u as a vector in $\mathbb{R}^n \subset \mathbb{R}^{1+n}$. Then $u \in \mathcal{O}_3$. Notice that

$$a_t u = (\sinh(t), 0, \dots, 0, \cosh(t))^t.$$

Let $v = (s, y) \in \mathcal{O}_3$. By multiplying by $\lambda = (-\beta(v, v))^{-1/2}$, we may assume that $\beta(v, v) = -1$. Then $\|y\|^2 - s^2 = 1$. Hence we can find t such that

$$\|y\| = \cosh(t) \quad \text{and} \quad s = \sinh(t).$$

In particular, both $\cosh(t)u$ and y are in $S_{\cosh(t)}(0) = \{x \in \mathbb{R}^n \mid \|x\| = \cosh(t)\} \subset \mathbb{R}^{1+n}$. Recall that $\text{SO}_o(n)$ acts transitively on $S_{\cosh(t)}(0)$. Choose $k \in \text{SO}_o(n) \subset \text{SO}_o(1, n)$ with $k(\cosh(t)u) = y$. Then $ka_t u = v$. Thus $\text{SO}_o(1, n)$ acts transitively.

Let $g = \lambda \begin{pmatrix} a & x^t \\ y & A \end{pmatrix} \in H$ satisfy $gu = u$. Note first that $\lambda = 1$ and thus

$$gu = \begin{pmatrix} x_n \\ a_{1n} \\ \vdots \\ a_{nn} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

Hence $x \in \mathbb{R}^{n-1}$ and A has the form

$$A = \begin{pmatrix} A' & 0 \\ 0 & 1 \end{pmatrix}.$$

That the last row is $(0, \dots, 0, 1)$ follows from the fact that $g \in \text{SO}(1, n)$. Since $g \in \text{SO}_o(1, n)$, one has

$$\begin{pmatrix} a & 0 \\ 0 & A' \end{pmatrix} = g' \in \text{SO}_o(1, n-1) \subset \text{SO}_o(1, n).$$

Hence the claim. □

From this it follows, that we can decompose $L^2(\mathbb{R}^{1+n})$ as a representation of $G = H \times_{\pi} \mathbb{R}^{n+1}$ into irreducible parts as

$$L^2(\mathbb{R}^n) \simeq_G L_{\mathcal{O}_1}^1(\mathbb{R}^n) \oplus L_{\mathcal{O}_2}^1(\mathbb{R}^n) \oplus L_{\mathcal{O}_3}^1(\mathbb{R}^n).$$

The first two spaces have G -wavelets. If we replace H by $\mathbb{R}^* \text{SO}_o(n+1)$, then this decomposition simplifies to

$$L^2(\mathbb{R}^n) \simeq_G L_{\mathcal{O}_1 \cup \mathcal{O}_2}^1(\mathbb{R}^n) \oplus L_{\mathcal{O}_3}^1(\mathbb{R}^n)$$

where the first space has a dense subspace of wavelets. To deal with the orbit \mathcal{O}_3 we show that $H^{e_{n+1}}$ is in fact a symmetric subgroup of H . To that end define an involution τ on H by

$$(5.1) \quad \begin{aligned} \tau \left(\lambda \begin{pmatrix} A & x \\ v^t & a \end{pmatrix} \right) &= \frac{1}{\lambda} \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} A & x \\ v^t & a \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix} \\ &= \frac{1}{\lambda} \begin{pmatrix} A & -x \\ -v^t & a \end{pmatrix}. \end{aligned}$$

Then $(H^\tau)_o = \text{SO}_o(1, n-1)$.

Lemma 5.2. $\mathbb{R}^+ \text{SO}_o(1, n-1)$ is a symmetric subgroup of H .

We notice that \mathfrak{a} is maximal abelian in \mathfrak{g} , $\mathfrak{g} \cap \mathfrak{q}$, and \mathfrak{q} . Define $\alpha \in \mathfrak{a}^*$ by $\alpha(tX) = t$ and $\alpha(I_{n+1}) = 0$. Then $\Delta = \{\alpha, -\alpha\}$. We choose α as the positive root. Then

$$\mathfrak{g}_\alpha = \left\{ X(v) = \begin{pmatrix} 0 & v^t & 0 \\ v & 0_{n-1} & -v \\ 0 & v^t & 0 \end{pmatrix} \mid v \in \mathbb{R}^{n-1} \right\}$$

and

$$N = \left\{ n(v) = \begin{pmatrix} 1 + \frac{1}{2} \|v\|^2 & v^t & -\frac{1}{2} \|v\|^2 \\ v & I_{n-1} & -v \\ \frac{1}{2} \|v\|^2 & v^t & 1 - \frac{1}{2} \|v\|^2 \end{pmatrix} \mid v \in \mathbb{R}^{n-1} \right\}.$$

In this case $\mathfrak{m}_2 = \{0\}$ and

$$\mathfrak{m} = \mathfrak{b} \simeq \mathfrak{so}(n-1) \ni X \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & 0 \end{pmatrix} \subset \mathfrak{so}(n).$$

The group M is given by

$$M \simeq \text{SO}_o(n-1) \ni k \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{pmatrix} \subset \text{SO}_n(n) \subset \text{SO}_o(1, n)$$

and

$$A = \left\{ a(\lambda, t) = \lambda \begin{pmatrix} \cosh(t) & 0 & \sinh(t) \\ 0 & I_{n-1} & 0 \\ \sinh(t) & 0 & \cosh(t) \end{pmatrix} \mid t \in \mathbb{R}, \lambda > 0 \right\}.$$

Each of the groups M , A , and N act on \mathbb{R}^n in the following way:

- M : Rotation in the v_2, \dots, v_{n-1} coordinates;
- A : Dilations and hyperbolic rotations:

$$a(\lambda, t)v = \lambda(\cosh(t)v_1 + \sinh(t)v_{n+1}, v_2, \dots, v_n, \sinh(t)v_1 + \cosh(t)v_{n+1})^t$$

- N : Write a vector $v \in \mathbb{R}^{n+1}$ as $v = (a, x, b)$ with $a, b \in \mathbb{R}$ and $x \in \mathbb{R}^{n-1}$. Then

$$n(u)v = v + (a - b) \begin{pmatrix} \frac{1}{2} \|u\|^2 \\ u \\ \frac{1}{2} \|u\|^2 \end{pmatrix} + (x | v) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

The Weyl group W consists of two elements $W = \{1, w\}$ where w is given by $X \rightarrow -X$. It is realized by conjugating by s where $s = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & I_{n-2} & \\ & & & -1 \end{pmatrix}$. Notice that $s \in H^\tau \setminus (H^\tau)_o$ by (5.1), and that $s(e_{n+1}) = -e_{n+1}$. It follows that \mathcal{O}_3 decomposes into two open dense P -orbits $\mathcal{O}_{3,1}$, $\mathcal{O}_{3,2}$ and orbits of lower dimension.

Theorem 5.3. *Let $n(v) \in N$, $a(\lambda, t) \in A$, and $k \in M$. Then*

$$\begin{aligned} n(v)a(\lambda, t)k \cdot e_{n+1} &= \lambda \left(\sinh(t) - \frac{e^{-t}}{2} \|v\|^2, -e^{-t}v, \cosh(t) - \frac{e^{-t}}{2} \|v\|^2 \right)^t \\ n(v)a(\lambda, t)k \cdot (-e_{n+1}) &= \lambda \left(-\sinh(t) + \frac{e^{-t}}{2} \|v\|^2, e^{-t}v, -\cosh(t) + \frac{e^{-t}}{2} \|v\|^2 \right)^t \end{aligned}$$

In particular $\mathcal{O}_{3,2} = -\mathcal{O}_{3,1}$ for

$$\begin{aligned} \mathcal{O}_{3,1} &= Pe_{n+1} = \{(a, v, b) \mid \|v\|^2 > a^2 - b^2 \text{ and } a < b\} \text{ and} \\ \mathcal{O}_{3,2} &= P(-e_{n+1}) = \{(a, v, b) \mid \|v\|^2 > a^2 - b^2 \text{ and } a > b\}. \end{aligned}$$

To avoid the problem of having to change the group as one goes from the orbits \mathcal{O}_1 and \mathcal{O}_2 to the orbit \mathcal{O}_3 we notice that $H = PK$, where $K = \text{SO}_o(n) = H^{\pm e_1}$. Hence P actually acts transitively on the orbits \mathcal{O}_1 and \mathcal{O}_2 and in both cases we have a compact stabilizers. Thus

Theorem 5.4. *Under the action of the group $P \times_\pi \mathbb{R}^{1+n}$, the space $L^2(\mathbb{R}^{1+n})$ decomposes into four irreducible parts*

$$L^2(\mathbb{R}^{1+n}) \simeq L^2(\mathcal{O}_1) \oplus L^2(\mathcal{O}_2) \oplus L^2(\mathcal{O}_{3,1}) \oplus L^2(\mathcal{O}_{3,2}).$$

If we replace the central subgroup \mathbb{R}^+ by \mathbb{R}^* , then there are only two irreducible parts:

$$L^2(\mathbb{R}^{1+n}) \simeq L^2(\mathcal{O}_1 \cup \mathcal{O}_2) \oplus L^2(\mathcal{O}_{3,1} \cup \mathcal{O}_{3,2}).$$

All the irreducible subrepresentations are square integrable, i.e., allow for dense subspaces of wavelet vectors.

Example $(\mathbb{R}^+ \text{SL}(n, \mathbb{R}), \text{Symm}(n, \mathbb{R}))$:

In the last example one could do the continuous wavelet transforms uniformly by replacing the group $H = \mathbb{R}^+ \text{SO}_o(n, \mathbb{R})$ by the parabolic subgroup P . We include the next example to show that this in general is *not* the case, and in particular to show that the

group R used in the reconstruction of f from its wavelet transform depends in general on the orbit \mathcal{O}_j .

Let V be the $\frac{n(n+1)}{2}$ -dimensional Euclidean vector space

$$\text{Symm}(n, \mathbb{R}) = \{X \in M_n(\mathbb{R}) \mid X^t = X\}$$

with the inner product $(X, Y) = \text{Tr}(XY^t)$. The group $H = \mathbb{R}^+ \text{SL}(n, \mathbb{R})$ acts on V by

$$g \cdot X = gXg^t.$$

The orbits are open if they contain nondegenerate bilinear forms and in this case are parameterized by their signature. Specifically, let $p, q \in \mathbb{N}_0$ be such that $p + q = n$. Set

$$I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \text{ and let}$$

$$\mathcal{O}_{p,q} = H \cdot I_{p,q} = \{\text{symmetric matrices of signature } (p, q)\}.$$

Take $\text{SO}(p, q) = \{g \in \text{SL}(n, \mathbb{R}) \mid gI_{p,q}g^t = I_{p,q}\}$, and define $\tau_{p,q} : H \rightarrow H$ by

$$\tau_{p,q}(\lambda g) = \lambda^{-1} I_{p,q}(g^t)^{-1} I_{p,q}, \quad \lambda > 0, \quad g \in \text{SL}(n, \mathbb{R}).$$

Then $\tau_{p,q} : H \rightarrow H$ is an involution and $H^\tau = \text{SO}(p, q)$. The following is now clear:

Theorem 5.5. *Let the notation be as above. Then the following hold:*

- (a) *Each orbit $\mathcal{O}_{p,q}$ is open in V .*
- (b) *$\mathcal{O}_{p,q} = H/\text{SO}(p, q)$. In particular it follows that the stabilizer of a point in $\mathcal{O}_{p,q}$ is compact if and only if $n = p$ or $n = q$.*
- (c) *$V \setminus \bigcup_{p+q=n} \mathcal{O}_{p,q} = \{X \in V \mid \det(X) = 0\}$. In particular it follows that $\bigcup_{p+q=n} \mathcal{O}_{p,q}$ is dense and open in V .*

We note that the parabolics $P_{p,q}$ and the corresponding groups $R_{p,q}$ depend on the orbits $\mathcal{O}_{p,q}$. These can be worked out in detail but we leave the details to the reader.

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