SEGAL-BARGMANN TRANSFORMS OF ONE-MODE INTERACTING FOCK SPACES ASSOCIATED WITH GAUSSIAN AND POISSON MEASURES

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ABSTRACT. Let μ_g and μ_p denote the Gaussian and Poisson measures on $\mathbb{R}$, respectively. We show that there exists a unique measure $\widetilde{\mu}_g$ on $\mathbb{C}$ such that under the Segal-Bargmann transform S_{μ_g} the space $L^2(\mathbb{R}, \mu_g)$ is isomorphic to the space $\mathcal{H}L^2(\mathbb{C}, \widetilde{\mu}_g)$ of analytic L^2-functions on $\mathbb{C}$ with respect to $\widetilde{\mu}_g$. We also introduce the Segal-Bargmann transform S_{μ_p} for the Poisson measure μ_p and prove the corresponding result. As a consequence, when μ_g and μ_p have the same variance, $L^2(\mathbb{R}, \mu_g)$ and $L^2(\mathbb{R}, \mu_p)$ are isomorphic to the same space $\mathcal{H}L^2(\mathbb{C}, \widetilde{\mu}_g)$ under the S_{μ_g} and S_{μ_p}-transforms, respectively. However, we show that the multiplication operators by x on $L^2(\mathbb{R}, \mu_p)$ and on $L^2(\mathbb{R}, \mu_g)$ act quite differently on $\mathcal{H}L^2(\mathbb{C}, \widetilde{\mu}_g)$.

1. INTRODUCTION

Let μ be a probability measure on $\mathbb{R}$ having finite moments of all orders. In the paper [1] Accardi and Bożejko discovered a canonical unitary isomorphism between the Hilbert space $L^2(\mu)$ and the one-mode interacting Fock space $\Gamma(\lambda)$ associated with a sequence λ arising from μ. Under this isomorphism the number vectors $\Phi_n, n \geq 0$, conrrespond to the orthogonal polynomials $P_n(x)$ associated with μ and the modified field operator on $\Gamma(\lambda)$ corresponds to the multiplication operator by x on $L^2(\mu)$.

Being motivated by Accardi-Bożejko's discovery, Asai has recently introduced in [3] coherent vectors which are used to define the S_μ-transform. The S_μ-transform is shown in [3] to be a unitary operator from $L^2(\mu)$ onto a Hilbert space $\mathcal{H}_\lambda$ of analytic functions on a disk $\Omega_\lambda \subset \mathbb{C}$, where λ is determined by μ. The composition of the Accardi-Bożejko isomorphism and the S_μ-transform gives the interacting Fock space counterpart of the well-known Segal-Bargmann transform (cf. [8, 9, 15, 19, 20]).

The purpose of this paper is to apply the results of Accardi-Bożejko [1] and Asai [3] to the cases of Gaussian measure μ_g and Poisson measure μ_p. In paticular, when μ_g and μ_p have the same variance, we will see that by Theorems 4.6 and 4.8 the Segal-Bargmann transforms S_{μ_g} and S_{μ_p} take $L^2(\mu_g)$ and $L^2(\mu_p)$, respectively, to the same space $\mathcal{H}L^2(\mathbb{C}, \widetilde{\mu}_g)$ of analytic L^2-functions, where $\widetilde{\mu}_g$ is the Gaussian measure on $\mathbb{C}$. However, the Segal-Bargmann representation of multiplication by x on $L^2(\mu_g)$ is quite different from that on $L^2(\mu_p)$.

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2. One-mode Interacting Fock space

Let μ be a probability measure on $\mathbb{R}$ having finite moments of all orders. It is well-known [22] that there exist (1) a complete orthogonal system $\{P_n(x)\}_{n=0}^{\infty}$ of polynomials for $L^2(\mu)$ with $P_0 = 1$, (2) a sequence $\{\omega_n\}_{n=1}^{\infty}$ of nonnegative real numbers, and (3) a sequence $\{\alpha_n\}_{n=0}^{\infty}$ of real numbers such that the following equalities hold for all $n \geq 0$:

$$\begin{aligned}(x-\alpha_n)P_n(x) &= P_{n+1}(x) + \omega_n P_{n-1}(x), \\ \langle P_n, P_m\rangle_{L^2(\mu)} &= \delta_{n,m}\omega_1\cdots\omega_n,\end{aligned} \tag{2.1}$$

where $\omega_0 P_{-1} = 0$ by convention. We have the fact that the sequence $\alpha_n = 0$ for all n if and only if μ is symmetric.

For a probability measure μ with the associated sequence $\{\omega_n\}_{n=1}^{\infty}$, we define a sequence $\{\lambda_n\}_{n=0}^{\infty}$ by

$$\lambda_0 = 1, \quad \lambda_n = \omega_1\omega_2\cdots\omega_n, \quad n \geq 1. \tag{2.2}$$

Assume that the sequence $\{\lambda_n\}_{n=0}^{\infty}$ satisfies the condition:

$$(\star) \qquad \inf_{n\geq 0} \lambda_n^{\frac{1}{n}} > 0. \tag{2.3}$$

With such a sequence $\lambda = \{\lambda_n\}_{n=0}^{\infty}$, we define $\Gamma(\lambda)$ by

$$\Gamma(\lambda) = \Big\{(a_0, a_1, \ldots, a_n, \ldots) : a_n \in \mathbb{C}, \sum_{n=0}^{\infty} \lambda_n |a_n|^2 < \infty\Big\}$$

and a norm $\|\cdot\|_\lambda$ on $\Gamma(\lambda)$ by

$$\|(a_n)\|_\lambda = \Big(\sum_{n=0}^{\infty} \lambda_n |a_n|^2\Big)^{1/2}.$$

Then $\Gamma(\lambda)$ is a Hilbert space with norm $\|\cdot\|_\lambda$. It is called the *one-mode interacting Fock space* associated with λ [1, 2].

Define a *number vector* $\Phi_n, n \geq 0$, by

$$\Phi_n = (0, \ldots, 0, \overset{\overset{n+1}{\vee}}{1}, 0, \ldots).$$

The vector Φ_0 is called a *vacuum vector*. Let A be a densely defined operator on $\Gamma(\lambda)$ such that

$$A\Phi_0 = 0, \quad A\Phi_n = \omega_n \Phi_{n-1}, \quad n \geq 1.$$

The adjoint operator A^* of A is easily checked to be given by

$$A^*\Phi_n = \Phi_{n+1}, \quad n \geq 0.$$

The operators A and A^* are called the *annihilation* and *creation operators* on $\Gamma(\lambda)$, respectively. The number operator N is defined by

$$N\Phi_n = n\Phi_n, \quad n \geq 0.$$

In addition, we define an operator α_N on $\Gamma(\lambda)$ by

$$\alpha_N \Phi_n = \alpha_n \Phi_n, \quad n \geq 0.$$

Now we can state the result of Accardi and Bożejko [1]: There exists an unitary isomorphism $U : \Gamma(\lambda) \to L^2(\mu)$ satisfying the following conditions:

(1) $U\Phi_0 = 1$,

(2) $UA^*U^*P_n = P_{n+1}$,
(3) $U(A + A^* + \alpha_N)U^* = Q$, where Q is the multiplication operator by x on $L^2(\mu)$.

3. Segal-Bargmann Transform

Let $\{\lambda_n\}_{n=0}^{\infty}$ be the sequence defined in Equation (2.2) and consider the following series of a complex number z:

$$G_\lambda(z) = \sum_{n=0}^{\infty} \frac{1}{\lambda_n} z^n. \tag{3.1}$$

Note that by Condition $(\star)$ in Equation (2.3) this series has a positive radius of convergence, denoted by r_λ .

Let $\Omega_\lambda = \{z \in \mathbb{C} : |z| < \sqrt{r_\lambda}\}$. For each $z \in \Omega_\lambda$, Asai [3] has introduced a *coherent vector* $E_\lambda(\cdot, z)$ with respect to the family $\{P_n\}$ in Equation (2.1) by

$$E_\lambda(x, z) = \sum_{n=0}^{\infty} \frac{P_n(x)}{\lambda_n} z^n, \quad x \in \mathbb{R}. \tag{3.2}$$

It is easy to see

$$\|E_\lambda(\cdot, z)\|_{L^2(\mu)} = G_\lambda(|z|^2)^{1/2}$$

and so $E_\lambda(\cdot, z) \in L^2(\mu)$ for all $z \in \Omega_\lambda$. Moreover, the set $\{E_\lambda(\cdot, z) : z \in \Omega_\lambda\}$ is linearly independent and spans a dense subspace of $L^2(\mu)$.

For $f \in L^2(\mu)$, let $S_\mu f$ be the function defined by:

$$(S_\mu f)(z) = \langle E_\lambda(\cdot, \overline{z}), f\rangle_{L^2(\mu)} = \int_{\mathbb{R}} E_\lambda(x, z) f(x)\, d\mu(x), \quad z \in \Omega_\lambda. \tag{3.3}$$

The mapping S_μ defined on $L^2(\mu)$ is called the *Segal-Bargmann transform*. Asai has shown in [3] that S_μ is a unitary operator from $L^2(\mu)$ onto $\mathcal{H}_\lambda$. Here $\mathcal{H}_\lambda$ is given by

$$\mathcal{H}_\lambda = \left\{ F(z) = \sum_{n=0}^{\infty} a_n z^n : F \text{ is analytic on } \Omega_\lambda \text{ and } \sum_{n=0}^{\infty} \lambda_n |a_n|^2 < \infty \right\}. \tag{3.4}$$

It is a Hilbert space with the norm

$$\|F\|_{\mathcal{H}_\lambda} = \left(\sum_{n=0}^{\infty} \lambda_n |a_n|^2 \right)^{1/2}. \tag{3.5}$$

Now, let us introduce operators $\widetilde{A}$ and $\widetilde{A}^*$ acting on $\mathcal{H}_\lambda$ by

$$\widetilde{A}1 = 0, \quad \widetilde{A}z^n = \omega_n z^{n-1}, \quad n \geq 1$$

and

$$\widetilde{A}^* z^n = z^{n+1}, \quad n \geq 0.$$

Operators $\widetilde{A}$ and $\widetilde{A}^*$ satisfy the commutation relation $[\widetilde{A}, \widetilde{A}^*]z^n = (\omega_{n+1} - \omega_n)z^n$.

The number operator $\widetilde{N}$ acting on $\mathcal{H}_\lambda$ is defined by

$$\widetilde{N} z^n = n z^n, \quad n \geq 0$$

In addition, we can define an operator $\widetilde{\alpha}_N$ acting on $\mathcal{H}_\lambda$ by

$$\widetilde{\alpha}_N z^n = \alpha_n z^n, \quad n \geq 0.$$

The operators $\widetilde{A}$, $\widetilde{A}^*$, $\widetilde{N}$ and $\widetilde{\alpha}_N$ correspond to the operators A, A, N and α_N on $\Gamma(\lambda)$, respectively.

4. Main results

Our main results (Theorems 4.6 and 4.8 below) in this papers are concerned with the special cases when μ is a Gaussian or Poisson measure.

Case 1: $\mu =$ Gaussian measure μ_g with mean m and variance σ^2.

From [17, 22] we have the following equalities:

$$H_{n+1}(x;\sigma^2) - xH_n(x;\sigma^2) + \sigma^2 n H_{n-1}(x;\sigma^2) = 0,$$
$$\int_{\mathbb{R}} H_n(x;\sigma^2) H_m(x;\sigma^2) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}\, dx = \delta_{n,m}\sigma^{2n} n!,$$

where $H_n(x;\sigma^2)$ is the Hermite polynomial of degree n with parameter σ^2. Thus the three quantities for the Gaussian measure μ_g in Equation (2.1) are given by

$$P_n(x) = H_n(x-m;\sigma^2),$$
$$\omega_n = \sigma^2 n,$$
$$\alpha_n = m \quad \forall n \geq 1.$$

Moreover, we have the associated quantities in Equations (2.2), (3.1), and (3.2):

$$\lambda_n = \sigma^{2n} n!, \tag{4.1}$$
$$G_\lambda(z) = \exp\left[\frac{z}{\sigma^2}\right], \quad r_\lambda = \infty, \tag{4.2}$$
$$E_\lambda(x,z) = \exp\left[\frac{z}{\sigma^2}(x-m) - \frac{z^2}{2\sigma^2}\right].$$

Obviously, Condition $(\star)$ in Equation (2.3) is satisfied.

Case 2: $\mu =$ Poisson measure μ_p with parameter a.

From [10] we have the following equalities:

$$C_{n+1}(x;a) = (x-n-a)C_n(x;a) - anC_{n-1}(x;a),$$
$$\int_{\mathbb{R}} C_n(x;a)C_m(x;a)\, d\mu_p(x) = \delta_{n,m} a^n n!,$$

where $C_n(x;a)$ is the Charlier polynomial of degree n with parameter a. Thus the three quantities for the Poisson measure μ_p in Equation (2.1) are given by

$$P_n(x) = C_n(x;a),$$
$$\omega_n = an,$$
$$\alpha_n = n + a.$$

Moreover, we have the associated quantities in Equations (2.2), (3.1), and (3.2):

$$\lambda_n = a^n n!,$$
$$G_\lambda(z) = \exp\left[\frac{z}{a}\right], \quad r_\lambda = \infty,$$
$$E_\lambda(x,z) = e^{-z}\left(1 + \frac{z}{a}\right)^x.$$

Obviously, Condition $(\star)$ in Equation (2.3) is satisfied.

For the Poisson case, we have

$$\alpha_N = \frac{1}{a}A^*A + a = N + a, \tag{4.3}$$

which implies that

$$A^* + A + \alpha_N = \Big(\frac{1}{\sqrt{a}}A^* + \sqrt{a}\Big)\Big(\frac{1}{\sqrt{a}}A + \sqrt{a}\Big). \tag{4.4}$$

Proposition 4.1. *For the Gaussian measure μ_g on $\mathbb{R}$ with mean m and variance σ^2 , the following equalities hold:*

(a) $S_{\mu_g}H_n(\cdot - m;\sigma^2) = z^n$,
(b) $S_{\mu_g}UAU^*H_n(\cdot - m;\sigma^2) = \sigma^2 n z^{n-1}$,
(c) $S_{\mu_g}UA^*U^*H_n(\cdot - m;\sigma^2) = z^{n+1}$,
(d) $S_{\mu_g}\big((x-m)H_n(\cdot - m;\sigma^2)\big) = z^{n+1} + \sigma^2 n z^{n-1}$.

Proof. Conclusion (a) follows from Equations (3.2), (3.3), (4), and (4.1). (b) follows from (a) and the fact that $UAU^*H_n(\cdot - m;\sigma^2) = \sigma^2 n H_{n-1}(\cdot - m;\sigma^2)$. (c) follows from (a) and the fact that $UA^*U^*H_n(\cdot - m;\sigma^2) = H_{n+1}(\cdot - m;\sigma^2)$. (d) follows from (a) and Equation (4). □

Proposition 4.2. *For the Poisson measure μ_p on $\mathbb{R}$ with parameter a, the following equalities hold:*

(a) $S_{\mu_p}C_n(\cdot;a) = z^n$,
(b) $S_{\mu_p}UAU^*C_n(\cdot;a) = anz^{n-1}$,
(c) $S_{\mu_p}UA^*U^*C_n(\cdot;a) = z^{n+1}$,
(d) $S_{\mu_p}\big(xC_n(\cdot;a)\big) = z^{n+1} + (n+a)z^n + anz^{n-1}$.

Proof. The idea is similar to the proof of Proposition 4.1 with the Hermite polynomials being replaced by the Charlier polynomials. □

We point out that for the Poisson case, we have the equalities for operators acting on $\mathcal{H}_\lambda$

$$\widetilde{\alpha}_N = \frac{1}{a}\widetilde{A}^*\widetilde{A} + a = \widetilde{N} + a$$

and

$$\widetilde{A}^* + \widetilde{A} + \widetilde{\alpha}_N = \Big(\frac{1}{\sqrt{a}}\widetilde{A}^* + \sqrt{a}\Big)\Big(\frac{1}{\sqrt{a}}\widetilde{A} + \sqrt{a}\Big)$$

which correspond to the Equations (4.3) and (4.4), respectively.

Then we can apply Propositions 4.1 and 4.2 to get the next result.

Corollary 4.3. *Let μ_g be the Gaussian measure on $\mathbb{R}$ with mean m and variance σ^2. Let μ_p be the Poisson measure on $\mathbb{R}$ with the parameter a. Then*

(1) $S_{\mu_g}\big((x-m)H_n(\cdot - m;\sigma^2)\big) = (\widetilde{A}^* + \widetilde{A})S_{\mu_g}H_n(\cdot - m;\sigma^2)$.
(2) $S_{\mu_p}\big(xC_n(\cdot;a)\big) = (\widetilde{A}^* + \widetilde{A} + \widetilde{\alpha}_N)S_{\mu_p}C_n(\cdot;a)$.

Now, suppose μ is a probability measure on $\mathbb{R}$ with finite moments of all orders. Let λ be the sequence associated with μ as given in Equation (2.2). With this λ, we have a Hilbert space $\mathcal{H}_\lambda$ of analytic functions on Ω_λ in Equation (3.4) with norm $\|\cdot\|_{\mathcal{H}_\lambda}$ in Equation (3.5). Consider the following

Question: Does there exist a unique measure $\widetilde{\mu}$ on Ω_λ such that $F \in \mathcal{H}_\lambda$ if and only if F is analytic on Ω_λ and $F \in L^2(\Omega_\lambda, \widetilde{\mu})$ with

$$\|F\|^2_{\mathcal{H}_\lambda} = \int_{\Omega_\lambda} |F(z)|^2 \, d\widetilde{\mu}(z) \quad ? \tag{4.5}$$

Bargmann [8] considered the equality in Equation (4.5) for the multidimensional standard Gaussian case. See also the paper by Gross and Malliavin [12].

Our main results answer the above question for Gaussian and Poisson measures. For convenience, let $\mathcal{H}L^2(\Omega_\lambda, \widetilde{\mu})$ denote the Hilbert space of analytic functions F on Ω_λ which are square integrable with respect to $\widetilde{\mu}$. The norm on $\mathcal{H}L^2(\Omega_\lambda, \widetilde{\mu})$ is the $L^2(\widetilde{\mu})$-norm.

To answer the above question, we consider a criterion to check whether a measure ν satsifying the equation

$$\int_{\mathbb{C}} \overline{z}^m z^n d\nu(z) = \gamma_{m,n}. \tag{4.6}$$

is unique for given moments $\{\gamma_{m,n}\}$.

Proposition 4.4. *Suppose* $\{\gamma_{m,n}\}$ *satsifies the condition*

$$\lim_{n\to\infty} \frac{\gamma_{n,n}^{1/n}}{n^2} = 0. \tag{4.7}$$

Then the measure ν *satisfying Equation* (4.6) *is unique.*

Proof. Apply the Schwartz inequality to get

$$|\gamma_{m,n}| = \left| \int_{\mathbb{C}} \overline{z}^m z^n d\nu(z) \right| \le (\gamma_{m,m}\gamma_{n,n})^{\frac{1}{2}}.$$

Therefore the function

$$g(t,s) = \sum_{m,n=0}^{\infty} \gamma_{m,n} \frac{t^m s^n}{m! n!}$$

converges absolutely for $t, s \in \mathbb{C}$ by Equation (4.7). In fact, we see that

$$\begin{aligned} \int_{\mathbb{C}} \sum_{m,n=0}^{\infty} \frac{|(t\overline{z})^m (sz)^n|}{m!n!} d\nu(z) &\le \sum_{m,n=0}^{\infty} \frac{(\gamma_{m,m}\gamma_{n,n})^{\frac{1}{2}}}{m!n!} |t|^m |s|^n \\ &\le \left(\sum_{n=0}^{\infty} \frac{\gamma_{n,n}^{\frac{1}{2}}}{n!} R^n \right)^2 < \infty \end{aligned} \tag{4.8}$$

for $t, s \in \mathbb{C}$, $|t|, |s| \le R$. Equation (4.8) implies that $\exp[t\overline{z} + sz]$ is integrable with respect to a measure ν and

$$\int_{\mathbb{C}} \exp[t\overline{z} + sz] d\nu(z) = g(t,s)$$

holds. Therefore the characteristic function of a measure $d\nu(x,y) = d\nu(z)$, $z = x + iy$, satisfies the equality

$$\int_{\mathbb{R}^2} \exp[i\xi x + i\eta y] d\nu(x,y) = g\left(\frac{i\xi + \eta}{2}, \frac{i\xi - \eta}{2} \right)$$

for any $\xi, \eta \in \mathbb{R}$. Hence ν is unique. □

Remark 4.5. We are unable to find any literature dealing with the moment problem for measures on the complex plane $\mathbb{C}$. The above proposition gives a sufficient condition for the uniqueness of a measure on $\mathbb{C}$ in the moment problem. On the other hand, many authors have studied the moment problem for measures on $\mathbb{R}$.

Theorem 4.6. *Let μ_g be the Gaussian measure with mean m and variance σ^2. Let $\mathcal{H}_\lambda$ be the Hilbert space associated with μ_g as in Equation (3.4), i.e., $\Omega_\lambda = \mathbb{C}$ and $\lambda_n = \sigma^{2n} n!$. Then there exists a unique measure $\widetilde{\mu}$ on $\mathbb{C}$ such that $\mathcal{H}_\lambda = \mathcal{H}L^2(\mathbb{C}, \widetilde{\mu})$ and Equation (4.5) holds.*

Proof. Consider the above question for the Gaussian measure μ_g with mean m and variance σ^2. In this case, we have $\Omega_\lambda = \mathbb{C}$ by Equation (4.2) and $\lambda_n = \sigma^{2n} n!$ by Equation (4.1).

The uniqueness of $\widetilde{\mu}$ follows from Proposition 4.4. Thus we only need to find a measure $\widetilde{\mu}$ on $\mathbb{C}$ such that

$$\int_{\mathbb{C}} \overline{z}^m z^n \, d\widetilde{\mu}(z) = \delta_{m,n} \sigma^{2n} n!. \tag{4.9}$$

Suppose $\widetilde{\mu}$ is given by $d\widetilde{\mu}(z) = \frac{1}{2\pi} d\rho(r) d\theta$ for $z = re^{i\theta}$. Then Equation (4.9) becomes

$$\frac{1}{2\pi} \int_0^\infty \left(\int_0^{2\pi} e^{-i(m-n)\theta} \, d\theta \right) r^{m+n} \, d\rho(r) = \delta_{m,n} \sigma^{2n} n!.$$

This equality is obviously valid when $m \neq n$. Thus we are looking for ρ such that

$$\int_0^\infty r^{2n} \, d\rho(r) = \sigma^{2n} n!. \tag{4.10}$$

But it is easy to see that ρ is given by

$$d\rho(r) = \frac{2}{\sigma^2} r \exp\left[-\frac{r^2}{\sigma^2} \right] dr. \tag{4.11}$$

The resulting measure

$$d\widetilde{\mu}_g(z) = \frac{1}{\pi\sigma^2} r \exp\left[-\frac{r^2}{\sigma^2} \right] dr d\theta \quad z = re^{i\theta}$$

is a *Gaussian measure* on $\mathbb{C}$. Hence we have proved the existence of a measure $\widetilde{\mu}$ satisfying Equation (4.5). □

Remark 4.7. The measure given in Equation (4.11) is the unique measure satisfying Equation (4.10) since the sequence $\lambda_n = \sigma^{2n} n!$ satisfies the condition

$$\sum_{n=1}^{\infty} (\lambda_n)^{-\frac{1}{2n}} = \infty$$

in Theorem 1.11 of the book by Shohat and Tamarkin [21]. However, in general it is not true that a measure $d\widetilde{\mu}(z)$ on $\mathbb{C}$ can be written as $\frac{1}{2\pi} d\rho(r) d\theta$. Thus we really need Proposition 4.4 to show the uniqueness of $\widetilde{\mu}$.

Observe that we can also apply the above arguments to the Poisson measure μ_p. Thus we have the corresponding theorem for μ_p.

Theorem 4.8. *Let μ_p be the Poisson measure with parameter a. Let $\mathcal{H}_\lambda$ be the Hilbert space associated with μ_p as in Equation (3.4), i.e., $\Omega_\lambda = \mathbb{C}$ and $\lambda_n = a^n n!$. Then there exists a unique measure $\widetilde{\mu}$ on $\mathbb{C}$ such that $\mathcal{H}_\lambda = \mathcal{H}L^2(\mathbb{C}, \widetilde{\mu})$ and Equation (4.5) holds.*

In fact, it is easy to see that the unique measure $\widetilde{\mu}$ for μ_p is also a Gaussian measure on $\mathbb{C}$ with a replacing σ^2 in Equation (4.11). In particular, we see that when $a = \sigma^2$ (i.e., μ_g and μ_p have the same variance) the Segal-Bargmann representing spaces for μ_g and μ_p are the same space, namely, the space $\mathcal{H}L^2(\mathbb{C}, \widetilde{\mu}_g)$. However, the multiplication operators for μ_g and μ_p acting on $\mathcal{H}L^2(\mathbb{C}, \widetilde{\mu}_g)$ are quite different.

Remark 4.9. Hudson-Parthasarathy [13] and Ito-Kubo [14] obtained similar results presented in this section from the viewpoint of quantum stochastic calculus and white noise calculus (cf. [17]), respectively. On the relationship between infinite dimensional Gaussian analysis and the Segal-Bargmann transform, see the papers by Asai-Kubo-Kuo [4, 5, 6, 7], Cochran-Kuo-Sengupta [11], Kubo-Yokoi [16], Lee [18], Yokoi [23] and the references therein.

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