ESTIMATES FOR DERIVATIVES OF THE GREEN FUNCTIONS FOR THE NONCOERCIVE DIFFERENTIAL OPERATORS ON HOMOGENEOUS MANIFOLDS OF NEGATIVE CURVATURE

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Abstract. We consider the Green functions $\mathcal{G}$ for a second order noncoercive differential operators on homogeneous manifolds of negative curvature, being a semi-direct product of a nilpotent Lie group N and $A = \mathbb{R}^+$. Estimates for derivatives of the Green functions $\mathcal{G}$ with respect to the N and A-variables are obtained.

1. Introduction and the main results.

In this paper we study the Green function for a second order noncoercive differential operator $\mathcal{L}$ on a connected, simply connected homogeneous manifold of negative curvature. Such a manifold is a solvable Lie group $S = NA$, a semi-direct product of a nilpotent Lie group N and an abelian group $A = \mathbb{R}^+$. Moreover, for an H belonging to the Lie algebra $\mathfrak{a}$ of A, the real parts of the eigenvalues of $\mathrm{Ad}_{\exp H}|_{\mathfrak{n}}$, where $\mathfrak{n}$ is the Lie algebra of N, are all greater than 0. Conversely, every such a group equipped with a suitable left-invariant metric becomes a homogeneous Riemannian manifold with negative curvature (see [6]).

On S we consider a second order left-invariant operator

$$\mathcal{L} = \sum_{j=0}^{m} Y_j^2 + Y.$$

We assume that $Y_0, Y_1, \dots, Y_m$ generate the Lie algebra $\mathfrak{s}$ of S. We can always make $Y_0, \dots, Y_m$ linearly independent and moreover, we can choose $Y_0, Y_1, \dots, Y_m$ so that $Y_1(e), \dots, Y_m(e)$ belong to $\mathfrak{n}$. Let $\pi : S \to A = S/N$ be the canonical homomorphism. Then the image of $\mathcal{L}$ under π is a second order left-invariant operator on $\mathbb{R}^+$,

$$(a\partial_a)^2 - \gamma a\partial_a,$$

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where $\gamma \in \mathbb{R}$. $\mathcal{L} = \mathcal{L}_\gamma$ is *noncoercive* (there is no $\varepsilon > 0$ such that $\mathcal{L} + \varepsilon I$ admits the Green function) if and only if $\gamma = 0$.

Finally, the operator we are interested in, i.e. noncoercive one, can be written in the form

$$\mathcal{L} = \sum_j \Phi_a(X_j)^2 + \Phi_a(X) + a^2\partial_a^2 + a\partial_a, \tag{1.1}$$

where $X, X_1, \dots, X_m$ are left-invariant vector fields on N, $X_1, \dots, X_m$ linearly independent and generate $\mathfrak{n}$, $\Phi_a = \mathrm{Ad}_{\exp(\log a)Y_0} = e^{(\log a)\,\mathrm{ad}_{Y_0}} = e^{(\log a)D}$. $D = \mathrm{ad}_{Y_0}$ is a derivation of the Lie algebra $\mathfrak{n}$ of the Lie group N such that the real parts d_j of the eigenvalues λ_j of D are positive. By multiplying $\mathcal{L}_\gamma$ by a constant, i.e. changing Y_0, we can make d_j arbitrarily large (see [3]).

Let $\mathcal{G}(xa, yb)$ be the *Green function* for $\mathcal{L}$. $\mathcal{G}$ is (uniquely) defined by two conditions:

i) $\mathcal{L}\mathcal{G}(\cdot, yb) = -\delta_{yb}$ as distributions (functions are identified with distributions via the right Haar measure),
ii) for every $yb \in S$, $\mathcal{G}(\cdot, yb)$ is a potential for $\mathcal{L}$.

Let

$$\mathcal{G}(x, a) = \mathcal{G}(xa; e), \tag{1.2}$$

where e is the identity element of the group S. In this paper we call $\mathcal{G}(x, a)$ the Green function for $\mathcal{L}$.

The main goal of this paper is to prove the following estimates for derivatives of the Green function (1.2) for $\mathcal{L} = \mathcal{L}_0$, i.e. in the noncoercive case, with respect to x-variables (Theorem 6.1) and a-variable (Theorem 6.2). For every neighborhood $\mathcal{U}$ of the identity e of NA there is a constant C such that we have

$$|\mathcal{X}^I\mathcal{G}(x,a)| \le \begin{cases} C(|x|+a)^{-\|I\|-Q} & \\ \times(1+|\log(|x|+a)^{-1}|)^{\|I\|_0} & \text{for } (x,a) \in (\mathcal{Q}\cup\mathcal{U})^c, \\ C & \text{for } (x,a) \in \mathcal{Q}\setminus\mathcal{U}, \end{cases} \tag{1.3}$$

where $|\cdot|$ stands for a "homogeneous norm" on N, $\mathcal{Q} = \{|x| \le 1, a \le 1\}$, $\|I\|$ is a suitably defined length of the multi-index I and $\|I\|_0$ is a certain number depending on I and the nilpotent part of the derivation D. In particular, $\|I\|_0$ is equal to 0 if the action of $A = \mathbb{R}^+$ on N, given by Φ_a, is diagonal or, if $I = 0$. $\mathcal{X}_1, \dots, \mathcal{X}_n$ is an appropriately chosen basis of $\mathfrak{n}$. For the precise definitions of all the notions that have appeared here see Section 2.

For $\partial_a^k \mathcal{G}(x,a)$, $k \geq 0$ we have the following estimate

$$|\partial_a^k \mathcal{G}(x,a)| \leq \begin{cases} Ca^{-k}(|x|+a)^{-Q} & \text{for } (x,a) \in (\mathcal{Q} \cup \mathcal{U})^c, \\ Ca^{-k} & \text{for } (x,a) \in \mathcal{Q} \setminus \mathcal{U}. \end{cases} \tag{1.4}$$

Some comments should be made at this moment. First of all, it should be said that the estimate (1.3), also from below, for $I = 0$ was proved by the author in [13] but at that time it was impossible to prove analogous estimate for derivatives. The reason was that we did not have sufficient estimates for the derivatives of the transition probabilities of the evolution on N generated by an appropriate operator which appears as the "horizontal" component of the diffusion on $N \times \mathbb{R}^+$ generated by $a^{-2}\mathcal{L}$ (cf. [3]). These estimates have been obtained by the author in [10] and eventually led up to the estimate (1.3) that we are going to present here.

Another remark is that the case $\gamma = 0$ is essentially different from the case $\gamma > 0$ since the operator becomes noncoercive and, in particular, the Ancona theory [1] is of no use.

The proofs of (1.3) and (1.4) require both analytic and probabilistic techniques. Some of them have been introduced in [4, 3] and [13].

Guide to the paper. The structure of the paper is as follows. In Section 2 we state precisely notation and all necessary definitions.

In Section 3 we recall a definition of the Bessel process which appears as the "vertical" component of the diffusion generated by $a^{-2}\mathcal{L}$ on $N \times \mathbb{R}^+$ (cf. [3]). Moreover, we state some lemmas about its properties (without the proofs but we give references where they can be found).

In Section 4 we state the main estimates of the transition probabilities of the evolution on N generated by an appropriate operator which appears as the "horizontal" component of the diffusion on $N \times \mathbb{R}^+$ mentioned above, as well as their derivatives. All results from this sections are taken from [10] and [3].

In Section 5 we prove the main probabilistic lemmas, which are crucial in the proofs of (1.3) and (1.4). This section heavily depends on Sections 3 and 4.

Finally, in Section 6 we state precisely the estimates (1.3) (see Theorem 6.1) and (1.4) (see Theorem 6.2) and we give their proofs.

2. Preliminaries.

Let N be a connected and simply connected nilpotent Lie group. Let D be a derivation of the Lie algebra $\mathfrak{n}$ of N. For every $a \in \mathbb{R}^+$ we define an automorphism Φ_a of $\mathfrak{n}$ by the formula

$$\Phi_a = e^{(\log a)D}.$$

Writing $x = \exp X$ we put

$$\Phi_a(x) := \exp \Phi_a(X).$$

Let $\mathfrak{n}^{\mathbb{C}}$ be the complexification of $\mathfrak{n}$. Define

$$\mathfrak{n}^{\mathbb{C}}_\lambda = \{X \in \mathfrak{n}^{\mathbb{C}} : \exists k > 0 \text{ such that } (D - \lambda I)^k = 0\}.$$

Then

$$\mathfrak{n} = \bigoplus_{\mathrm{Im}\lambda \geq 0} V_\lambda, \tag{2.1}$$

where

$$V_\lambda = \begin{cases} V_{\bar\lambda} = (\mathfrak{n}^{\mathbb{C}} \oplus \mathfrak{n}^{\mathbb{C}}_{\bar\lambda}) \cap \mathfrak{n} & \text{if } \mathrm{Im}\lambda \neq 0, \\ \mathfrak{n}^{\mathbb{C}}_\lambda \cap \mathfrak{n} & \text{if } \mathrm{Im}\lambda = 0. \end{cases}$$

We assume that the real parts d_j of the eigenvalues λ_j of the matrix D are strictly greater than 0. We define the number

$$Q = \sum_j \mathrm{Re}\ \lambda_j = \sum_j d_j \tag{2.2}$$

and we refer to this as a *"homogeneous dimension"* of N. In this paper $D = \mathrm{ad}_{Y_0}$ (see Introduction). Under the assumption on positivity of d_j, (2.1) is a gradation of $\mathfrak{n}$.

We consider a group S which is a *semi-direct* product of N and the multiplicative group $A = \mathbb{R}^+ = \{\exp tY_0 : t \in \mathbb{R}\}$:

$$S = NA = \{xa : x \in N, a \in A\}$$

with multiplication given by the formula

$$(xa)(yb) = (x\Phi_a(y)\ ab).$$

In N we define a *"homogeneous norm"*, $|\cdot|$ (cf. [4, 3]) as follows. Let $(\cdot,\cdot)$ be a fixed inner product in $\mathfrak{n}$. We define a new inner product

$$\langle X, Y\rangle = \int_0^1 \Big(\Phi_a(X), \Phi_a(Y)\Big)\frac{da}{a} \tag{2.3}$$

and the corresponding norm

$$\|X\| = \langle X, X\rangle^{1/2}.$$

We put

$$|X| = \left(\inf\{a > 0 : \|\Phi_a(X)\| \geq 1\}\right)^{-1}.$$

One can easily show that for every $Y \neq 0$ there exists precisely one $a > 0$ such that $Y = \Phi_a(X)$ with $|X| = 1$. Then we have $|Y| = a$.

Finally, we define the homogeneous norm on N. For $x = \exp X$ we put

$$|x| = |X|.$$

Notice that if the action of $A = \mathbb{R}^+$ on N (given by Φ_a) is diagonal the norm we have just defined is the usual homogeneous norm on N and the number Q in (2.2) is simply the homogeneous dimension of N (see [5]).

Having all that in mind we define appropriate derivatives (see also [4]). We fix an inner product (2.3) in $\mathfrak{n}$ so that V_{λ_j}, $j = 1, \dots, k$ are mutually orthogonal and an orthonormal basis $\mathcal{X}_1, \dots, \mathcal{X}_n$ of $\mathfrak{n}$. The enveloping algebra $\mathfrak{U}(\mathfrak{n})$ of $\mathfrak{n}$ is identified with the polynomials in $\mathcal{X}_1, \dots, \mathcal{X}_n$. In $\mathfrak{U}(\mathfrak{n})$ we define $\langle \mathcal{X}_1 \otimes \dots \otimes \mathcal{X}_r, \mathcal{Y}_1 \otimes \dots \otimes \mathcal{Y}_r \rangle = \prod_{j=1}^r \langle \mathcal{X}_j, \mathcal{Y}_j \rangle$. Let V_j^r be the symmetric tensor product of r copies of V_{λ_j}. For $I = (i_1, \dots, i_k) \in (\mathbb{N} \cup \{0\})^k$ let

$$\mathcal{X}^I = \mathcal{X}_1^{(i_1)} \dots \mathcal{X}_k^{(i_k)}, \text{ where } \mathcal{X}_j^{(i_j)} \in V_j^{i_j}.$$

Then for $\mathcal{X} \in V_{\lambda_j}$

$$\|\Phi_a(\mathcal{X})\| \leq c \exp(d_j \log a + D_j \log(1 + |\log a|)),$$

where $d_j = \mathrm{Re}\lambda_j$ and $D_j = \dim V_{\lambda_j} - 1$, and so

(2.4)
$$\|\Phi_a(\mathcal{X}^I)\| \leq \exp\left(\sum_{j=1}^k i_j(d_j \log a + D_j \log(1 + |\log a|))\right) \prod_{j=1}^k \|\mathcal{X}_j^{(i_j)}\|.$$

Notation. The letter C or c, possibly with a subscript number, occurs in inequalities as a positive constant which is independent of the important parameters of the formula, and may vary from statement to statement, even in the same calculation.

3. Bessel process.

Let b_t denotes the Bessel process with a parameter $\alpha \geq 0$ (cf. [8]), i.e. a continuous Markov process with the state space $[0, +\infty)$ generated by $\partial_a^2 + \frac{2\alpha+1}{a}\partial_a$.

The transition function with respect to the measure $y^{2\alpha+1}dy$ is given, e.g. in [2, 8], by:

$$(3.1) \qquad p_t(x,y) = \begin{cases} \frac{1}{2t} \exp\left(\frac{-x^2-y^2}{4t}\right) I_\alpha\left(\frac{xy}{2t}\right) \frac{1}{(xy)^\alpha} & \text{for } x, y > 0, \\ \frac{1}{2^\alpha (2t)^{\alpha+1} \Gamma(\alpha+1)} \exp\left(\frac{-y^2}{4t}\right) & \text{for } x = 0,\ y > 0, \end{cases}$$

where

$$I_\alpha(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k+\alpha}}{k!\Gamma(k+\alpha+1)}$$

is the Bessel function (see [7]). Therefore for $x \geq 0$ and a measurable set $B \subset (0,\infty)$:

$$\mathbf{P}_x(b_t \in B) = \int_B p_t(x,y) y^{2\alpha+1} dy.$$

If b_t is the Bessel process with a parameter α starting from x, i.e. $b_0 = x$, then we will write that $b_t \in \mathrm{BESS}_x(\alpha)$ or simply $b_t \in \mathrm{BESS}(\alpha)$ if the starting point is not important or is clear from the context.

The following lemmas concerning some properties of the Bessel process are very well known and their proofs are rather standard. Their proofs can be found e.g. in [3, 11, 12].

Lemma 3.1. *Let $b_t \in \mathrm{BESS}(\alpha)$, $\alpha \geq 0$. Let $D, \gamma, x \geq 0$. There exists a positive constant C such that for every $t > s \geq 0$,*

$$\mathbf{E}_x \left(\int_s^t b_s^\gamma ds \right)^{-D} \leq C(t-s)^{-D(1+\gamma/2)}.$$

Proof. See e.g. Theorem 3.8 in [11]. □

Lemma 3.2. *Let $b_t \in \mathrm{BESS}(\alpha)$, $\alpha \geq 0$. There exist constants c_1, c_2 such that for every $x \geq 0$, for every $\lambda > 0$ and for every $t > 0$,*

$$\mathbf{P}_x(\sup_{s\in[0,t]} b_s \geq x + \lambda) \leq c_1 e^{-c_2\lambda^2/t}.$$

Proof. See e.g. Lemma 3.7 in [11]. □

Lemma 3.3. *Let $b_t \in \mathrm{BESS}(\alpha)$, $\alpha \geq 0$. There exist constants c_1, c_2 such that for every $x \geq 0$, for every $\lambda > 0$ and for every $t > 0$,*

$$\mathbf{P}_x(\inf_{s\in[0,t]} b_s < x - \lambda) \leq c_1 e^{-c_2\lambda^2/t}.$$

Proof. Imitate the proof of Lemma 3.7 in [11]. □

Lemma 3.4. *Let $b_t \in \mathrm{BESS}(\alpha)$, $\alpha \geq 0$. There exist positive constants c_1, c_2 such that for every for every $x \geq 0$, $r > 0$ and $t > 0$,*

$$\mathbf{P}_x(\sup_{s\in[0,t]} b_s \leq r) \leq c_1 e^{-c_2\frac{\sqrt{t}}{r}}.$$

Proof. See Lemma 3.5 in [11] or Lemma 2.3 in [3]. □

By a straightforward computation, using the definition of the transition function $p_t(x, y)$ of the Bessel process (3.1) and the asymptotic behavior of the Bessel function (see [7]):

$$I_\alpha(x) \asymp \begin{cases} \frac{x^\alpha}{2^\alpha \Gamma(1+\alpha)}, & x \to 0, \\ \frac{\exp(x)}{(2\pi x)^{1/2}}, & x \to \infty, \end{cases} \tag{3.2}$$

we get

Lemma 3.5. *Let $b_t \in \mathrm{BESS}(\alpha)$, $\alpha \geq 0$. There exists a constant C independent of x such that*

$$\mathbf{P}_x(a - \eta \leq b_t \leq a + \eta) \leq C t^{-(\alpha+1)} m([a - \eta, a + \eta]),$$

where $m(B) = \int_B y^{2\alpha+1} dy$.

4. Evolutions.

Let $X, X_1, \ldots, X_m$ be as in (1.1). Let $\sigma : [0, \infty) \longrightarrow [0, \infty)$ be a continuous function such that $\sigma(t) > 0$ for every $t > 0$. We consider the family of evolutions operators $L_{\sigma(t)} - \partial_t$, where

$$L_{\sigma(t)} = \sigma(t)^{-2} \bigg(\sum_j \Phi_{\sigma(t)}(X_j)^2 + \Phi_{\sigma(t)}(X) \bigg). \tag{4.1}$$

Since we may assume that $X_1, \ldots, X_m$ are linearly independent we select $X_{m+1}, \ldots, X_n$ so that $X_1, \ldots, X_n$ form a basis of $\mathfrak{n}$. For a multi-index $I = (i_1, \ldots, i_n)$, $i_j \in \mathbb{Z}^+$ and the basis $X_1, \ldots, X_n$ of the Lie algebra $\mathfrak{n}$ of N we write: $X^I = X_1^{i_1} \ldots X_n^{i_n}$ and $|I| = i_1 + \ldots + i_n$. For $k = 0, 1, \ldots, \infty$ we define:

$$C^k = \{ f : X^I f \in C(N), \text{ for } |I| < k + 1 \}$$

and

$$C^k_\infty = \{ f \in C^k : \lim_{x \to \infty} X^I f(x) \text{ exists for } |I| < k + 1 \}.$$

For $k < \infty$ the space C^k_∞ is a Banach space with the norm

$$\|f\|_{C^k_\infty} = \sum_{|I| \leq k} \|X^I f\|_{C(N)}.$$

Let $\{U^\sigma(s, t) : 0 \leq s \leq t\}$ be the unique family of bounded operators on $C_\infty = C^0_\infty$ which satisfy

i) $U^\sigma(s, s) = I$,
ii) $U^\sigma(s, r) U^\sigma(r, t) = U^\sigma(s, t)$, $s < r < t$,
iii) $\partial_s U^\sigma(s, t) f = -L_{\sigma(s)} U^\sigma(s, t) f$ for every $f \in C_\infty$,
iv) $\partial_t U^\sigma(s, t) f = U^\sigma(s, t) L_{\sigma(t)} f$ for every $f \in C_\infty$,
v) $U^\sigma(s, t) : C^2_\infty \longrightarrow C^2_\infty$.

$U^\sigma(s,t)$ is a convolution operator. Namely, $U^\sigma(s,t)f = f * p^\sigma(t,s)$, where $p^\sigma(t,s)$ is a smooth density of a probability measure. By ii) we have $p^\sigma(t,r) * p^\sigma(r,s) = p^\sigma(t,s)$ for $t > r > s$. Existence of the family $U^\sigma(s,t)$ follows from [9].

For $s < t$ let

$$\xi^\sigma(s,t) = \sup_{u\in[s,t]} \|\Phi_{\sigma(u)}\|_{\mathfrak{n}\to\mathfrak{n}}, \tag{4.2}$$

$$\eta^\sigma(s,t) = \sup_{u\in[s,t]} \|\Phi^{-1}_{\sigma(u)}\|_{\mathfrak{n}\to\mathfrak{n}}. \tag{4.3}$$

It is not difficult to show (see [4]) that there exist $\beta_1, \beta_2 > 2$ and $C > 0$ such that for every $a > 0$,

$$\|\Phi_a\|_{\mathfrak{n}\to\mathfrak{n}} \le C(a^{\beta_1} + a^{\beta_2}). \tag{4.4}$$

Therefore,

$$\begin{aligned} \xi^\sigma(s,t) \le& C((\sup_{u\in[s,t]} \sigma(u))^{\beta_1} + (\sup_{u\in[s,t]} \sigma(u))^{\beta_2}), \\ \eta^\sigma(s,t) \le& C((\inf_{u\in[s,t]} \sigma(u))^{-\beta_1} + (\inf_{u\in[s,t]} \sigma(u))^{-\beta_2}). \end{aligned} \tag{4.5}$$

We define:

$$A(s,t) = \int_s^t \sigma^{\beta_1-2}(u) + \sigma^{\beta_2-2}(u)du, \tag{4.6}$$

where β_1 and β_2 are as in (4.4).

Now we are ready to state the main estimates for the derivatives of the evolution kernels $p^\sigma(t,0)$.

Theorem 4.1 ([10], Theorem 4.9)**.** *Let $0 < T_1 < T_2 < t$. For every multi-index I such that $\|X^I\| \le 1$ and for every compact set $K \subset N$ which does not contain the identity e of N, there exist positive constants ξ, C, M, ϑ such that for every $x \in K$ and for every $t > 0$,*

$$\begin{aligned} |X^I p^\sigma(t,0)(x)| \le& C \max\{(T_2 - T_1)^{-\vartheta}, 1\} \\ & \times (1 + \xi^\sigma(T_1,T_2) + \eta^\sigma(T_1,T_2))^M e^{-\xi/A(0,t)}. \end{aligned}$$

where $A(s,t)$ is as in (4.6)*.*

The constant D which is appearing in the next theorems is an arbitrary constant between the local dimension and the dimension at infinity of $(N, X_1, \ldots, X_m)$ (see [14]).

Theorem 4.2 ([10], Theorem 4.10)**.** *Let $0 < T_1 < T_2 < t$. For every multi-index I such that $\|X^I\| \le 1$ there exist positive constants C, M, ϑ*

and $D \le Q$ *such that*

$$\|X^I p^\sigma(t,0)\|_{L^\infty} \le C \max\{(T_2-T_1)^{-\vartheta}, 1\} \left(\int_{T_2}^t \sigma^{-2(1-Q/D)}(u) du \right)^{-D/4} \times (1 + \xi^\sigma(T_1,T_2) + \eta^\sigma(T_1,T_2))^M.$$

By the proof of Theorem 4.9 in [10], taking limit in (4.12) as α tends to zero we immediately get the following corollary.

Corollary 4.3. *Let* $0 < T_1 < T_2 < t$. *For every multi-index* I *such that* $\|X^I\| \le 1$ *and for every compact set* $K \subset N$, *there exist positive constants* C, M, ϑ *such that for every* $x \in K$ *and for every* $t > 0$,

$$|X^I p^\sigma(t,0)(x)| \le C \max\{(T_2-T_1)^{-\vartheta}, 1\} \times (1 + \xi^\sigma(T_1,T_2) + \eta^\sigma(T_1,T_2))^M.$$

The above estimates are not by any means optimal but they allow us to proceed further.

The next two theorems will be used in the proof of Theorem 6.2.

Theorem 4.4 ([3], Theorem 4.1)**.** *For every compact set* $K \subset N$ *which does not contain the identity* e *of* N, *there exist positive constants* C *and* ξ *such that for every* $x \in K$ *and for every* t,

$$p^\sigma(t,0)(x) \le C \left(\int_0^t \sigma_u^{-2(1-Q/D)} du \right)^{-D/2} e^{-\xi/A(0,t)}.$$

In the proof of the above theorem the following estimate of the norm $\|p^\sigma(t,s)\|_{L^\infty(N)}$ has been obtained:

Theorem 4.5. *There exist positive constant* C *and* $D \le Q$ *such that for every* $s < t$,

$$\|p^\sigma(t,s)\|_{L^\infty(N)} \le C \left(\int_s^t \sigma_u^{-2(1-Q/D)} du \right)^{-D/2}.$$

5. Probabilistic lemmas.

From now on we consider the Bessel process σ_t with a parameter $\alpha = 0$ (i.e. $\sigma_t \in \mathrm{BESS}(0)$). In this case $\sigma_t = \|w_t\|$, where w_t is a Brownian motion on $\mathbb{R}^2$.

In this section we prove some lemmas, which are our main tools in writing estimates for derivatives of the Green function (1.2). The first five lemmas will be used to prove Theorem 6.1. The next five are for the proof of Theorem 6.2

In the whole section we use the following notation. For $\eta > 0$, $I_{a,\eta}$ stands for the interval $[a-\eta, a+\eta]$. p^σ is the evolution kernel defined in Section 4 corresponding to the operator (4.1). Finally, $dm(a) = ada$.

Lemma 5.1. *For every $1 > \delta > 0$ and for every multi-index I such that $|I| > 0$ there exists a constant C such that for every $(x, a) \in \{a \leq 1-\delta, \delta \leq |x| \leq 1\}$*

$$\sup_{0<\eta<\delta/2} |\int_0^1 \mathbf{E}_1 X^I p^\sigma(t,0)(x) m(I_{a,\eta})^{-1} 1_{I_{a,\eta}}(\sigma) dt| \leq C. \tag{5.1}$$

Proof. Denote by I the integral in (5.1). We divide the set of all trajectories of the Bessel process $\sigma_t \in \mathrm{BESS}_1(0)$ into two subsets:

$$A = \{\sigma : \sup_{s\in[0,t]} \sigma_s \leq 2\}, \quad B = \{\sigma : \sup_{s\in[0,t]} \sigma_s > 2\}.$$

First we consider the set A. Define two stopping times:

$$\begin{aligned} T_1 &= \inf\{s \leq t : \sigma_s = 1 - \delta/2\} \\ T_2 &= \inf\{T_1 < s \leq t : \sigma_s = 1 - \delta/4 \text{ or } \sigma_s = 1 - 3\delta/4\}. \end{aligned}$$

By Theorem 4.1 applied to the stopping times defined above and the fact that on the set A, $A(0,t) \leq ct$, we have:

$$|I| \leq \int_0^1 e^{-c\xi/t} \mathbf{E}_1 \max\{(T_2 - T_1)^{-\vartheta}, 1\} m(I_{a,\eta})^{-1} 1_{I_{a,\eta}}(\sigma_t) dt, \tag{5.2}$$

where ξ is adjusted to the set $\{x \in N : \delta \leq |x| \leq 1\}$. Now we are going to deal with the expected value in (5.2). To do it, for $n \geq 1$, let $A_n = \{t/2^n < T_2 \leq t/2^{n-1}\}$. Then by the Markov property and Lemma 3.5,

$$\begin{aligned} &\mathbf{E}_1 \max\{(T_2 - T_1)^{-\vartheta}, 1\} m(I_{a,\eta})^{-1} 1_{I_{a,\eta}}(\sigma_t) \\ &\leq \sum_{n=1}^\infty \mathbf{E}_1 \max\{(T_2 - T_1)^{-\vartheta}, 1\} 1_{\{T_2 \leq t/2^{n-1}\}}(\sigma) m(I_{a,\eta})^{-1} 1_{I_{a,\eta}}(\sigma_t) \\ &= \sum_{n=1}^\infty \mathbf{E}_1 \max\{(T_2 - T_1)^{-\vartheta}, 1\} 1_{\{T_2 \leq t/2^{n-1}\}}(\sigma) \\ &\quad \times \mathbf{E}_{\sigma_{t/2^{n-1}}} m(I_{a,\eta})^{-1} 1_{I_{a,\eta}}(\sigma_{t - t/2^{n-1}}) \\ &\leq C t^{-1} \sum_{n=2}^\infty \mathbf{E}_1 \max\{(T_2 - T_1)^{-\vartheta}, 1\} 1_{\{T_2 \leq t/2^{n-1}\}}(\sigma). \end{aligned} \tag{5.3}$$

By the Schwarz inequality we have

$$(5.4)\quad \mathbf{E}_1 \max\{(T_2-T_1)^{-\vartheta},1\}1_{\{T_2\le t/2^{n-1}\}}(\sigma)$$
$$\le (\mathbf{E}_1 \max\{(T_2-T_1)^{-2\vartheta},1\}1_{\{T_2\le t/2^{n-1}\}}(\sigma))^{1/2}(\mathbf{E}_1 1_{\{T_2\le t/2^{n-1}\}}(\sigma))^{1/2}$$

Writing

$$\{T_2\le t/2^{n-1}\} = \bigcup_{l\ge[\log_2(2^{n-1}/t)]}^{\infty}\{1/2^{l+1}\le T_2-T_1<1/2^l\}$$

we have

$$\mathbf{E}_1(T_2-T_1)^{-2\vartheta}1_{\{T_2\le t/2^{n-1}\}}$$
$$\le \sum_{l\ge[\log_2(2^{n-1}/t)]}^{\infty} 2^{(l+1)2\vartheta}\mathbf{P}_1(1/2^{l+1}\le T_2-T_1<1/2^l).$$

Now we have to consider two cases. The first one when $\sigma_{T_2}=1-\delta/4$ and the second when $\sigma_{T_2}=1-3\delta/4$. In the first case, by Lemma 3.2

$$\begin{aligned}\mathbf{P}_1(1/2^{l+1}\le T_2-T_1<1/2^l) \le&\mathbf{P}_{1-\delta/2}(\sup_{s\in[0,1/2^l]}\sigma_s\ge 1-\delta/4)\\ \le& c_1e^{-c_2 2^l}.\end{aligned}$$

In the second case, by Lemma 3.3,

$$\begin{aligned}\mathbf{P}_1(1/2^{l+1}\le T_2-T_1<1/2^l) \le&\mathbf{P}_{1-\delta/2}(\inf_{s\in[0,1/2^l]}\sigma_s\le 1-3\delta/4)\\ \le& c_1e^{-c_2 2^l}.\end{aligned}$$

In any case

$$(5.5)\qquad \mathbf{E}_1(T_2-T_1)^{-2\vartheta}1_{\{T_2\le t/2^{n-1}\}}\le C.$$

Now we are left with the second expectation on the right hand side of (5.4). But by Lemma 3.3

$$(5.6)\quad \mathbf{E}_1 1_{\{T_2\le t/2^{n-1}\}}(\sigma)\le \mathbf{P}_1(\inf_{[0,t/2^{n-1}]}\sigma_s\le 1-\delta/2)\le c_2e^{-c_2 2^{n-1}/t}.$$

Now (5.6), (5.5), (5.4) and (5.3) prove that (5.2) is finite.

Now we consider the set B. Define two stopping times

$$T_1=\inf\{s\le t:\sigma_s=2\},\quad T_2=\inf\{T_1<s\le t:\sigma_s=3/2\}.$$

For $k\ge 0$, let $A_k=\{\sigma: 2+k<\sup_{s\in[0,T_2]}\sigma_s\le 3+k\}$. Then clearly, $B\subset\bigcup_{k=0}^\infty A_k$. Therefore,

$$|\int_0^1 \mathbf{E}_1 X^I p^\sigma(t,0)(x)m(I_{a,\eta})^{-1}1_{I_{a,\eta}}(\sigma_t)1_B(\sigma)dt|\le\sum_{k=0}^\infty |I_k|,$$

where

$$I_k = \int_0^1 \mathbf{E}_1 X^I p^\sigma(t,0)(x) m(I_{a,\eta})^{-1} 1_{I_{a,\eta}}(\sigma_t) 1_{A_k}(\sigma) dt.$$

Our aim is to estimate $|I_k|$. By Theorem 4.1 for $\sigma \in A_k$ we have

$$|X^I p^\sigma(t,0)(x)| \leq C k^{\max\{\beta_1,\beta_2\}M} (T_2 - T_1)^{-\vartheta}.$$

Therefore, it is enough to estimate:

$$\begin{aligned}(5.7)\quad & m(I_{a,\eta})^{-1}\mathbf{E}_1(T_2-T_1)^{-\vartheta}1_{I_{a,\eta}}(\sigma_t)1_{A_k}(\sigma) \\ &= m(I_{a,\eta})^{-1}\mathbf{E}_1(T_2-T_1)^{-\vartheta}1_{A_k}(\sigma)\mathbf{E}_{3/2}1_{I_{a,\eta}}(b_{t-T_2}) \\ &\leq C\mathbf{E}_1(T_2-T_1)^{-\vartheta}(t-T_2)^{-1}1_{A_k}(\sigma).\end{aligned}$$

To obtain the above inequality we have used the Markov property. Now we are going to estimate the right hand side of (5.7).

For $n \geq 1$ and $l \geq 0$, let $B_n = \{\sigma : t - t/2^{n-1} < T_2(\sigma) \leq t - t/2^n\}$ and $C_l = \{1/2^{l+1} < T_2 - T_1 \leq 1/2^l\}$. First, given k, n, l we estimate (5.7) on the set $B_n \cap C_l$.

$$\begin{aligned}& \mathbf{E}_1(T_2-T_1)^{-\vartheta}(t-T_2)^{-1}1_{A_k}(\sigma)1_{B_n}(\sigma)1_{C_l}(\sigma) \\ (5.8)\qquad & \leq 2^{(l+1)\vartheta}2^n\mathbf{E}_1 1_{A_k}(\sigma)1_{B_n}(\sigma)1_{C_l}(\sigma) \\ & \leq 2^{(l+1)\vartheta}2^n(\mathbf{E}_1 1_{A_k}(\sigma))^{1/2}(\mathbf{E}_1 1_{B_n}(\sigma))^{1/4}(\mathbf{E}_1 1_{C_l}(\sigma))^{1/4}.\end{aligned}$$

By Lemma 3.2

$$(5.9)\qquad \mathbf{E}_1 1_{A_k}(\sigma) \leq c_1 e^{-c_2 k^2/t}.$$

By Lemma 3.3 we have

$$(5.10)\qquad \mathbf{E}_1 1_{B_n}(\sigma) \leq \mathbf{P}_{3/2}(\inf_{s\in[0,t/2^{n-1}]} \sigma_s \leq 1) \leq c_1 e^{-c_2 2^n/t}$$

and

$$(5.11)\qquad \mathbf{E}_1 1_{C_l}(\sigma) \leq \mathbf{P}_2(\inf_{s\in[0,1/2^l]} \sigma_s \leq 3/2) \leq c_1 e^{-c_2 2^l}.$$

Summing up (5.8) over n and l and using estimates (5.9), (5.10) and (5.11) we get that

$$(5.12)\qquad \mathbf{E}_1(T_2-T_1)^{-\vartheta}(t-T_2)^{-1}1_{A_k}(\sigma) \leq c_1 e^{-c_2 k^2/t}.$$

Finally, by (5.7) and (5.12) it follows that

$$|I_k| \leq C k^{\max\{\beta_1,\beta_2\}M} \int_0^1 e^{-c_2 k^2/t} dt,$$

which shows that $\sum_{k=0}^\infty |I_k|$ is finite. □

Lemma 5.2. *For every $1 > \delta > 0$ and for every multi-index I such that $|I| > 0$ there exists a constant C such that for every $(x, a) \in \{a \le 1-\delta\}$*

$$\sup_{0<\eta<\delta/2} \left| \int_1^\infty \mathbf{E}_1 X^I p^\sigma(t,0)(x) m(I_{a,\eta})^{-1} 1_{I_{a,\eta}}(\sigma) dt \right| \le C. \tag{5.13}$$

Proof. We will consider two cases. The first one, when $\inf_{s\in[0,t/2]} \sigma_s \le 1-\delta$ and the second one when $\inf_{s\in[0,t/2]} \sigma_s > 1-\delta$.

Case 1. We define two stopping times:

$$\begin{aligned} T_1 =& \inf\{s < t/2 : \sigma_s = 1 - \delta/2\}, \\ T_2 =& \inf\{T_1 < s < t/2 : \sigma_s = 1 - \delta/4 \text{ or } \sigma_s = 1 - 3\delta/4\}. \end{aligned}$$

First we want to estimate the left hand side of (5.13) on the set $\Omega_1 = \{\inf_{s\in[0,t/2]} \sigma_s \le 1-\delta\}$, i.e. the absolute value of the following integral,

$$I = \int_1^\infty \mathbf{E}_1 X^I p^\sigma(t,0)(x) m(I_{a,\eta})^{-1} 1_{I_{a,\eta}}(\sigma_t) 1_{\Omega_1}(\sigma) dt.$$

Since on Ω_1 we have $T_2 \le t/2$, by Theorem 4.2 and (4.5) we get that

$$|X^I p^\sigma(t,0)(x)| \le C \max\{(T_2-T_1)^{-\vartheta}, 1\} \left(\int_{t/2}^{3t/4} \sigma_u^{-2(1-Q/D)} du \right)^{-D/4},$$

where C depends on β_1 and β_2. Hence, by the Markov property and Lemma 3.5,

$$\begin{aligned} |I| \le& C \int_1^\infty \mathbf{E}_1 \max\{(T_2-T_1)^{-\vartheta}, 1\} 1_{\Omega_1}(\sigma) \\ &\times \left(\int_{t/2}^{3t/4} \sigma_u^{-2(1-Q/D)} du \right)^{-D/4} \mathbf{E}_{\sigma_{3t/4}} m(I_{a,\eta})^{-1} 1_{I_{a,\eta}}(b_{t/4}) dt \\ \le& C \int_1^\infty t^{-1} \mathbf{E}_1 \max\{(T_2-T_1)^{-\vartheta}, 1\} 1_{\Omega_1}(\sigma) \\ &\times \left(\int_{t/2}^{3t/4} \sigma_u^{-2(1-Q/D)} du \right)^{-D/4} dt. \end{aligned}$$

Now, by the Schwarz inequality and Lemma 3.1 we obtain

$$\begin{aligned}|I| \leq & C \int_1^\infty t^{-1} (\mathbf{E}_1 \max\{(T_2 - T_1)^{-4\vartheta}, 1\})^{1/4} \\ & \times \mathbf{P}_1(\Omega_1)^{1/4} \left(\mathbf{E}_1 \left(\int_{t/2}^{3t/4} \sigma_u^{-2(1-Q/D)} du \right)^{-D/2} \right)^{1/2} \\ \leq & C \int_1^\infty t^{-1-Q/2} (\mathbf{E}_1 \max\{(T_2 - T_1)^{-4\vartheta}, 1\})^{1/4} dt.\end{aligned}$$

But

$$\begin{aligned}\mathbf{E}_1 \max\{(T_2 - T_1)^{-4\vartheta}, 1\} & \leq 1 + \mathbf{E}_1 (T_2 - T_1)^{-4\vartheta} \\ & \leq 2 + \sum_{l=0}^\infty 2^{(l+1)4\vartheta} \mathbf{P}_1(1/2^{l+1} \leq T_2 - T_1 \leq 1/2^l)\end{aligned}$$

and

$$\begin{aligned}& \mathbf{P}_1(1/2^{l+1} \leq T_2 - T_1 \leq 1/2^l) \\ & \quad \leq \mathbf{P}_{1-\delta/2}(\inf_{s\in[0,1/2^l]} \sigma_s \leq 1 - 3\delta/4) + \mathbf{P}_{1-\delta/2}(\sup_{s\in[0,1/2^l]} \sigma_s \geq 1 - \delta/4).\end{aligned}$$

Therefore, by Lemma 3.3 and Lemma 3.2,

$$\mathbf{E}_1 \max\{(T_2 - T_1)^{-4\vartheta}, 1\} \leq 2 + c_1 \sum_{l=0}^\infty 2^{(l+1)4\vartheta} e^{-c_2 2^l} \leq C.$$

Thus, $|I|$ is finite and the proof of the first case is done.

Case 2. Now we are on the set $\Omega_2 = \{\inf_{s\in[0,t/2]} > 1 - \delta\}$. For $k \geq 0$ define the following sets:

$$A_k = \begin{cases} \{\sigma : \sup_{s\in[0,t/2]} \sigma_s \geq 2\} & \text{for } k = 0, \\ \{\sigma : 1 + 1/(k+1) \leq \sup_{s\in[0,t/2]} \sigma_s < 1 + 1/k\} & \text{for } k \geq 1. \end{cases}$$

Let

$$\begin{aligned}T_1 = T_1(k) &= \inf\{0 < s < t/2 : \sigma_s = 1 + 1/2(k+1)\}, \\ T_2 = T_2(k) &= \inf\{T_1 < s < t/2 : \sigma_s = 1 + 1/(k+1) \\ & \qquad \text{or } \sigma_s = 1 + 1/4(k+1)\}.\end{aligned}$$

Now we want to estimate (5.13) on the set $\Omega_2 \cap A_k$, i.e. the absolute value of the following integral:

$$I = \int_1^\infty \mathbf{E}_1 X^I p^\sigma(t,0)(x) m(I_{a,\eta})^{-1} 1_{I_{a,\eta}}(\sigma_t) 1_{\Omega_2}(\sigma) 1_{A_k}(\sigma) dt.$$

Since on A_k we have $T_2 \le t/2$, by Theorem 4.2 and (4.5) we get that

$$|X^I p^\sigma(t,0)(x)| \le C \max\{(T_2-T_1)^{-2\vartheta},1\}(k+1)^{M\beta} \times \left(\int_{t/2}^{3t/4} \sigma_u^{-2(1-Q/D)} du\right)^{-D/4},$$

where $\beta = \max\{\beta_1, \beta_2\}$.

Hence, by the Markov property and Lemma 3.5,

$$\begin{aligned} |I| \le & C(k+1)^{M\beta} \int_1^\infty \mathbf{E}_1 \max\{(T_2-T_1)^{-2\vartheta},1\} 1_{A_k}(\sigma) 1_{\Omega_2}(\sigma) \\ & \times \left(\int_{t/2}^{3t/4} \sigma_u^{-2(1-Q/D)} du\right)^{-D/4} \mathbf{E}_{\sigma_{3t/4}} m(I_{a,\eta})^{-1} 1_{I_{a,\eta}}(b_{t/4}) dt \\ \le & C(k+1)^{M\beta} \int_1^\infty t^{-1} \mathbf{E}_1 \max\{(T_2-T_1)^{-2\vartheta},1\} 1_{A_k}(\sigma) \\ & \times \left(\int_{t/2}^{3t/4} \sigma_u^{-2(1-Q/D)} du\right)^{-D/4} dt. \end{aligned}$$

Now by the Schwarz inequality and Lemma 3.1 we obtain

$$\begin{aligned} |I_k| \le & C(k+1)^{M\beta} \int_1^\infty t^{-1} (\mathbf{E}_1 \max\{(T_2-T_1)^{-4\vartheta},1\})^{1/4} \\ & \times \mathbf{P}_1(A_k)^{1/4} \left(\mathbf{E}_1 \left(\int_{t/2}^{3t/4} \sigma_u^{-2(1-Q/D)} du\right)^{-D/2}\right)^{1/2} \\ \le & C(k+1)^{M\beta} \int_1^\infty t^{-1-Q/2} (\mathbf{E}_1 \max\{(T_2-T_1)^{-4\vartheta},1\})^{1/4} \\ & \times \mathbf{P}_1(A_k)^{1/4} dt. \end{aligned}$$

But, as in the proof of the previous case,

$$\mathbf{E}_1 \max\{(T_2-T_1)^{-4\vartheta},1\} \le 2 + \sum_{l=0}^\infty 2^{(l+1)4\vartheta} \mathbf{P}_1(1/2^{l+1} \le T_2 - T_1 \le 1/2^l)$$

and

$$\begin{aligned} & \mathbf{P}_1(1/2^{l+1} \le T_2 - T_1 \le 1/2^l) \\ & \le \mathbf{P}_{1+1/2(k+1)}\big(\inf_{s\in[0,1/2^l]} \sigma_s \le 1 + 1/4(k+1)\big) \\ & \quad + \mathbf{P}_{1+1/2(k+1)}\big(\sup_{s\in[0,1/2^l]} \sigma_s \ge 1 + 1/(k+1)\big). \end{aligned}$$

Therefore, by Lemma 3.3 and Lemma 3.2,

$$\mathbf{E}_1 \max\{(T_2-T_1)^{-4\vartheta}, 1\} \le 2+c_1 \sum_{l=0}^{\infty} 2^{(l+1)4\vartheta} e^{-c_2 2^l/(k+1)^2} \le C_1(k+1)^{C_2\vartheta}.$$

On the other hand by Lemma 3.4, $\mathbf{P}_1(A_k) \le c_1 e^{-c_2\sqrt{t}k}$ so we obtain that

$$|I_k| \le C(k+1)^{M\beta+C_2\vartheta} \int_1^\infty t^{-1-Q/2} e^{-c_2\sqrt{t}k} dt.$$

Since the left hand side of (5.13) is equal to $|\sum_{k=0}^\infty I_k|$ the proof is complete. □

Lemma 5.3. *For every $1 > \delta > 0$ and for every multi-index I such that $|I| > 0$ there exists a constant C such that for every $(x, a) \in \{a \le 1-\delta, |x| \le \delta\}$*

$$\sup_{0<\eta<\delta/2} \left| \int_0^1 \mathbf{E}_1 X^I p^\sigma(t,0)(x) m(I_{a,\eta})^{-1} 1_{I_{a,\eta}}(\sigma) dt \right| \le C. \tag{5.14}$$

Proof. Define two stopping times:

$$T_1 = \inf\{s \le t : \sigma_s = 1-\delta/2\},$$
$$T_2 = \inf\{T_1 < s \le t : \sigma_s = 1-\delta/4 \text{ or } \sigma_s = 1-3\delta/4\}.$$

By Corollary 4.3 and (4.5) we obtain

$$I := \int_0^1 \mathbf{E}_1 |X^I p^\sigma(t,0)(x)| m(I_{a,\eta})^{-1} 1_{I_{a,\eta}}(\sigma_t) dt$$
$$\le C \int_0^1 \mathbf{E}_1 (T_2-T_1)^{-\vartheta} m(I_{a,\eta})^{-1} 1_{I_{a,\eta}}(\sigma_t) dt.$$

For $n \ge 1$, $l \ge 0$, let $B_n = \{\sigma : t - t/2^{n-1} < T_2(\sigma) \le t - t/2^n\}$ and $C_l = \{1/2^{l+1} \le T_2 - T_1 \le 1/2^l\}$. Then by the Markov property and Lemma 3.5, we get

$$\begin{aligned}
I \le & C \int_0^1 m(I_{a,\eta})^{-1} \mathbf{E}_1 (T_2-T_1)^{-\vartheta} \mathbf{E}_{\sigma_{T_2}} 1_{I_{a,\eta}}(b_{t-T_2}) dt \\
\le & C \int_0^1 \mathbf{E}_0 (T_2-T_1)^{-\vartheta} (t-T_2)^{-1} dt \\
\le & C \sum_{n\ge1, l\ge0} 2^n 2^{(l+1)\vartheta} \int_0^1 t^{-1} \mathbf{P}_1(B_n \cap C_l) dt \\
\le & C \sum_{n\ge1, l\ge0} 2^n 2^{(l+1)\vartheta} \int_0^1 t^{-1} \mathbf{P}_1(B_n)^{1/2} \mathbf{P}_1(C_l)^{1/2} dt.
\end{aligned}$$

But

$\mathbf{P}_1(B_n) \le \mathbf{P}_{1-\delta/4}(\inf_{s\in[0,t/2^{n-1}]} \sigma_s \le 1-\delta) + \mathbf{P}_{1-3\delta/4}(\inf_{s\in[0,t/2^{n-1}]} \sigma_s \le 1-\delta)$

and

$\mathbf{P}_1(C_l) \le \mathbf{P}_{1-\delta/2}(\inf_{s\in[0,1/2^l]} \sigma_s \le 1-3\delta/4) + \mathbf{P}_{1-\delta/2}(\sup_{s\in[0,1/2^l]} \sigma_s \ge 1-\delta/4).$

Therefore, by Lemma 3.2 and Lemma 3.3,

$$I \le C \sum_{n\ge 1, l\ge 0}^{\infty} 2^n 2^{(l+1)\vartheta} \int_0^1 t^{-1} e^{-c_2 2^{n-1}/t} e^{-c_2 2^l} dt \le C.$$

□

Lemma 5.4. *For every $0 < \chi_0 \le 1$, $0 < r_0 \le 1$ and for every multi-index I such that $|I| > 0$ there exists a constant C such that for every $\chi \le \chi_0$ and for every $(x,a) \in \{0 < a \le 1, r_0 \le |x| \le 1\}$,*

$$\sup_{0<\eta<1} \Big| \int_0^\infty \mathbf{E}_\chi X^I p^\sigma(t,0)(x) m(I_{a,\eta})^{-1} 1_{I_{a,\eta}}(\sigma) dt \Big| \le C. \tag{5.15}$$

Proof. First we consider small time, i.e. $t \le 1$. Thus we have to show that (5.15) holds with $\int_0^\infty$ replaced by $\int_0^1$. To do this we divide the set of all trajectories of the Bessel process $\sigma_t \in \mathrm{BESS}_\chi(0)$ into two subsets:

$$A = \{\sigma : \sup_{s\in[0,t/2]} \sigma_s \le \chi + 2\}, \quad B = \{\sigma : \sup_{s\in[0,t/2]} \sigma_s > \chi + 2\}.$$

Consider the set A. For $k \ge -1$, let

$$A_k = \{\sigma : \chi + 1/2^{k+1} < \sup_{s\in[0,t/2]} \sigma_s \le \chi + 1/2^k\}.$$

Then clearly, $A = \bigcup_{k=-1}^\infty A_k$. Therefore,

$$\int_0^1 \mathbf{E}_\chi X^I p^\sigma(t,0)(x) m(I_{a,\eta})^{-1} 1_{I_{a,\eta}}(\sigma_t) 1_A(\sigma) dt = \sum_{k=-1}^\infty I_k,$$

where

$$I_k = \int_0^1 \mathbf{E}_\chi X^I p^\sigma(t,0)(x) m(I_{a,\eta})^{-1} 1_{I_{a,\eta}}(\sigma_t) 1_{A_k}(\sigma) dt.$$

Define two stopping times

$T_1 = T_1(k) = \inf\{s \le t/2 : \sigma_s = \chi + 1/2^{k+2}\},$

$T_2 = T_2(k) = \inf\{T_1 < s \le t/2 : \sigma_s = \chi + 1/2^{k+1} \text{ or } \sigma_s = \chi + 1/2^{k+3}\}.$

Now, since $\chi \le \chi_0 \le 1$ all the calculations from the proof of Lemma 5.1 in [10], for A_k, T_1 and T_2 defined as above remain valid.

On the set B we proceed by the same token as in the proof of Lemma 5.1 in [10]. We only need to replace the stopping times T_1 and T_2 defined there by the following ones:

$$T_1 = \inf\{s \leq t/2 : \sigma_s = \chi + 1\},$$
$$T_2 = \inf\{T_1 < s \leq t/2 : \sigma_s = \chi + 3/2 \text{ or } \sigma_s = \chi + 1/2\}.$$

Now we are left with $t \geq 1$. So, we have to prove (5.15) with $\int_0^\infty$ replaced by $\int_1^\infty$. In order do this it is enough to imitate the proof of Lemma 5.2 in [10] with the following slight modifications of A_k, T_1 and T_2 :

$$A_k = \begin{cases} \{\sigma : \sup_{s\in[0,t/2]} \sigma_s \geq \chi + 1\} & \text{for } k = 0, \\ \{\sigma : \chi + 1/(k+1) \leq \sup_{s\in[0,t/2]} \sigma_s < \chi + 1/k\} & \text{for } k \geq 1. \end{cases}$$

The corresponding stopping times now are defined as

$$T_1 = T_1(k) = \inf\{0 < s < t/2 : \sigma_s = \chi + 1/2(k+1)\},$$
$$T_2 = T_2(k) = \inf\{T_1 < s < t/2 : \sigma_s = \chi + 1/(k+1) \text{ or } \sigma_s = \chi + 1/3(k+1)\}.$$

□

Lemma 5.5. *For every $0 < \delta < 1/2$ and for every multi-index I such that $|I| > 0$ there exists a constant C such that for every $\chi \leq 1/2 - \delta$ and for every $(x, a) \in \{(1-\delta)/2 \leq a \leq 1/2\}$,*

$$\sup_{0<\eta<\delta/4} \Big| \int_0^\infty \mathbf{E}_\chi X^I p^\sigma(t,0)(x) m(I_{a,\eta})^{-1} 1_{I_{a,\eta}}(\sigma) dt \Big| \leq C. \tag{5.16}$$

Proof. First we prove (5.16) with $\int_0^\infty$ replaced by $\int_0^1$. The proof is virtually the same as the proof of Lemma 5.3. The only difference is in the definition of the stopping times T_1 and T_2 :

$$T_1 = \inf\{s \leq t : \sigma_s = 1/2 - 3\delta/4\},$$
$$T_2 = \inf\{T_1 < s \leq t : \sigma_s = 1/2 - 5\delta/8 \text{ or } \sigma_s = 1/2 - 7\delta/8\}.$$

In order to prove (5.16) with $\int_0^\infty$ replaced by $\int_1^\infty$ it is enough to rewrite the proof of Lemma 5.2 in [10] with new A_k, T_1 and T_2 :

$$A_k = \begin{cases} \{\sigma : \sup_{s\in[0,t/2]} \sigma_s \geq \chi + 1/2\} & \text{for } k = 0, \\ \{\sigma : \chi + 1/2(k+1) \leq \sup_{s\in[0,t/2]} \sigma_s < \chi + 1/2k\} & \text{for } k \geq 1 \end{cases}$$

and

$$T_1 = T_1(k) = \inf\{0 < s < t/2 : \sigma_s = \chi + 1/4(k+1)\},$$
$$T_2 = T_2(k) = \inf\{T_1 < s < t/2 : \sigma_s = \chi + 1/2(k+1)$$
$$\text{or } \sigma_s = \chi + 1/8(k+1)\}.$$

□

Lemma 5.6. *Let $k \geq 1$ be fixed. For every $1 > \delta > 0$ there exists a constant C such that for every $(x,a) \in \{a \leq 1-\delta\}$ and for every $0 \leq l \leq k-1$,*

$$\sup_{0<\eta<\delta/2} \left| \int_1^\infty \mathbf{E}_1 p^\sigma(t,0)(x) \partial_a^l m(I_{a,\eta})^{-1} \partial_a^{k-l} 1_{I_{a,\eta}}(\sigma) dt \right| \leq C a^{-k}.$$

Proof. We have

$$\partial_a^l m(I_{a,\eta})^{-1} = (-1)^l l! m(I_{a,\eta})^{-(l+1)} (2\eta)^l, \quad l \geq 0 \tag{5.17}$$

and for every $\chi > 0$,

$$\begin{aligned} \mathbf{E}_\chi \partial_a 1_{I_{a,\eta}}(\sigma_t) = &\lim_{h\to 0} h^{-1}(\mathbf{P}_\chi(a+\eta \leq \sigma_t \leq a+\eta+h) \\ &- \mathbf{P}_\chi(a-\eta+h \leq \sigma_t \leq a-\eta)) \\ =&p_t(\chi, a+\eta)(a+\eta) - p_t(\chi, a-\eta)(a-\eta), \end{aligned} \tag{5.18}$$

where p_t is the transition function (3.1). Formula (5.18) together with (3.1) allow us to calculate $\mathbf{E}_\chi \partial_a^l 1_{I_{a,\eta}}(\sigma_t)$ for $l \geq 2$,

(5.19)
$$\begin{aligned} \mathbf{E}_\chi \partial_a^l 1_{I_{a,\eta}}(\sigma_t) =&(2t)^{-1} e^{-\chi^2/4t} \partial_a^{l-1} (e^{-(a+\eta)^2/4t} I_0(\chi(a+\eta)/2t)(a+\eta) \\ &- e^{-(a-\eta)^2/4t} I_0(\chi(a-\eta)/2t)(a-\eta)). \end{aligned}$$

We have to estimate the absolute value of

$$I_l^{\text{large}} := \lim_{\eta\to 0} \int_1^\infty \mathbf{E}_1 p^\sigma(t,0)(x) \partial_a^l m(I_{a,\eta})^{-1} \partial_a^{k-l} 1_{I_{a,\eta}}(\sigma_t) dt. \tag{5.20}$$

Since $\lim_{\eta\to 0} \frac{(2\eta)^l}{m(I_{a,\eta})^l} = C a^{-l}$, by Theorem 4.5, the Markov property and Lemma 3.1 we get,

$$\begin{aligned} |I_l^{\text{large}}| \leq &C \lim_{\eta\to 0} \int_1^\infty \mathbf{E}_1 \left(\int_0^{t/2} \sigma_u^{-2(1-Q/D)} du \right)^{-D/2} \frac{(2\eta)^l}{m(I_{a,\eta})^l} \\ &\times |\mathbf{E}_{\sigma_{t/2}} m(I_{a,\eta})^{-1} \partial_a^{k-l} 1_{I_{a,\eta}}(b_{t/2})| dt \\ =&C a^{-l} \int_1^\infty t^{-Q/2} \lim_{\eta\to 0} |\mathbf{E}_{\sigma_{t/2}} m(I_{a,\eta})^{-1} \partial_a^{k-l} 1_{I_{a,\eta}}(b_{t/2})| dt. \end{aligned}$$

Using (5.18), (5.19) and the following formulae (cf. [7]):

$$\frac{d}{dx}I_\alpha(x) = I_{\alpha-1}(x) - \frac{\alpha}{x}I_\alpha(x) = I_{\alpha+1}(x) + \frac{\alpha}{x}I_\alpha(x)$$

we get, after not difficult but a little tedious computation, that

$$\lim_{\eta\to 0} |\mathbf{E}_\chi m(I_{a,\eta})^{-1}\partial_a^{k-l}1_{I_{a,\eta}}(b_{t/2})| \le Ct^{-k+l-2}a^{-k+l}e^{-\chi^2/4t}e^{-a^2/4t} \times \sum_{(w_1,w_2,w_3,w_4)\in W} c_{w_1,w_2,w_3,w_4}\chi^{w_1}a^{w_2}t^{w_3}I_{w_4}(\chi a/2t), \tag{5.21}$$

where

$$W = \{(w_1,w_2,w_3,w_4) : 0 \le w_1 \le k-l+1, 0 \le w_2 \le 2(k-l)+1, 0 \le w_3 \le k-l, w_4 \in \{0,1\} \text{ and } w_1/2 + w_3 < k-l+1\}$$

and c_{w_1,w_2,w_3,w_4} are nonnegative real numbers. Assume now that $a \le 1$. Then, by $w_1/2 + w_3 < k-l+1$, using asymptotic behavior (3.2) of I_0 and I_1, we can estimate (5.21), independently of χ, as follows:

$$\lim_{\eta\to 0} |\mathbf{E}_\chi m(I_{a,\eta})^{-1}\partial_a^{k-l}1_{I_{a,\eta}}(b_{t/2})| \le Ct^{-k+l-2}a^{-k+l} \sum_{(w_1,w_2,w_3,w_4)\in A} c_{w_1,w_2,w_3,w_4}a^{w_2}t^{w_1/2+w_3} \le Ct^{-1}a^{-k+l}. \tag{5.22}$$

Thus,

$$|I_l^{\text{large}}| \le Ca^{-k}\int_1^\infty t^{-Q/2-1}dt \le Ca^{-k}.$$

So, the proof is complete. □

Lemma 5.7. *Let $k \ge 1$ be fixed. For every $1 > \delta > 0$ there exists a constant C such that for every $(x,a) \in \{a \le 1-\delta, \delta \le |x| \le 1\}$ and for every $0 \le l \le k-1$,*

$$\sup_{0<\eta<\delta/2} |\int_0^1 \mathbf{E}_1 p^\sigma(t,0)(x)\partial_a^l m(I_{a,\eta})^{-1}\partial_a^{k-l}1_{I_{a,\eta}}(\sigma)dt| \le Ca^{-k}.$$

Proof. Define

$$I_l^{\text{small}} := \sup_{0<\eta<\delta/2}\int_0^1 \mathbf{E}_1 p^\sigma(t,0)(x)\partial_a^l m(I_{a,\eta})^{-1}\partial_a^{k-l}1_{I_{a,\eta}}(\sigma)dt.$$

Divide the set of all trajectories of the Bessel process $\text{BESS}_1(0)$ into two subsets:

$$A = \{\sigma : \sup_{s\in[0,t]}\sigma_s > 2\} \text{ and } B = \{\sigma : \sup_{s\in[0,t]}\sigma_s \le 2\}.$$

Consider the set A. Let $T = \inf\{s : \sigma_s = 2\}$. Let, for $n \geq 1$, $A_n = \{\sigma : t/2^n < T \leq t/2^{n-1}\}$. Then, by the Markov property and Theorem 4.5,

$$
\begin{aligned}
&|I_l^{\mathrm{small}}| \\
\leq &C|\lim_{\eta\to 0}\sum_{n=1}^{\infty}\int_0^1 \mathbf{E}_1\left(\int_0^{t/2^n}\sigma_u^{-2(1-Q/D)}du\right)^{-D/2} \\
&\times \frac{(2\eta)^l}{m(I_{a,\eta})^l} m(I_{a,\eta})^{-1}\partial_a^{k-l}1_{I_{a,\eta}}(\sigma_t)1_A(\sigma)dt| \\
\leq &Ca^{-l}\sum_{n=1}^{\infty}\lim_{\eta\to 0}\int_0^1 |\mathbf{E}_1\left(\int_0^{t/2^n}\sigma_u^{-2(1-Q/D)}du\right)^{-D/2} \\
&\times m(I_{a,\eta})^{-1}\partial_a^{k-l}1_{I_{a,\eta}}(\sigma_t)1_{\{T\leq t/2^{n-1}\}}(\sigma)|dt \\
= &Ca^{-l}\sum_{n=1}^{\infty}\lim_{\eta\to 0}\int_0^1 \mathbf{E}_1\left(\int_0^{t/2^n}\sigma_u^{-2(1-Q/D)}du\right)^{-D/2} \\
&\times 1_{\{\sup_{s\in[0,t/2^{n-1}]}\sigma_s\geq 2\}}(\sigma)|\mathbf{E}_{\sigma_{t/2^{n-1}}} m(I_{a,\eta})^{-1}\partial_a^{k-l}1_{I_{a,\eta}}(b_{t-t/2^{n-1}})|dt.
\end{aligned}
$$

By (5.22) and the Schwarz inequality we get that

$$
\begin{aligned}
|I_l^{\mathrm{small}}| \leq Ca^{-k+l}\sum_{n=1}^{\infty}\int_0^1 t^{-1}\mathbf{E}_1&\left[\left(\int_0^{t/2^n}\sigma_u^{-2(1-Q/D)}du\right)^{-D}\right]^{1/2} \\
&\times \mathbf{E}_1\left[1_{\{\sup_{s\in[0,t/2^{n-1}]}\sigma_s\geq 2\}}(\sigma)\right]^{1/2} dt.
\end{aligned}
$$

By Theorem 3.1 and Lemma 3.2 we obtain

$$
|I_l^{\mathrm{small}}| \leq Ca^{-k+l}\sum_{n=1}^{\infty}\int_0^1 t^{-1}(t/2^n)^{-Q/2}e^{-c2^{n-1}/t}dt \leq Ca^{-k+l} \leq Ca^{-k}.
$$

Now we consider the set B. Notice that $\sup_{s\in[0,t]}\sigma_s \leq 2$ implies that $A(0,t)$ in (4.6) is less than or equal to Ct. Thus, using Theorem 4.4, the Markov property, Lemma 3.1 and (5.22),

$$\begin{aligned}|I_l^{\text{small}}| \leq & Ca^{-l} \lim_{\eta\to 0} | \int_0^1 \mathbf{E}_1 \left(\int_0^t \sigma_u^{-2(1-Q/D)} du \right)^{-D/2} \\ & \times e^{-c/t} m(I_{a,\eta})^{-1} \partial_a^{k-l} 1_{I_{a,\eta}}(\sigma_t) | dt \\ \leq & Ca^{-l} \lim_{\eta\to 0} \int_0^1 \left(\int_0^{t/2} \sigma_u^{-2(1-Q/D)} du \right)^{-D/2} \\ & \times e^{-c/t} m(I_{a,\eta})^{-1} |\mathbf{E}_{\sigma_{t/2}} \partial_a^{k-l} 1_{I_{a,\eta}}(\sigma_t)| dt \\ \leq & Ca^{-k+l} \int_0^1 e^{-c/t} t^{-Q/2-1} dt \leq Ca^{-k}.\end{aligned}$$

□

Lemma 5.8. *Let $k \geq 1$ be fixed. For every $1 > \delta > 0$ there exists a constant C such that for every $(x, a) \in \{a \leq 1 - \delta, |x| \leq \delta\}$ and for every $0 \leq l \leq k - 1$,*

$$\sup_{0<\eta<\delta/2} | \int_0^1 \mathbf{E}_1 p^\sigma(t,0)(x) \partial_a^l m(I_{a,\eta})^{-1} \partial_a^{k-l} 1_{I_{a,\eta}}(\sigma) dt | \leq Ca^{-k}.$$

Proof. Define the following stopping time:

$$T = \inf\{s \leq t : \sigma_s = 1 - \delta/2\}.$$

Let I_l^{small} be defined as in the previous lemma. Then by the Markov property and Theorem 4.5,

$$\begin{aligned}|I_l^{\text{small}}| = & \lim_{\eta\to 0} | \int_0^1 \mathbf{E}_1 p^\sigma(t,0)(x) \partial_a^l m(I_{a,\eta})^{-1} \partial_a^{k-l} 1_{I_{a,\eta}}(\sigma) dt| \\ \leq & Ca^{-l} \lim_{\eta\to 0} \int_0^1 \mathbf{E}_1 (\int_0^T \sigma_u^{-2(1-Q/D)} du)^{-D/2} \\ & \times |\mathbf{E}_{\sigma_T} m(I_{a,\eta})^{-1} \partial_a^{k-l} 1_{I_{a,\eta}}(\sigma)| dt.\end{aligned}$$

By (5.22) and Lemma 3.1 we get

$$|I_l^{\text{small}}| \leq Ca^{-k} \int_0^1 t^{-1} \mathbf{E}_1 T^{-Q/2} dt. \tag{5.23}$$

Now we are going to estimate the expectation above. For $n \geq 0$, let $A_n = \{\sigma : t/2^{n+1} < T \leq t/2^n\}$. Then, by Lemma 3.3,

$$
\begin{aligned}
\mathbf{E}_1 T^{-Q/2} &\leq \sum_{n=1}^{\infty} 2^{(n+1)Q/2} \mathbf{P}_1(A_n) \\
&\leq \sum_{n=1}^{\infty} 2^{(n+1)Q/2} \mathbf{P}_1(\inf_{s \in [0,t/2^n]} \sigma_s \leq 1 - \delta/2) \qquad (5.24) \\
&\leq c_1 \sum_{n=1}^{\infty} 2^{(n+1)Q/2} e^{-c_2 2^n/t}.
\end{aligned}
$$

Now (5.23) and (5.24) complete the proof. □

The next two lemmas have in fact been proved between the lines during the course of proving the previous lemmas. However, in order to make the paper more transparent and to avoid complicated formulation we decided to state them separately and to explain how they can be deduced from the previous considerations.

Lemma 5.9. *Let $k \geq 1$ be fixed. For every $0 < \chi_0 \leq 1$, $0 < r_0 \leq 1$ there exists a constant C such that for every $\chi \leq \chi_0$ and for every $(x, a) \in \{0 < a \leq 1, r_0 \leq |x| \leq 1\}$ and for every $0 \leq l \leq k-1$,*

$$\sup_{0<\eta<1} \left| \int_0^\infty \mathbf{E}_\chi p^\sigma(t,0)(x) \partial_a^l m(I_{a,\eta})^{-1} \partial_a^{k-l} 1_{I_{a,\eta}}(\sigma) dt \right| \leq C a^{-k}.$$

Proof. For small time ($t \leq 1$) we proceed as in the proof of Lemma 5.7 changing the definitions of the sets A and B into

$$A = \{\sigma : \sup_{s\in[0,t]} \sigma_s > \chi + 2\} \quad B = \{\sigma : \sup_{s\in[0,t]} \sigma_s \leq \chi + 2\}$$

and setting $T = \inf\{s \leq t : \sigma_s = 2 + \chi\}$ on A.

For large time ($t > 1$) the proof of Lemma 5.6 works. □

Lemma 5.10. *For every $0 < \delta < 1/2$ and for every multi-index I such that $|I| > 0$ there exists a constant C such that for every $\chi \leq 1/2 - \delta$ and for every $(x, a) \in \{(1-\delta)/2 \leq a \leq 1/2\}$,*

$$\sup_{0<\eta<\delta/4} \left| \int_0^\infty \mathbf{E}_\chi p^\sigma(t,0)(x) \partial_a^l m(I_{a,\eta})^{-1} \partial_a^{k-l} 1_{I_{a,\eta}}(\sigma) dt \right| \leq C a^{-k}.$$

Proof. For small time imitate the proof of Lemma 5.8 with a new $T = \inf\{s \leq t : \sigma_s = 1/2 - 3\delta/4\}$. Of course in (5.24) we have to put $\mathbf{P}_\chi(\sup_{s\in[0,t/2^n]} \sigma_s \geq 1/2 - \delta/2)$ instead of $\mathbf{P}_1(\inf_{s\in[0,t/2^n]} \sigma_s \leq 1 - \delta/2)$.

For large t, notice that proof of Lemma 5.6 works. □

6. Estimates for derivatives of the Green function $\mathcal{G}$

In this section we obtain pointwise estimates for derivatives of the Green function (1.2) in the noncoercive case.

For a positive $\delta < 1/2$ define

$$\begin{aligned} T_\delta =& \{(x,a) \in N \times \mathbb{R}^+ : 1-\delta < a < 1+\delta, |x| < \delta\}, \\ \mathcal{Q} =& \{(x,a) \in N \times \mathbb{R}^+ : |x| \le 1, a \le 1\}. \end{aligned}$$

Theorem 6.1. *For a multi-index $I = (i_1, \dots, i_k)$ and all operators $\mathcal{X}^I = \mathcal{X}_1^{(i_1)} \dots \mathcal{X}_k^{(i_k)}$, where $\mathcal{X}_j^{(i_j)} \in V_j^{i_j}$, with $\|\mathcal{X}^I\| \le 1$, there are constants C such that*

$$|\mathcal{X}^I G(x,a)| \le \begin{cases} C(|x|+a)^{-\|I\|-Q} & \\ \times(1 + |\log(|x|+a)^{-1}|)^{\|I\|_0} & \text{for } (x,a) \in (\mathcal{Q} \cup T_\delta)^c, \\ C & \text{for } (x,a) \in \mathcal{Q} \setminus T_\delta, \end{cases}$$

where $\|I\| = \sum_{j=1}^k i_j d_j$, $d_j = Re\lambda_j$, and $\|I\|_0 = \sum_{j=1}^k i_j D_j$, $D_j = dim V_{\lambda_j} - 1$.

Theorem 6.2. *For every nonnegative integer k, there is a constant C such that*

$$|\partial_a^k \mathcal{G}(x,a)| \le \begin{cases} Ca^{-k}(|x|+a)^{-Q} & \text{for } (x,a) \in (\mathcal{Q} \cup T_\delta)^c, \\ Ca^{-k} & \text{for } (x,a) \in \mathcal{Q} \setminus T_\delta. \end{cases}$$

Along with the operator $\mathcal{L}$ defined in (1.1) we consider the corresponding operator L,

$$L = a^{-2}\mathcal{L} = a^{-2} \sum_j \Phi_a(X_j)^2 + a^{-2}\Phi_a(X) + \partial_a^2 + a^{-1}\partial_a. \tag{6.1}$$

The Green function G for L is given by

$$G(x,a;y,b) = \int_0^\infty p_t(x,a;y,b)dt,$$

where $T_t f(x,a) = \int f(y,b) p_t(x,a;y,b) dybdb$ is the heat semigroup on $L^2(N \times \mathbb{R}^+, dybdb)$ with the infinitesimal generator L.

On $N \times \mathbb{R}^+$ we define *dilations*:

$$D_t(x,a) = (\Phi_t(x), ta), \qquad t > 0.$$

It is not difficult to check that although the operator L is not left-invariant it has some homogeneity with respect to the family of dilations introduced above:

$$L(f \circ D_t) = t^2 Lf \circ D_t.$$

This implies that

$$G(x,a;y,b) = t^{-Q} G(D_{t^{-1}}(x,a); D_{t^{-1}}(y,b)). \tag{6.2}$$

It turns out (see (1.17) in [3]) that

$$\mathcal{G}(x,a) = G(x,a;e,1) = G^*(e,1;x,a),$$

where G^* is the Green function for the operator

$$L^* = a^{-2} \sum \Phi_a(X_j)^2 - a^{-2}\Phi_a(X) + \partial_a^2 + a^{-1}\partial_a$$

conjugate to L with respect to the measure $adxda$. Moreover,

$$G^*(e,1;x,a) = \lim_{\eta \to 0} \int_0^\infty \mathbf{E}_1 p^\sigma(t,0)(x) m(I_{a,\eta})^{-1} 1_{I_{a,\eta}}(\sigma_t) dt, \tag{6.3}$$

where the expectation is taken with respect to the distribution of the Bessel process starting from 1 on the space $C((0,\infty),(0,\infty))$ and $I_{a,\eta} = [a-\eta, a+\eta]$. All the above facts are proved in [3].

Now we are ready to give

Proof of Theorem 6.1. For $r \geq 0$, define

$$V_r = \{(x,a) \in N \times \mathbb{R}^+ : |(x,a)| = r\},$$

where $|(x,a)| = |x| + a$.

Let $0 < \delta < 1/2$ and a multi-index I be fixed.

Case 1. We consider the set

$$S_1 = \mathcal{Q} \setminus T_\delta.$$

We have to show that there exists a positive constant C such that

$$|\mathcal{X}^I \mathcal{G}(x,a)| = |\mathcal{X}^I G^*(e,1;x,a)| \leq C \tag{6.4}$$

for every $(x,a) \in S_1$.

By (6.3) and results of Section 5: Lemmas 5.1, 5.2 and 5.3 it follows immediately that we have (6.4) on $\tilde{S}_1 = S_1 \cap \{(x,a) \in N \times \mathbb{R}^+ : a \leq 1-\delta\}$. Therefore we are left with $(x,a) \in S_1 \setminus \tilde{S}_1$. But $S_1 \setminus \mathrm{Int}\tilde{S}_1$ is a compact set. Since G^* is a continuous function we get (6.4) on S_1.

Case 2. We consider the set

$$S_2 = \{(x,a) \in N \times \mathbb{R}^+ : |x| \geq 1, |x| \geq a\}.$$

(Of course, $S_2 \cap T_\delta = \emptyset$.)

Every element $(x,a) \in N \times \mathbb{R}^+$ can be written as

$$(x,a) = D_t(y,b), \text{ where } (y,b) \in V_1 \text{ and } t = |(x,a)| = |x| + a.$$

By homogeneity of G (see (6.2)) and (2.4) we get

$$
\begin{aligned}
|\mathcal{X}^I G^*(e,1;x,a)| =& |\mathcal{X}^I G^*(D_t(e,t^{-1}),D_t(y,b))| \\
=& |\Phi_{t^{-1}}(\mathcal{X}^I)(G^*\circ D_t)(e,t^{-1};y,b)| \\
\le& t^{-\|I\|}(1+|\log t^{-1}|)^{\|I\|_0} \\
& \times \sup_{\|\mathcal{Y}\|\le 1} |\mathcal{Y}^I(G^*\circ D_t)(e,t^{-1};y,b)| \\
\le& (|x|+a)^{-\|I\|-Q}(1+|\log|(x,a)|^{-1}|)^{\|I\|_0} \\
& \times \sup_{\|\mathcal{Y}\|\le 1} |\mathcal{Y}^I G^*(e,|(x,a)|^{-1};y,b)|.
\end{aligned} \tag{6.5}
$$

If $(x,a)\in S_2$ then the corresponding $(y,b)\in V_1$ has the property $|y|\ge b$. Indeed, $x=\Phi_t(y)$ and $a=tb$ thus $t|y|=|x|\ge a=tb$. The above property and $|y|+b=1$ imply that $b\le 1/2$. Therefore,

$$
(y,b)\in V_1\cap\{(x,a)\in N\times\mathbb{R}^+ : a\le 1/2\}. \tag{6.6}
$$

Let $\beta=|(x,a)|^{-1}$. For $(x,a)\in S_2$ we have $\beta\le 1$. Thus by (6.3) and Lemma 5.4 we get

$$
\sup_{\|\mathcal{Y}\|\le 1}|\mathcal{Y}^I G^*(e,\beta;y,b)|\le C
$$

for (y,b) as in (6.6). Thus by (6.5) we are done in this case.

Case 3. Finally we consider the set

$$
S_3=\{(x,a)\notin T_\delta : a\ge |x|, a\ge 1\}.
$$

Because $V_1\cap T_\delta\ne\emptyset$ we write every element $(x,a)\in N\times\mathbb{R}^+$ as a dilation of some element from $V_{1/2}$:

$$
(x,a)=D_t(y,b),\ \text{where } (y,b)\in V_{1/2} \text{ and } t=2|(x,a)|=2|x|+2a.
$$

By homogeneity, we can write analogously to (6.5),

$$
\begin{aligned}
|\mathcal{X}^I G^*(e,1;x,a)| \le& 2^{-\|I\|-Q}(|x|+a)^{-\|I\|-Q} \\
& \times (1+|\log|(x,a)|^{-1}|)^{\|I\|_0}\sup_{\|\mathcal{Y}\|\le 1}|\mathcal{Y}^I G^*(e,\tilde\beta;y,b)|.
\end{aligned} \tag{6.7}
$$

where $\tilde\beta=2^{-1}(|x|+a)^{-1}$. If $(x,a)\in S_3$ then the corresponding $(y,b)\in V_{1/2}$ has the property $|y|\le b$. Indeed, $|x|=t|y|\le a=tb$. This, together with $|y|+b=1/2$ implies that b belongs to the interval $[1/4,1/2]$.

For $(x,a)\in S_3$ we have $\tilde\beta\le(2+2\delta)^{-1}:=1/2-\tilde\delta$. Indeed, this is clear if $a\ge 1+\delta$. But if $a<1+\delta$ then $|x|\ge\delta$. Thus by (6.3), using Lemma 5.5 if $b\ge(1-\tilde\delta)/2$ or Lemma 5.4 if $b\le(1-\tilde\delta)/2$ (then $|y|\ge\tilde\delta/2$) we get that there exists a constant C such that $\sup_{\|\mathcal{Y}\|\le 1}|\mathcal{Y}^I G^*(e,\tilde\beta;y,b)|$ in (6.7) is less than or equal to C. Thus the proof is complete. □

Proof of Theorem 6.2. We may assume that $k > 0$ since for $k = 0$ this is exactly Theorem 1.4 in [13]. Let $g_{e,\beta}(x,a) := G^*(e,\beta;x,a)$. By formula (6.3), for every nonnegative integer k,

$$(6.8)\quad \partial_a^k g_{e,\beta}(x,a) = \sum_{l=0}^{k}\binom{k}{l}\lim_{\eta\to 0}\int_0^\infty \mathbf{E}_\beta p^\sigma(t,0)(x)\partial_a^l m(I_{a,\eta})^{-1}\partial_a^{k-l}1_{I_{a,\eta}}(\sigma_t)dt.$$

Let V_r, S_1, S_2 and S_3 be defined as in the proof of Theorem 6.1.

Case 1. We consider the set S_1. We have to show that for every integer $k \ge 1$ there is a constant C such that

$$(6.9)\qquad |\partial_a^k \mathcal{G}(x,a)| \le Ca^{-k} \quad \text{for every } (x,a)\in S_1.$$

By (5.17), the last term in (6.8) corresponding to $l = k$ for $\beta = 1$ is equal to

$$(6.10)\qquad \lim_{\eta\to 0}(-1)^k k!\frac{(2\eta)^k}{m(I_{a,\eta})^k}\mathcal{G}(x,a) = (-1)^k Ca^{-k}\mathcal{G}(x,a).$$

But by Theorem 1.4 in [13] $\mathcal{G}(x,a) \le C$ on S_1. The remaining terms in (6.8), i.e. for $l = 0,1,\dots,k-1$ are also estimated by Ca^{-k} on $S_1 \cap \{a \le 1-\delta\}$ by results of Section 5: Lemmas 5.6, 5.7 and 5.8. Since $\partial_a^k\mathcal{G}(x,a)$ is a continuous function we get the estimate (6.9).

Case 2. Now we consider the set S_2. By homogeneity of G (see (6.2)) we get

$$(6.11)\qquad \begin{aligned}(\partial_b^k G^*)(x,a;y,b) =& t^{-Q}\partial_b^k G^*(D_{t^{-1}}(x,a);\delta_{t^{-1}}y,tb)\\ =& t^{-Q-k}(\partial_b G^*)(D_{t^{-1}}(x,a);\delta_{t^{-1}},t^{-1}b).\end{aligned}$$

As in the previous proof we write an arbitrary element $(x,a)\in N\times\mathbb{R}^+$ as

$$(x,a) = D_t(y,b), \text{ where } (y,b)\in V_1 \text{ and } t = |(x,a)| = |x|+a.$$

By (6.11),

$$(6.12)\qquad \begin{aligned}|\partial_a^k\mathcal{G}(x,a)| =& |(\partial_a^k G^*)(e,1;x,a)| = |(\partial_a^k G^*)(e,1;\delta_t y,tb)|\\ =& |(x,a)|^{-Q-k}|(\partial_a^k G^*)(e,|(x,a)|^{-1};y,b)|.\end{aligned}$$

If $(x,a)\in S_2$ then the corresponding $(y,b)\in V_1$ belongs to the set

$$(6.13)\qquad V_1\cap\{(x,a)\in N\times\mathbb{R}^+ : a\le 1/2\}$$

(see the previous proof). Let $\beta = |(x,a)|^{-1}$. For $(x,a)\in S_2$ we have $\beta \le 1$. Thus, reasoning as in the Case 1, by (6.3) and Lemma 5.9 we get

$$|(\partial_a^k G^*)(e,|(x,a)|^{-1};y,b)| \le Cb^{-k}$$

for (y,b) as in (6.13). Thus by (6.12) this case is done.

Case 3. We consider the set S_3. We write every element $(x,a) \in N \times \mathbb{R}^+$ as:

$$(x,a) = D_t(y,b), \text{ where } (y,b) \in V_{1/2} \text{ and } t = 2|(x,a)| = 2|x| + 2a.$$

Again, homogeneity gives us

$$|\partial_a^k G(x,a)| \le 2^{-Q-k} |(x,a)|^{-Q-k} |(\partial_a^k G^*)(e, \tilde{\beta}; y, b)|. \tag{6.14}$$

where $\tilde{\beta} = 2^{-1}|(x,a)|^{-1}$. If $(x,a) \in S_3$ then the corresponding $(y,b) \in V_{1/2}$ has the property that b belongs to the interval $[1/4, 1/2]$ (see proof of Theorem 6.1).

Then the argument is the same as in the third case of the proof of Theorem 6.1. The only difference is that in order to show that $|(\partial_a^k G^*)(e, \tilde{\beta}; y, b)| \le Ca^{-k}$ we use (6.8) and Theorems 5.9 and 5.10 instead of 5.4 and 5.5 respectively. □

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