Impossibility of C^∞ variation or formal power series variation in solutions to Hilbert's 17th problem [⋆]

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Abstract

No matter how a positive semidefinite polynomial $f \in \mathbb{R}[X_1, \ldots, X_n]$ is represented (according to E. Artin's 1926 solution to Hilbert's 17th problem) in the form $f = \sum p_i r_i^2$ (with $0 \leq p_i \in \mathbb{R}$ and $r_i \in \mathbb{R}(X_1, \ldots, X_n)$), the p_i and the coefficients of the r_i cannot be chosen to depend in a C^∞ (i.e., infinitely differentiable) manner upon the coefficients of f (unless $\deg f \leq 2$); formal powers series variation is also impossible. This answers a question we had raised in 1990 (Contemp. Math., Vol. 155, AMS, 1994, pp. 107–17), where we had already shown that real analytic variation was impossible; and Gonzalez-Vega and Lombardi (Math. Z. 225(3) (1997), 427–51) then showed that for every fixed, finite $r \in \mathbb{N}$, C^r variation is possible, improving upon their and the author's result that continuous, piecewise-polynomial variation is possible.

Key words: C^∞ functions, sums of squares, basic closed semianalytic sets, positive semidefinite polynomials, Hilbert's 17th problem, formal power series, Weierstraß polynomials
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1. Introduction

Suppose $n \in \mathbb{N} := \{0, 1, \ldots\}$, $X := (X_1, \ldots, X_n)$ are indeterminates, and $f \in \mathbb{R}[X]$ is psd (positive semidefinite), i.e., $\forall x := (x_1, \ldots, x_n) \in \mathbb{R}^n$, $f(x) \geq 0$.

[⋆] See `http://at.yorku.ca/cgi-bin/amca/cacv-60` for an abstract of this paper (dated June 1999).

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Hilbert's 17th problem [Hi1900] was to prove that we can always write such an f in the form

$$f = \sum_i r_i^2,$$

for some $r_i \in \mathbb{R}(X)$.

E. Artin solved this problem in [A1926], and went on to prove that if $K \subseteq \mathbb{R}$ is a subfield, $f \in K[X]$, and f is psd, then we can write

$$f = \sum p_i r_i^2, \tag{1.0.1}$$

for some $r_i \in K(X)$ and $p_i \in K$ such that $p_i \geq 0$.

Parametrization of Hilbert's 17th problem: Now let $d \in \mathbb{N}$, let $m := m_{nd} = \binom{n+d}{n}$, let $C := (C_1, \ldots, C_m)$ be indeterminates, and let $f_{nd} := f_{nd}(C; X) \in \mathbb{Z}[C; X]$ be the general polynomial of degree d in X with coefficients C:

$$f_{nd} = \sum_{|\alpha| \leq d} C_{j(\alpha)} X^\alpha,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \sum \alpha_i$, $X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$, and j is any fixed bijection:

$$j := j_{nd} : \{ \alpha \mid |\alpha| \leq d \} \to \{1, \ldots, m\}.$$

Writing $c = (c_1, \ldots, c_m) \in \mathbb{R}^m$, let

$$P_{nd} = \{ c \in \mathbb{R}^m \mid f_{nd}(c; X) \text{ is psd in } X \}.^{2} \tag{1.0.2}$$

Let B be any subring of the ring of all functions $\mathbb{R}^m \to \mathbb{R}$.

Question 1.1. *For which subrings B can we solve Hilbert's 17th problem so that the p_i and the coefficients of the r_i in* (1.0.1) *are functions in B?* (Obviously, we seek B as small as possible.)

Precisely, for which B is the following true?

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[2] P_{nd} is a closed, convex, semialgebraic cone; its interior P_{nd}° consists of those $c \in P_{nd}$ such that the X-homogenization (in $\mathbb{R}[X_1, \ldots, X_n, X_{n+1}]$) of $f(c; X)$ is positive at all $(x_1, \ldots, x_n, x_{n+1}) \in \mathbb{R}^{n+1} \setminus \{(0, \ldots, 0)\}$; and $P_{nd} = \overline{P_{nd}^\circ}$. But we do not need these facts here.

For all $n, d \in \mathbb{N}$, there exist $s \in \mathbb{N}$, $p_1, \ldots, p_s \in B$, and $g_1, \ldots, g_s, h_1, \ldots, h_s \in B[X]$ such that

$$\forall c \in \mathbb{R}^m,\ f_{nd}(c; X) = \sum_{i=1}^{s} p_i(c) \left(\frac{g_i(c; X)}{h_i(c; X)} \right)^2; \tag{1.1.1}$$

$$\forall c \in P_{nd},\ \ \forall i,\ \ p_i(c) \geq 0;\ \text{and} \tag{1.1.2}$$

$$\forall c \in P_{nd},\ \text{if } f_{nd}(c; X) \neq 0 \in \mathbb{R}[X],\ ^{3}\ \text{then } \forall i,\ \ h_i(c; X) \neq 0. \tag{1.1.3}$$

Note 1.2. If we write $g_i = \sum_\alpha g_{i,\alpha} X^\alpha$ with $g_{i,\alpha} \in B$, for finitely many $\alpha \in \mathbb{N}^n$, then $g_i(c; X)$ means $\sum_\alpha g_{i,\alpha}(c) X^\alpha \in \mathbb{R}[X]$. Similarly for $h_i(c; X)$.

When seeking function rings B that satisfy the conditions of Question 1.1, we prefer those B such that for all $p \in B$ and for all subfields $K \subseteq \mathbb{R}$, p takes values in K at K-rational points, i.e., $p(K^m) \subseteq K$; such B give a parametrized version of the full force of Artin's result (1.0.1), "uniformly" for all $K \subseteq \mathbb{R}$. On the other hand, if we show that some function ring B does *not* satisfy the conditions of (1.1), then it is immaterial whether all functions in B take values in K at K-rational points.

Remark 1.3. (1.1.1) is equivalent to:

$$\forall c \in \mathbb{R}^m,\ \ h(c; X)^2 f_{nd}(c; X) = \sum_{i=1}^{s} p_i(c) g_i'(c; X)^2, \tag{1.1.1$'$}$$

where $h = h_1 \cdots h_s \in B[X]$ is a common denominator, and

$$g_i' = g_i \prod_{j \neq i} h_j.$$

With this alternative notation, (1.1.3) is then equivalent to

$$\forall c \in P_{nd},\ \text{if } f_{nd}(c; X) \neq 0,\ \text{then } h(c; X) \neq 0. \tag{1.1.3$'$}$$

The main result of this paper is that the ring B in Question 1.1 may *not* be taken to be $C^\infty(\mathbb{R}^m)$, the ring of C^∞ (i.e., infinitely differentiable) functions $p : \mathbb{R}^m \to \mathbb{R}$:

Theorem 1.4. *For $n \geq 1$ and $d \geq 4$, there exists* **no** *C^∞ varying solution to the 17th problem. I.e., there exist no $s \in \mathbb{N}$, no $p_1, \ldots, p_s \in C^\infty(\mathbb{R}^m)$, and no $g_1, \ldots, g_s, h \in C^\infty(\mathbb{R}^m)[X]$ such that (1.1.1$'$), (1.1.2), and (1.1.3$'$) hold.*

This will follow immediately from Theorem 1.5 below, which deals with the following simpler situation:

[3] Note that for all $c \in \mathbb{R}^m$, $f_{nd}(c; X) \neq 0 \in \mathbb{R}[X]$ if and only if $c \neq (0, \ldots, 0) \in \mathbb{R}^m$.

From now on let $m = 2$, let $C := (C_1, C_2)$, let $c := (c_1, c_2) \in \mathbb{R}^2$, and let

$$\begin{aligned} f(C; X_1) &= X_1^4 + C_1 X_1^2 + C_2 \\ &= \Big(X_1^2 + \frac{1}{2}C_1\Big)^2 + \Big(C_2 - \frac{1}{4}C_1^2\Big). \end{aligned} \tag{1.4.1}$$

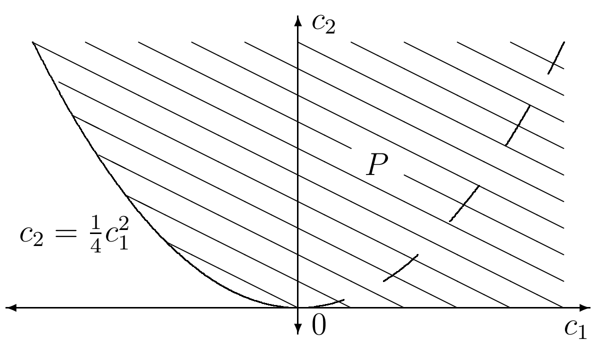
Let $P = \{\, c \in \mathbb{R}^2 \mid f(c; X_1) \text{ is psd in } X_1 \,\}$; this is a 2-dimensional cross-section of the 5-dimensional set $P_{1,4}$ of (1.0.2). Then

$$P = \left\{ c \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} [c_1 \geq 0 \;\wedge\; c_2 \geq 0] \text{ or} \\ [c_1 \leq 0 \;\wedge\; c_2 \geq c_1^2/4] \end{array} \right\}, \tag{1.4.2}$$

since, e.g.,

$$\min_{x \in \mathbb{R}} f(c; x) = \begin{cases} c_2 & \text{if } c_1 \geq 0 \\ c_2 - \frac{1}{4}c_1^2 & \text{if } c_1 \leq 0, \end{cases} \tag{1.4.3}$$

by (1.4.1). (These minima are achieved for $x = 0$ or $x = \sqrt{-c_1/2} \in \mathbb{R}$, respectively.)



Let $U \subseteq \mathbb{R}^2$ be any open neighborhood of $(0, 0)$, and let $C^\infty(U) = \{C^\infty \text{ functions } U \to \mathbb{R}\}$. We now restrict (1.1.1′), (1.1.2), and (1.1.3′) to U, and replace f_{nd} by f, obtaining, respectively:

$$\forall c \in U, \quad h(c; X_1)^2 f(c; X_1) = \sum_{i=1}^{s} p_i(c) g_i(c; X_1)^2; \tag{1.1.1″}$$

$$\forall c \in P \cap U, \quad \forall i, \quad p_i(c) \geq 0; \text{ and} \tag{1.1.2″}$$

$$\forall c \in P \cap U, \quad h(c; X_1) \neq 0.^4 \tag{1.1.3″}$$

[4] In (1.1.3′) we required (for any given $c \in P_{nd}$) the hypothesis that $f_{nd}(c; X) \neq 0 \in \mathbb{R}[X]$; when f_{nd} is replaced by f in (1.1.3″), this hypothesis becomes $f(c; X_1) \neq 0 \in \mathbb{R}[X_1]$, which is satisfied for all $c \in \mathbb{R}^2$.

Theorem 1.5. *There exist no open neighborhood $U \subseteq \mathbb{R}^2$ of $(0,0)$, no $s \in \mathbb{N}$, no $p_1, \ldots, p_s \in C^\infty(U)$, and no $g_1, \ldots, g_s, h \in C^\infty(U)[X_1]$ satisfying (1.1.1″), (1.1.2″), and (1.1.3″). I.e., there are no* germs *at $(0,0)$ of C^∞ functions of c_1, c_2 that provide the weights and the coefficients of a representation of $X_1^4 + c_1X_1^2 + c_2$ as a weighted sum of squares in $\mathbb{R}(X_1)$ (as in (1.1.1″)), where the weights are nonnegative for (c_1, c_2) in (the germ at $(0,0)$ of) P.*

(Taking $U = \mathbb{R}^2$ in (1.5), we prove (1.4).)

We shall prove Theorem 1.5 in section 6 below, after first proving (5.2) that formal power series variation is also impossible in Artin's theorem. Both of these results require some review of well-known facts about C^∞ functions (§3) and formal power series (§4). First, however, we review earlier work on Question 1.1 in §2 below.

2. Review of earlier answers to Question 1.1

Artin's theorem itself (1.0.1) may be considered to be a trivial "parametrization," for it says that we may take B in (1.1) to be the ring (here denoted by $\mathbb{R}_K^{(\mathbb{R}^m)}$) of *all* functions from $\mathbb{R}^m$ to $\mathbb{R}$ with values in K at K-rational points [5]; in other words, Artin did not consider whether the variation of the sum-of-squares representation in (1.0.1) (or in (1.1.1)) could be continuous, or could have other interesting properties.

The first non-trivial parametrization of (1.0.1) was by Henkin [Hen1960], who found that the variation can be given by $\mathbb{Z}$-*piecewise-polynomial* functions, here denoted by PWP($\mathbb{R}^m$); i.e., P_{nd} (or $\mathbb{R}^m$) can be decomposed into $\mathbb{Z}$-semialgebraic "pieces" $S_1 \cup \cdots \cup S_k$, on each of which the p_i and the coefficients of the r_i in (1.0.1) are given by functions in $\mathbb{Z}[C]$. van den Dries gave another proof of this in [vdD1977].

Shortly thereafter, Kreisel [Kre1962] asked whether this variation could even be polynomial (i.e., with only one piece S_1), or at least continuous. We showed [D1982] that for $d \geq 4$, polynomial variation is impossible, i.e., that

[5] Actually, the usual statement of Artin's theorem (1.0.1) gives only functions $p : P_{nd} \to \mathbb{R}$ (with $p(P_{nd} \cap K^m) \subseteq K$); but these functions may be extended to the rest of the desired domain $\mathbb{R}^m$ (i.e., for $c \in \mathbb{R}^m \setminus P_{nd}$) so as to continue to satisfy (1.1.1′), using the identity $f = (\frac{f+1}{2})^2 + (-1)(\frac{f-1}{2})^2$ (which also appeared in [A1926]). Condition (1.1.2) is still (vacuously) satisfied with $p_2(c) = -1$ for $c \in \mathbb{R}^m \setminus P_{nd}$, since the p_i need be nonnegative only for $c \in P_{nd}$; and (1.1.3′) is also obvious under this extension (where $h(c; X) = 2 \neq 0$ for $c \in \mathbb{R}^m \setminus P_{nd}$).

we may not take $B = \mathbb{R}[C]$. [6] We also showed [D1980–84] that for all $d \geq 0$, continuous (even continuous "$\mathbb{Z}$-semialgebraic") variation is possible; i.e., we may take $B = \mathrm{CSA}_{P_{nd}}(\mathbb{R}^m)$, the ring of $\mathbb{Z}$-semialgebraic functions $p : \mathbb{R}^m \to \mathbb{R}$ *whose restrictions to* P_{nd} *are continuous.* [7] A ($\mathbb{Z}$-)semialgebraic function is defined to be one with a ($\mathbb{Z}$-)semialgebraic graph; such a function will usually not take values in K at K-rational points $c \in P_{nd} \cap K^m$ (where $K \subseteq \mathbb{R}$ is as in (1.0.1)), unless K is *real closed* (e.g., when K is $\mathbb{R}$, or the real algebraic numbers). This makes semialgebraic functions unsatisfactory for parametrizing Artin's theorem over non-real-closed fields K, such as $\mathbb{Q}$ (Hilbert actually considered the case $K = \mathbb{Q}$ in his *Grundlagen der Geometrie* [Hi1899], and even emphasized the need to allow *arbitrary* $K \subseteq \mathbb{R}$ when he formulated the 17th problem in his famous list of 23 problems [Hi1900]).

Thus the above result answered Kreisel's question about continuous variation in Artin's theorem only over *real closed* subfields $K \subseteq \mathbb{R}$ [8]; the answer to his question over arbitrary $K \subseteq \mathbb{R}$ was not found until [D1989–93] (and re-discovered (using a different method) by Gonzalez-Vega and Lombardi [GL1993]; see also our joint article [DGL1993], and the treatments in [DM1995] and [PD2001]): the p_i and the coefficients of the r_i in (1.0.1) may be taken from the ring $\mathrm{SIPD}(\mathbb{R}^m)$ of "sup-inf-polynomially definable" functions

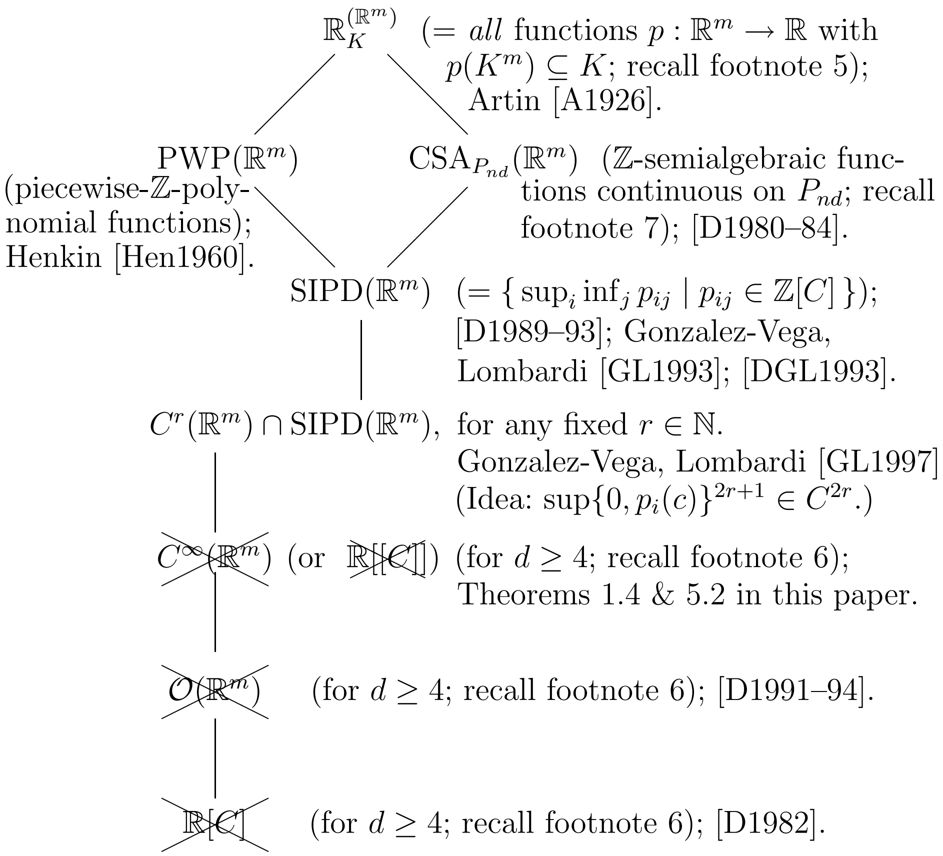
[6] For $d \leq 2$, however, we may even take $B = \mathbb{Z}[C]$ $(\subset \mathbb{R}[C] \subset \mathcal{O}(\mathbb{R}^m) \subset C^\infty(\mathbb{R}^m))$; see [D1980–82]. In fact, when $d = 2$, we may even take the common denominator h in (1.1.1′) to be in $\mathbb{Z}[C]$ (and not merely in $\mathbb{Z}[C; X]$); thus, psd *quadratic* forms can be continuously represented as nonnegatively weighted sums of squares of X-linear *forms* (and not merely rational functions as in (1.1.1)). Admittedly, the continuity result for $h \in Z[C]$ was established only for $c \in P_{n,2}$, and not necessarily for all $c \in \mathbb{R}^{\binom{n}{2}}$. Upper and lower bounds for the number of such continuously varying squared linear forms needed for this are in [D1987].

[7] Actually, [D1980–84] constructed only functions $p : P_{nd} \to \mathbb{R}$ (continuous and $\mathbb{Z}$-semialgebraic). To extend such p from P_{nd} to all of $\mathbb{R}^m$ (to conform to the formulation of the problem in Question 1.1) we need only add the trick, mentioned in footnote 5 above, for discontinuously extending the sum-of-squares representation (1.1.1) to the rest of $\mathbb{R}^m$. But the only way we know to get functions defined and continuous even outside of P_{nd} that answer Question 1.1 is to show that we may even take $B = \mathrm{SIPD}(\mathbb{R}^m)$ $(\subset \mathrm{CSA}_{P_{nd}}(\mathbb{R}^m))$, as in the next paragraph.

[8] Actually, a Boolean combination of $\mathbb{Z}$-polynomial inequalities defining a $\mathbb{Z}$-semialgebraic set such as P_{nd} over $\mathbb{R}$ also defines a corresponding set $P_{nd,R} \subseteq R^m$, where R is *any* real closed field. Likewise, the $\mathbb{Z}$-polynomial inequalities defining a $\mathbb{Z}$-semialgebraic function such as any $p \in \mathrm{CSA}(P_{nd})$ also define a corresponding function $p_R : P_{nd,R} \to R$. And for any subfield $K \subseteq R$, p_R will take values in K at K-rational points of $P_{nd,R}$ provided K, too, is real closed. Consequently, Question 1.1 has sometimes been formulated with $\mathbb{R}$ replaced by an arbitrary real closed field R; then the continuous semialgebraic variation in Artin's theorem constructed in [D1980–84] is seen to work uniformly over all real closed fields R. Similarly for the SIPD variation discussed later in the above sentence.

Summary of answers to Question 1.1

B can be any of the following rings (except those crossed out):



(i.e., functions of the form $\sup_i \inf_j p_{ij}(c)$, for finitely many polynomial functions $p_{ij} \in \mathbb{Z}[C]$). Such functions are not only continuous, but also piecewise-polynomial (with integer coefficients); thus for any $K \subset \mathbb{R}$, they take values in K at K-rational points $c \in K^m$, as Hilbert would have wanted. Thus this result combines the best features of [Hen1960] and [D1980–84] above.

In the 1990's, Question 1.1 was studied for various function-rings B *bigger* than $\mathbb{R}[C]$ (where the answer is no, as mentioned above), and/or *smaller* than $\mathrm{SIPD}(\mathbb{R}^m)$ (where the answer is yes). First, in [D1991–94] we showed that in (1.1), B can *not* be taken to be $\mathcal{O}(\mathbb{R}^m)$, the ring of real analytic functions on $\mathbb{R}^m$ [9]; we then asked whether B could be taken to be $C^\infty(\mathbb{R}^m)$. Meanwhile, Gonzalez-Vega and Lombardi [GL1997] showed that for each fixed $r \in \mathbb{N}$, B may be taken to be $C^r(\mathbb{R}^m) \cap \mathrm{SIPD}(\mathbb{R}^m)$ (where $C^r(\mathbb{R}^m)$ denotes the ring of

[9] This contrasts with the situation for Euler's theorem [E1754/55] that every positive rational number r is the sum of four squares of rationals: Heilbronn [Hei1964] showed that those four rational numbers can be chosen to vary analytically in r, answering another question of Kreisel; a similar result [D1985] holds for Siegel's generalization [Si1921] of Euler's theorem, that in *every* number field K, every totally positive element is the sum of four squares in K.

functions $p : \mathbb{R}^m \to \mathbb{R}$ all of whose rth-order partial derivatives exist and are continuous), improving upon the earlier result about SIPD($\mathbb{R}^m$). They also considered a weakening of Question 1.1 obtained by replacing "$\forall c \in P_{nd}$" with "$\forall c \in P_{nd}^\circ$" in (1.1.3′) (recall footnote 2); i.e., they considered allowing the "denominator" h in (1.1.1′) to vanish for c on the boundary ∂P_{nd} of P_{nd}, and outside P_{nd}. They then showed that B may be taken to be either (a) the subring of functions that are continuous and semialgebraic on $\mathbb{R}^m$, Nash on P_{nd}°, and zero outside P_{nd}°; or (b) the subring of functions that are C^∞ on $\mathbb{R}^m$, analytic on P_{nd}°, and zero outside P_{nd}°.

Our question about $C^\infty(\mathbb{R}^m)$ (with the original version of (1.1.3′)), however, remained unanswered until now; Theorem 1.4 above states that for $d \geq 4$, B may *not* be taken to be $C^\infty(\mathbb{R}^m)$ in (1.1).

At the moment we have no further candidates for function-rings B to consider in Question 1.1; so perhaps this line of investigation into the possible kinds of variation in Artin's theorem is finally complete.

Review of continuity results in other sum-of-squares representations. The real Nullstellensatz and Positivstellensatz (both due originally to Krivine [Kri1964], and re-discovered by Dubois, Prestel, Risler, and Stengle; or see, e.g., [PD2001]) also admit continuous versions in certain cases: [Sc1989], [D1989–93], [GL1993], [DGL1993]. While psd quartic polynomials in $\mathbb{R}[X_1, X_2]$ are sums of squares of quadratic polynomials in $\mathbb{R}[X_1, X_2]$,[10] such (denominator-free) sum-of-squares representations must vary discontinuously [D1980–82]. On the other hand, for every $d \geq 0$, a continuously varying representation of psd $f \in \mathbb{R}[X_1]$ of degree $\leq d$ as sums of squares in $\mathbb{R}[X_1]$ was explicitly constructed by M. Ziegler in 1988 (unpublished); see Cornelson's master's thesis [C1998] for an exposition. That thesis also presents Prestel's continuously varying representation of (most of) those $f \in \mathbb{R}[X_1]$ that are sums of $2m$th powers in $\mathbb{R}(X_1)$ (for any $m \geq 1$) as sums of $2m$th powers. Finally, Cornelsen's thesis also presents T. Backmeister's (unpublished) proof of continuous variation in the weak isotropy of torsion quadratic forms over $\mathbb{R}(X_1, \ldots, X_n)$ (weak isotropy is presented in [PD2001, §3.5]). Finally, Reznick [Re1995] considered certain *sub*sets of P_{nd}, in which even $\mathbb{Q}$-*linear* variation is possible in Artin's theorem.

[10] Hilbert [Hi1888]; modern expositions have been given by Choi and Lam [CL1977], Swan (1993, unpublished), and Rajwade [Ra1993].

3. Taylor series of C^∞ functions

For $m \geq 1$, let $U \subseteq \mathbb{R}^m$ be any open neighborhood of $\mathbf{0} := (0, \ldots, 0)$. $C^\infty(U)$ denotes the ring of functions $p : U \to \mathbb{R}$ whose partial derivatives of all orders exist on U. We write $\tau_C(p)$, or simply $\overline{p}$, for the Taylor (or Maclaurin) series of p at $\mathbf{0}$ in the indeterminates C, viz.,

$$\sum_{\beta \in \mathbb{N}^m} \frac{1}{\beta_1! \cdots \beta_m!} \frac{\partial^{|\beta|} p}{\partial c_1^{\beta_1} \cdots \partial c_m^{\beta_m}} (\mathbf{0})\, C^\beta.$$

τ_C is an $\mathbb{R}$-algebra homomorphism [11] $C^\infty(U) \to \mathbb{R}[[C]]$ (= the ring of formal, i.e., not necessarily convergent, power series in $C := (C_1, \ldots, C_m)$). Borel's lemma [12] states that τ_C is surjective; we shall not use this fact. We say that p is *flat* at $\mathbf{0}$ if p belongs to the (prime) ideal $\ker \tau_C$; i.e., if $\overline{p} = 0$. τ_C extends to an $\mathbb{R}$-algebra homomorphism $C^\infty(U)[X] \to \mathbb{R}[[C]][X]$ by

$$\tau_C\Big(\sum_\alpha p_\alpha X^\alpha\Big) = \sum_\alpha \overline{p_\alpha} X^\alpha \quad (p_\alpha \in C^\infty(U)).$$

For $m = 1$ we consider also *half*-open sets $U \subseteq \mathbb{R}^1$ of the form $U = [0, \delta)$, $\delta > 0$. Then $C^\infty([0,\delta))$ denotes the ring of functions $p : [0,\delta) \to \mathbb{R}$ all of whose derivatives exist on $[0,\delta)$, where, at $c_1 = 0$, we refer only to the *right*-hand derivatives of p. For $p \in C^\infty([0,\delta))$ we still have the "right-hand" Taylor series $\overline{p} \in \mathbb{R}[[C_1]]$ of p at $c_1 = 0$.

4. An ordering on $\mathbb{R}((T))$, and its real closure

Let T be a single indeterminate, and write a typical element $a := a(T) := \sum_{i=k}^\infty a_i T^i \in \mathbb{R}[[T]] \setminus \{0\}$, with $a_i \in \mathbb{R}$, $k \in \mathbb{N}$, and $a_k \neq 0$. We extend the unique field ordering $>$ on $\mathbb{R}$ to a ring-order on $\mathbb{R}[[T]]$ by defining $a > 0 \Leftrightarrow a_k > 0$. We further extend $>$ (uniquely) to the field of fractions

$$\mathbb{R}((T)) := \left\{ \sum_{i=k}^\infty a_i T^i \;\middle|\; k \in \mathbb{Z},\ a_i \in \mathbb{R} \right\},$$ and thence to the real closure

[11] Here, the fact that $\tau_C(pq) = \tau_C(p)\tau_C(q)$, for $p, q \in C^\infty(U)$, is the Leibniz product-rule for higher-order partial derivatives.

[12] Émile Borel proved this for $m = 1$ in [B1895, p. 44]; I thank Prof. Armand Borel for this reference. I do not know whether E. Borel ever stated this result for $m > 1$. At any rate, proofs (for all $m \geq 1$) can be found, e.g., in [N1985, p. 30].

$$\mathcal{R} := \bigcup_{e=1}^{\infty} \mathbb{R}((T^{1/e})). \quad \text{We write}$$

$$\mathcal{V} := \bigcup_{e=1}^{\infty} \mathbb{R}[[T^{1/e}]] \quad \text{(a valuation ring), with maximal ideal}$$

$$\mathcal{M} := \bigcup_{e=1}^{\infty} T^{1/e} \cdot \mathbb{R}[[T^{1/e}]]$$

$$= \{ a \in \mathcal{R} \mid a(\mathbf{0}) \text{ is defined and } a(\mathbf{0}) = 0 \}.$$

Lemma 4.1. *Suppose S is an indeterminate, $\varepsilon > 0$, $q \in C^\infty([0,\varepsilon))$, $e \in \mathbb{N}^+ := \{1,2,\ldots\}$, $\overline{q} := \tau_S(q) \in \mathbb{R}[[S]]$, and $0 < \overline{q}(T^{1/e})$ $(\in \mathcal{V})$. Then there exists a $\delta \in (0, \varepsilon^e)$ such that for all $t \in (0,\delta)$, $q(t^{1/e}) > 0$.*

Proof. Write $\overline{q} = \sum_{i=k}^{\infty} q_i S^i$, with $k \in \mathbb{N}$, $q_i \in \mathbb{R}$, $q_k \neq 0$; then in fact $q_k > 0$. Introducing the variable $s = t^{1/e}$, we get

$$\lim_{t\to 0^+} \frac{q(t^{1/e})}{t^{k/e}} = \lim_{s \to 0^+} \frac{q'(s)}{ks^{k-1}} = \cdots = \lim_{s\to 0^+} \frac{q^{(k)}(s)}{k!} = q_k > 0. \qquad \square$$

Lemma 4.2. *The order $>$ on $\mathcal{R}$ restricts to a dense order on $\mathcal{M}$. In fact, for each $e \in \mathbb{N}^+$, $T^{1/e}\mathbb{R}[T^{1/e}]$ is (order-)dense in $T^{1/e}\mathbb{R}[[T^{1/e}]]$.*

Proof. Given $a, b \in T^{1/e}\mathbb{R}[[T^{1/e}]]$ such that $a < b$, we are to find $\gamma_1, \gamma_2, \gamma_3 \in T^{1/e}\mathbb{R}[T^{1/e}]$ such that $\gamma_1 < a < \gamma_2 < b < \gamma_3$. For this, take truncations $\tilde{a}, \tilde{b} \in T^{1/e}\mathbb{R}[T^{1/e}]$ of a and b sufficiently long so that $\tilde{a} < \tilde{b}$, and then let $\gamma_1 = \tilde{a} - T$, $\gamma_2 = (\tilde{a} + \tilde{b})/2$, and $\gamma_3 = \tilde{b} + T$. $\square$

5. Impossibility of formal power series variation in solutions to Hilbert's 17th problem

Write

$$\mathbb{R}^2_{(0,0)} = \{ (T,\gamma), (-T,\gamma) \mid \gamma \in \mathcal{M} \} \cup \{ (0,T), (0,-T) \}.$$

Thus the elements $(\alpha_1, \alpha_2) \in \mathbb{R}^2_{(0,0)}$ are "standard" parametrizations of the algebroid curve germs of type $[0,\delta)$ at $(0,0)$, in which one of the coordinates (the first one, whenever possible) is chosen to be T or $-T$.

Recalling the ordering $>$ on $\mathcal{M} \subseteq \mathcal{R}$ (§4), it then makes sense to speak of those $(\alpha_1, \alpha_2) \in \mathbb{R}^2_{(0,0)}$ (or even those $(\alpha_1,\alpha_2) \in \mathcal{M} \times \mathcal{M}$) that satisfy some

Boolean combination of inequalities $p(\alpha_1, \alpha_2) \geq 0$, for various $p \in \mathbb{R}[[C_1, C_2]]$. Specifically, we write

$$\begin{aligned} P_{(0,0)} = & \{ (\ \ T, \gamma) \mid \gamma \in \mathcal{M},\ \gamma \geq 0 \} \cup \\ & \{ (-T, \gamma) \mid \gamma \in \mathcal{M},\ \gamma \geq T^2/4 \} \cup \\ & \{ (\ \ 0, T) \}. \end{aligned} \tag{5.0.1}$$

Recalling (1.4.2), $P_{(0,0)}$ is therefore the set of algebroid curve germs in $\mathbb{R}^2_{(0,0)}$ that "stay in P." The next lemma is the algebroid-curve-germ analog of (1.4.2).

Lemma 5.1. *With f as in* (1.4.1),

$$\begin{aligned} P_{(0,0)} &= \{ (\alpha_1, \alpha_2) \in \mathbb{R}^2_{(0,0)} \mid \forall \xi \in \mathcal{M},\ f(\alpha_1, \alpha_2; \xi) \geq 0 \} \\ &= \{ (\alpha_1, \alpha_2) \in \mathbb{R}^2_{(0,0)} \mid \forall \xi \in \mathcal{R},\ f(\alpha_1, \alpha_2; \xi) \geq 0 \}. \end{aligned}$$

Proof. As in (1.4.3), for any $(\alpha_1, \alpha_2) \in \mathbb{R}^2_{(0,0)}$,

$$\begin{aligned} \min_{\xi \in \mathcal{M}} f(\alpha_1, \alpha_2; \xi) &= \min_{\xi \in \mathcal{R}} f(\alpha_1, \alpha_2; \xi) \\ &= \begin{cases} \alpha_2 & \text{if } \alpha_1 \geq 0 \\ \alpha_2 - \frac{1}{4}\alpha_1^2 & \text{if } \alpha_1 \leq 0. \end{cases} \end{aligned}$$

(These minima are achieved for $\xi = 0$ or $\xi = \sqrt{-\alpha_1/2} \in \mathcal{M} \subset \mathcal{R}$, respectively.[13]) Thus the nonnegativity of this minimum in $\mathcal{R}$ is equivalent to the condition that $(\alpha_1, \alpha_2) \in P_{(0,0)}$, by (5.0.1). □

Theorem 5.2. *With f as in* (1.4.1), *there exist no $s \in \mathbb{N}$, no $p_1, \ldots, p_s \in \mathbb{R}[[C_1, C_2]]$, and no $g_1, \ldots, g_s, h \in \mathbb{R}[[C_1, C_2]][X_1]$ such that*

$$h(C_1, C_2; X_1))^2 f(C_1, C_2; X_1) = \sum_{i=1}^{s} p_i(C_1, C_2) g_i(C_1, C_2; X_1)^2, \tag{5.2.1}$$

$$\forall (\alpha_1, \alpha_2) \in P_{(0,0)},\ \forall i \in \{1, \ldots, s\},\ p_i(\alpha_1, \alpha_2) \geq 0 \in \mathcal{M},\ \textit{and} \tag{5.2.2}$$

$$h(0, 0; X_1) \neq 0 \in \mathbb{R}[X_1]. \tag{5.2.3}$$

The proof will use the following three lemmas.

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[13] And $\sqrt{-\alpha_1/2} = \sqrt{\frac{1}{2}} \cdot T^{1/2}$ if $\alpha_1 < 0$, since we are assuming that $(\alpha_1, \alpha_2) \in \mathbb{R}^2_{(0,0)}$.

Lemma 5.3. *Suppose* $0 \neq p \in \mathbb{R}[[C_1, C_2]]$*. Then*

$$p(C_1, C_2) = C_1^b \cdot u(C_1, C_2) \prod_{j=1}^{J} q_j(C_1, C_2)^{e_j}, \tag{5.3.1}$$

where $b, J \in \mathbb{N}$*;* $e_j \in \mathbb{N}^+$*;* $u \in \mathbb{R}[[C_1, C_2]]^\times$ *(i.e.,* $u(0,0) \neq 0$*); and the* q_j *are distinct irreducible "*C_2*-Weierstrass polynomials"* $\in \mathbb{R}[[C_1]][C_2]$ *(i.e.,* $q_j = C_2^{s_j} + \sum_{i=0}^{s_j-1} y_{ji}(C_1) C_2^i$*, some* $s_j \in \mathbb{N}^+$*,* $y_{ji} \in C_1 \cdot \mathbb{R}[[C_1]]$*). These data (except for the order of the* q_j *and* e_j*) are uniquely determined by* p*.*

Proof. First choose b maximal such that $C_1^b | p$ in (the UFD) $\mathbb{R}[[C_1, C_2]]$. Then $p' := p(C_1, C_2)/C_1^b \in \mathbb{R}[[C_1, C_2]]$ is "regular with respect to C_2," i.e., $p'(0, C_2) \neq 0 \in \mathbb{R}[[C_2]]$. Therefore we may apply the Weierstrass preparation theorem to p' to get $p' = up''$, for uniquely determined $u \in \mathbb{R}[[C_1, C_2]]^\times$ and C_2-Weierstrass polynomial $p'' \in \mathbb{R}[[C_1]][C_2]$.

Now write $p'' = \prod q_j^{e_j}$ for (up to order) uniquely determined C_2-monic, pairwise non-associate, irreducible $q_j \in \mathbb{R}((C_1))[C_2]$ (= a UFD), and $e_j \in \mathbb{N}^+$. Actually, we have $q_j \in \mathbb{R}[[C_1]][C_2]$, by Gauss' lemma; and every C_2-monic factor $v \in \mathbb{R}[[C_1]][C_2]$ of a C_2-Weierstrass polynomial is again a C_2-Weierstrass polynomial. □

Lemma 5.4. *Suppose* $0 \neq p \in \mathbb{R}[[C_1, C_2]]$*. Then*

$$p(\ \ T, C_2) = \ \ T^{\ b} u(\ \ T, C_2) \prod_{k=1}^{K^+} (C_2 - \zeta_k^+)^{e_k^+} \prod_{l=1}^{L^+} \left[(C_2 - \eta_l^+)^2 + \nu_l^{+2}\right]^{f_l^+}, \tag{5.4.1$^+$}$$

$$p(-T, C_2) = (-T)^b u(-T, C_2) \prod_{k=1}^{K^-} (C_2 - \zeta_k^-)^{e_k^-} \prod_{l=1}^{L^-} \left[(C_2 - \eta_l^-)^2 + \nu_l^{-2}\right]^{f_l^-}, \tag{5.4.1$^-$}$$

where $b, K^\pm, L^\pm \in \mathbb{N}$*;* $e_k^\pm, f_l^\pm \in \mathbb{N}^+$*;* $u \in \mathbb{R}[[C_1, C_2]]^\times$*; the* ζ_k^+ *are distinct elements of* $\mathcal{M}$*, as are the* ζ_k^-*; and the ordered pairs* (η_l^+, ν_l^+) *are distinct elements of* $\mathcal{M} \times (\mathcal{M} \setminus \{0\})$*, as are the* (η_l^-, ν_l^-)*. The above data are unique up to order. Finally, for each* C_2*-linear or irreducible* C_2*-quadratic factor* F *displayed in* (5.4.1$^\pm$)*, there exists a unique* $j \leq J$ *such that* $F | q_j(\pm T, C_2)$ *in* $\mathcal{R}[C_2]$*; moreover, for this* j*,* e_j *equals the multiplicity (*$= e_k^\pm$ *or* $f_l^\pm$*) of* F *in* (5.4.1$^\pm$)*. (Here,* J, q_j, e_j *are as in* (5.3.1)*.)*

Proof. (5.4.1$^\pm$) comes from (5.3.1) upon replacing C_1 by $\pm T$, and then factoring each $q_j(\pm T, C_2)$ into irreducible factors $\in \mathcal{R}[C_2]$; these new factors will have C_2-degree ≤ 2, since $\mathcal{R}$ is real closed.

The last statement of (5.4) (about multiplicities) follows from the separability of the extension $\mathcal{R}/\mathbb{R}((T))$. $\square$

Lemma 5.5$^{\pm}$. *Given the notation in* (5.4)*, we may* (*and shall*) *re-index the* $e_k^{\pm}$ *and* $\zeta_k^{\pm}$ *so that for some* $K'^{+} \in \{0, 1, \ldots, K^{+}\}$ *and some* $K'^{-} \in \{0, 1, \ldots, K^{-}\}$,

$$\begin{aligned}
&e_1^{\pm} \quad\quad\; , \ldots, e_{K'^{\pm}}^{\pm} \quad \text{are odd,}\\
&e_{K'^{\pm}+1}^{\pm}, \ldots, e_{K^{\pm}}^{\pm} \quad \text{are even, and}\\
&\zeta_1^{\pm} < \cdots < \zeta_{K'^{\pm}}^{\pm}.
\end{aligned} \tag{5.5.1$^{\pm}$}$$

Then

$$p(\pm T, C_2) = (\pm T)^b u(\pm T, C_2)[\alpha^{\pm}(C_2)^2 + \beta^{\pm}(C_2)^2] \prod_{k=1}^{K'^{\pm}} (C_2 - \zeta_k^{\pm}), \tag{5.5.2$^{\pm}$}$$

where $\alpha^{\pm}, \beta^{\pm} \in \mathcal{V}[C_2]$.

Proof. (5.5.2$^{\pm}$) follows from (5.4.1$^{\pm}$) via the two-square identity:

$$(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2. \qquad \square$$

Remark 5.6. In (5.5$^{\pm}$) we may actually choose $\alpha^{\pm}, \beta^{\pm} \in \mathcal{M}[C_2]$ provided $L^{\pm} > 0$.

Proof of (5.2). Suppose s, p_i, g_i, h satisfy (5.2.1–3); we seek a contradiction. We may assume each $p_i \neq 0$. Fix $i \leq s$, and apply (5.3) and (5.5$^{\pm}$) with p_i in place of p, obtaining the odd C_2-roots $\zeta_{i,1}^{\pm} < \cdots < \zeta_{i,K_i'^{\pm}}^{\pm} \in \mathcal{M}$ of $p_i(\pm T, C_2) \in \mathbb{R}[[T, C_2]]$, for some $K_i'^{\pm} \in \mathbb{N}$.

Claim 5.7. $\forall k^{-} \in \{1, \ldots, K_i'^{-}\}$, $\zeta_{i,k^{-}}^{-} \neq \frac{1}{4}T^2$.

Proof. Otherwise, there exists a unique $j \leq J$ such that $q_j(-T, \frac{1}{4}T^2) = 0$, and for this j, e_j ($= e_{k^{-}}^{-}$) is odd (where J, q_j, e_j, e_k^{-} are as in (5.3) and (5.4)). Then q_j must be the irreducible C_2-Weierstrass polynomial $C_2 - \frac{1}{4}C_1^2$. Then there exists a unique $k^{+} \in \{1, \ldots, K_i'^{+}\}$ such that $\zeta_{i,k^{+}}^{+} = \frac{1}{4}T^2$, and for this k^{+}, $e_{k^{+}}^{+}$ ($= e_j$) is odd (again by (5.4)). Pick $\gamma_1^{+}, \gamma_2^{+} \in \mathcal{M}$ such that

$$0 \leq \gamma_1^{+} \quad \text{and} \tag{5.7.1}$$

$$\zeta_{i,k'}^{+} < \gamma_1^{+} < \frac{1}{4}T^2 = \zeta_{i,k^{+}}^{+} < \gamma_2^{+} < \zeta_{i,k''}^{+} \tag{5.7.2}$$

for all $k' \in \{1, \ldots, k^+ - 1\}$ and for all $k'' \in \{k^+ + 1, \ldots, K_i'^+\}$; this is possible by (4.2) and (5.5.1$^+$). Then $p_i(T, \gamma_1^+)$ and $p_i(T, \gamma_2^+)$ have opposite signs, by (5.7.2) and (5.5.2$^+$); and (T, γ_1^+) and (T, γ_2^+) both belong to $P_{(0,0)}$, by (5.7.1–2) and (5.0.1). This violates (5.2.2), proving (5.7). □

Claim 5.8. $\forall k^- \in \{1, \ldots, K_i'\}, \;\; \zeta^-_{i,k^-} < \frac{1}{4}T^2$.

Proof. Otherwise, $\zeta^-_{i,k^-} > \frac{1}{4}T^2$ for some k^-, by (5.7); let k^- be minimal with respect to this property. Pick $\gamma_1^-, \gamma_2^- \in \mathcal{M}$ such that

$$\zeta^-_{i,k'} < \frac{1}{4}T^2 \leq \gamma_1^- < \zeta^-_{i,k^-} < \gamma_2^- < \zeta^-_{i,k''} \tag{5.8.1}$$

for all $k' \in \{1, \ldots, k^- - 1\}$ and for all $k'' \in \{k^- + 1, \ldots, K_i'^-\}$ ((4.2), (5.5.1$^-$)). Then $p_i(-T, \gamma_1^-)$ and $p_i(-T, \gamma_2^-)$ have opposite signs, by (5.8.1) and (5.5.2$^-$); and $(-T, \gamma_1^-)$ and $(-T, \gamma_2^-)$ both belong to $P_{(0,0)}$, by (5.8.1) and (5.0.1). This violates (5.2.2), proving (5.8). □

Returning to the proof of (5.2) itself, we "unfix" $i \in \{1, \ldots, s\}$ and choose $\gamma \in \mathcal{M}$ such that

$$\zeta^-_{i,K_i'^-} < \gamma < \frac{1}{4}T^2 \tag{5.9.1}$$

for all i such that $K_i'^- > 0$; this is possible by (4.2) and (5.8). Thus for all i, $p_i(-T, \gamma) \geq 0$, by (5.5.2$^-$), (5.0.1), and (5.2.2). Therefore for all $\xi \in \mathcal{R}$,

$$\sum_i p_i(-T, \gamma) g_i(-T, \gamma; \xi)^2 \geq 0. \tag{5.9.2}$$

As for the lefthand side of (5.2.1), note that $h(-T, \gamma; X_1) \neq 0 \in \mathcal{V}[X_1]$, by (5.2.3) (since $\gamma(0) = 0$). Therefore $h(-T, \gamma; X_1)$ has only finitely many X_1-roots $\xi \in \mathcal{R}$. Therefore for all $l \in \mathbb{N}^+$ sufficiently large,

$$h\Big(-T, \gamma; \sqrt{\frac{1}{2}T^{1/2} + T^l}\Big) \neq 0 \in \mathcal{V}.$$

It remains to examine the sign of f under these substitutions. Recalling (1.4.1),

$$f\Big(-T, \gamma; \sqrt{\frac{1}{2}T^{1/2} + T^l}\Big) = \Big[\Big(\sqrt{\frac{1}{2}T^{1/2} + T^l}\Big)^2 - \frac{1}{2}T\Big]^2 + \Big(\gamma - \frac{1}{4}T^2\Big)$$

$$= \Big(\frac{1}{2}T + \sqrt{2}T^{l+\frac{1}{2}} + T^{2l} - \frac{1}{2}T\Big)^2 + \Big(\gamma - \frac{1}{4}T^2\Big)$$
$$= \Big(2T^{2l+1} + 2\sqrt{2}T^{3l+\frac{1}{2}} + T^{4l}\Big) + \Big(\gamma - \frac{1}{4}T^2\Big),$$

an element of $\mathcal{M}$ that is negative for l sufficiently large, by (5.9.1).

Thus the lefthand side of (5.2.1) is negative under these substitutions, violating (5.9.2), and proving (5.2). □

6. Proof of Theorem 1.5

Recalling Theorem 1.5, suppose $U \subseteq \mathbb{R}^2$ is an open neighborhood of $(0,0)$, $s \in \mathbb{N}$, $p_1, \ldots, p_s \in C^\infty(U)$, and $g_1, \ldots, g_s, h \in C^\infty(U)[X_1]$; and suppose that all of these satisfy (1.1.1″), (1.1.2″), and (1.1.3″). We seek a contradiction. Taking Taylor series at $(0,0)$ in (1.1.1″), we get

$$\overline{h}(C_1, C_2; X_1)^2 f(C_1, C_2; X_1) = \sum \overline{p_i}(C_1, C_2)\overline{g_i}(C_1, C_2; X_1)^2$$

in $\mathbb{R}[[C_1, C_2]][X_1]$; this is of the form (5.2.1). Note that $\overline{h}(0,0;X_1) \neq 0 \in \mathbb{R}[X_1]$ (5.2.3), by (1.1.3″). The desired contradiction will then follow from (5.2) once we verify (5.2.2) for $\overline{p_i}$; i.e., that

$$\forall(\alpha_1, \alpha_2) \in P_{(0,0)},\ \forall i,\ \overline{p_i}(\alpha_1, \alpha_2) \geq 0 \in \mathcal{V}.$$

So suppose $\overline{p_i}(\alpha_1, \alpha_2) < 0$ for some i and some $(\alpha_1, \alpha_2) \in P_{(0,0)}$; we seek a contradiction. Pick $e \in \mathbb{N}^+$ such that $\alpha_2 \in \mathbb{R}[[T^{1/e}]]$.

Case 1: $\alpha_1 = -T$. Then $\alpha_2 \geq \frac{1}{4}T^2$ (5.0.1).

Subcase 1(a): $\alpha_2 = \frac{1}{4}T^2$. Then $\exists\delta > 0$ such that $\forall t \in (0,\delta)$, $p_i(-t, \frac{1}{4}t^2) < 0$, by (4.1), violating (1.1.2″).

Subcase 1(b): $\alpha_2 > \frac{1}{4}T^2$. There exists an open interval $I \subseteq \mathcal{M}$ containing α_2 such that $\forall\beta \in I$, $\overline{p_i}(-T, \beta(T)) < 0$, by (5.5.2⁻). Shrinking I if necessary, we may arrange that $\forall\beta \in I$, $\beta \geq \frac{1}{4}T^2$ (i.e., $(-T, \beta) \in P_{(0,0)}$), since $\alpha_2 > \frac{1}{4}T^2$. Pick $\beta \in I \cap \mathbb{R}[T^{1/e}]$ (4.2). Let $\gamma \in \mathbb{R}[S]$ (S a new indeterminate) such that $\beta(T) = \gamma(T^{1/e})$. For $s \in \mathbb{R}$, define $q(s) = p_i(-s^e, \gamma(s))$; then q is C^∞ in s. Moreover, $\overline{q}(S) = \overline{p_i}(-S^e, \gamma(S))$, by the chain-rule for higher-order partial derivatives. So

$$\begin{aligned}\overline{q}(T^{1/e}) &= \overline{p_i}(-T, \gamma(T^{1/e}))\\ &= \overline{p_i}(-T, \beta(T)) < 0.\end{aligned}$$

Apply (4.1) to q; we get $\delta > 0$ such that

$$\forall t \in (0, \delta), \quad 0 > q(t^{1/e}) = p_i(-t, \beta(t)). \tag{6.0.1}$$

Apply (4.1) similarly to $\gamma(s) - \frac{1}{4}s^{2e} \in C^\infty(\mathbb{R})$: $\beta(t) > \frac{1}{4}t^2$ for $t \in (0, \delta')$, some $\delta' \in (0, \delta)$; thus $(-t, \beta(t)) \in P$. This, together with (6.0.1), violates (1.1.2″).

Cases 2 and 3: $\alpha_1 = T$ and $\alpha_1 = 0$, respectively. These cases are easier than Case 1.

The three cases, taken together, prove (1.5). □

7. Postscript on work of Broglia and Pernazza

Theorem 1.4 above implies our earlier result [D1991–94] that analytic variation is impossible in Artin's theorem (i.e., that B in Question 1.1 cannot be taken to be $\mathcal{O}(\mathbb{R}^m)$). The latter result had originally been deduced (easily) from the fact that the closed semianalytic set $P \subset \mathbb{R}^2$ in (1.4.2) is not "basic" closed semianalytic (at the origin $(0,0)$); i.e., for every open neighborhood $U \subseteq \mathbb{R}^2$ of $(0,0)$, and for every $s \in \mathbb{N}$, and for every choice of $p_1, \dots, p_s \in \mathcal{O}(U)$:

$$P \cap U \neq \{\, c \in U \mid p_1(c) \geq 0, \dots, p_s(c) \geq 0 \,\}.$$ [14]

We had at one time hoped that Theorem 1.4 could be deduced in a similar (easy) way from the following statement. Suppose $U \subseteq \mathbb{R}^2$ is an open neighborhood of $(0,0)$, $s \in \mathbb{N}$, $p_1, \dots, p_s \in C^\infty(U)$, and

$$P \cap U = \{\, c \in U \mid p_1(c) \geq 0, \dots, p_s(c) \geq 0 \,\}; \tag{7.0.1}$$

then for at least one $i \leq s$, p_i is flat at $(0,0)$ (recall §3). I asked Francesca Acquistapace and Fabrizio Broglia whether this statement is true; they (together with Ludovico Pernazza) succeeded in proving a more general version of this statement (but for open rather than closed semianalytic sets, with the obvious analog for the definition of "basic" open semianalytic):

Theorem 7.1 ([BP2002]). *Let $S \subseteq \mathbb{R}^m$ be an open semianalytic set whose*

[14] This easy deduction goes as follows. We take as known the fact that P is not basic. Then P_{nd} is not basic (for $d \geq 4$). Now if B in (1.1) could be taken to be $\mathcal{O}(\mathbb{R}^m)$, we would conclude that $P = \{\, c \in \mathbb{R}^m \mid p_1(c) \geq 0, \dots, p_s(c) \geq 0 \,\}$, where the $p_i \in \mathcal{O}(\mathbb{R}^m)$ are given in (1.1.1–2) ($\subseteq$ by (1.1.2), and $\supseteq$ by (1.1.1)). I.e., P is basic, after all—contradiction.

germ $S_{\mathbf{0}}$ at $\mathbf{0} := (0,\ldots,0)$ is not basic open semianalytic, and suppose that for some open neighborhood $U \subseteq \mathbb{R}^m$ of $\mathbf{0}$, there are $\phi_i \in C^\infty(U)$ such that

$$S \cap U = \{\, c \in U \mid \phi_1(c) > 0, \ldots, \phi_s(c) > 0 \,\}.$$

Then for at least one $i \leq s$, ϕ_i is flat at $\mathbf{0}$.

Unfortunately, the fact for every choice of $p_i \in C^\infty(U)$ satisfying (7.0.1), p_i must be flat at $(0,0)$ for some i, does not seem to lead easily to the results of this paper. For example, if some of the p_i in (1.1.1$''$) are flat, then some of the summands on the righthand side of (1.1.1$''$) will be flat; but this does not seem to lead to the desired contradiction, since the lefthand side could also be flat (depending on h).

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References

[A1926] E. Artin, Über die Zerlegung definiter Funktionen in Quadrate, Abh. math. Sem. Hamburg 5 (1926) 100–15; and in: "Coll. Papers," S. Lang, J.T. Tate (eds.), Addison-Wesley 1965; Springer, 1982, pp. 273–88.

[B1895] E. Borel, Sur quelques points de la théorie des fonctions, Annales Sci. École Normale Sup. (3) 12 (Jan. 1895) 9–55.

[BP2002] F. Broglia, L. Pernazza, An Artin-Lang property for germs of C^∞ functions, J. reine angew. Math. 548 (2002) 129–47.

[CL1977] M.D. Choi, T.Y. Lam, Extremal positive semidefinite forms, Math. Ann. 231 (1977) 1–18.

[C1998] S. Cornelsen, "Stetigkeitsfragen rund ums 17. Hilbertsche Problem," Diplomarbeit, Fakultät Math. und Info., Univ. Konstanz, 1998. See `http://inf.uni-konstanz.de/~cornelse/ich.html`.

[D1980–82] C.N. Delzell, Continuous sums of squares of forms, in: "L.E.J. Brouwer Cent. Symp. (Noordwijkerhout, 1981)," Stud. Logic Found. Math., 110, North-Holland, 1982, pp. 65–75. Abstract in Univ. Microfilms International, Order No. 8024640; in Dissertation Abstracts International 41(5) (1980); and in Abstracts of Papers Presented to the Amer. Math. Soc. 2, Issue 8, Number 1 (Jan. 1981), #783-12-28, p. 51.

[D1982] C.N. Delzell, Case distinctions are necessary for representing polynomials as sums of squares, in: "Proc. Herbrand Symp. (Marseilles, 1981)," Stud. Logic Foundations Math., 107, North-Holland, 1982, pp. 87–103.

[D1980–84] C.N. Delzell, A continuous, constructive solution to Hilbert's 17th problem, Invent. Math. 76(3) (1984) 365–384. Abstract in Univ. Microfilms International, Order No. 8024640; in Dissertation Abstracts International 41(5) (1980); and in Abstracts of Papers Presented to the Amer. Math. Soc. 2, Issue 8, Number 1 (Jan. 1981), #783-12-28, p. 51.

[D1985] C.N. Delzell, Analytic right-inverses for quadratic forms over number fields, Bull. London Math. Soc. 17 (1985) 449–52.

[D1987] C.N. Delzell, Continuous Pythagoras numbers for rational quadratic forms, J. Number Theory 26 (1987) 257–73.

[D1989–93] C.N. Delzell, Continuous, piecewise-polynomial functions which solve Hilbert's 17th problem, J. reine angew. Math. 440 (1993) 157–73. Abstract in Abstracts of Papers Presented to the Amer. Math. Soc. 10, Issue 63, Number 10, (April 1989) #849-14-160, pp. 208–9.

[D1991–94] C.N. Delzell, Non-existence of analytically varying solutions to Hilbert's 17th problem, in: "Recent Advances in Real Algebraic Geometry and Quadratic Forms, Proc. RAGSQUAD Year, Berkeley, 1990–1991," W. Jacob, et al (eds.), Contemp. Math., Vol. 155, AMS, 1994, pp. 107–17. Abstract in Abstracts of Papers Presented to the Amer. Math. Soc. 12, Number 1, Issue 73, (Jan. 1991), #863-14-743, p. 47.

[DGL1993] C.N. Delzell, L. González-Vega, H. Lombardi, A continuous and rational solution to Hilbert's 17th problem and several cases of the Positivstellensatz, in: "Computational Algebraic Geometry," Eyssette, et al (eds.), Birkhäuser, Progr. Math., 109, 1993, pp. 61–75.

[DM1995] C.N. Delzell, James J. Madden, Lattice-ordered rings and semialgebraic geometry: I, in: "Real Analytic and Algebraic Geometry (Trento, 1992)," F. Broglia, et al (eds.), de Gruyter, 1995, pp. 103–29.

[vdD1977] L. van den Dries, Artin-Schreier theory for commutative regular rings, Ann. Math. Logic 12 (1977) 113–50.

[E1754/55] L. Euler, Demonstratio theorematis Fermatiani omnem numerum sive integrum sive fractum esse summam quatuor pauciorumve quadratorum, Commentatio 242 indicis Enestroemiani, Novi commentarii academiae scientiarum Petropolitanae 5 (1754/5), 1760, 13–58; and in: "Opera Omnia," ser. I, vol. 2, pp. 338–72.

[GL1993] L. González-Vega, H. Lombardi, A real Nullstellensatz and Positivstellensatz for the semi-polynomials over an ordered field, J. Pure Appl. Algebra 90 (1993) 167–88.

[GL1997] L. González-Vega, H. Lombardi, Smooth parametrizations for several cases of the Positivstellensatz, Math. Z. 225(3) (1997) 427–51.

[Hei1964] H. Heilbronn, On the representation of a rational as a sum of four squares by means of regular functions, Bull. London Math. Soc. 39 (1964) 72–6.

[Hen1960] L. Henkin, Sums of squares, in: "Summaries of Talks Presented at the Summer Institute for Symbolic Logic, Cornell Univ., 1957," Inst. for Defense Analyses, Princeton, 1960, pp. 284–91. Reviewed by A. Robinson, J. Symbolic Logic 31 (1966) 128.

[Hi1888] D. Hilbert, Über die Darstellung definiter Formen als Summe von Formenquadraten, Math. Ann. 32 (1888) 342–50; and in: "Ges. Abh.," vol. 2, 1933, pp. 154–61.

[Hi1899] D. Hilbert, "Grundlagen der Geometrie," Teubner, 1899. English translation by Townsend, Open Court, 1902; reprinted 1950. The second German edition was translated by Unger; this was revised and enlarged by Bernays, Open Court, 1971.

[Hi1900] D. Hilbert, Mathematische Probleme, Göttinger Nach. (1900) 253–97, and Arch. der Math. u. Physik (3rd ser.) 1 (1901) 44–53, 213–37; and in: "Ges. Abh.," vol. 3, 1935, pp. 290–329. Transl. in Bull. Amer. Math. Soc. 8 (1902) 437–79; and in F. Browder's "Mathematical Developments Arising from Hilbert Problems," Proc. Symp. Pure Math. 28, Amer. Math. Soc., 1976, pp. 1–34.

[Kre1962] G. Kreisel, Review of Goodstein, Math. Reviews 24A (1962) 336–7, #A1821.

[Kri1964] J.-L. Krivine, Anneaux préordonnés, J. Analyse Math. 12 (1964) 307–26.

[N1985] R. Narasimhan, "Analysis on Real and Complex Manifolds," North Holland Math. Library, Vol. 35, Elsevier Sci. Pub. B.V., 1968, 1973, 1985.

[PD2001] A. Prestel, C. Delzell, "Positive Polynomials: From Hilbert's 17th Problem to Real Algebra," Springer Monographs in Mathematics, 2001.

[Ra1993] A.R. Rajwade, "Squares," London Math. Soc. Lect. Notes Series 171, Cambridge Univ. Press, 1993.

[Re1995] B. Reznick, Uniform denominators in Hilbert's Seventeenth Problem, Math. Z. 220 (1995) 75–97.

[Sc1989] P. Scowcroft, Some continuous Positivstellensätze, J. Algebra 124 (1989) 521–32.

[Si1921] C.L. Siegel, Darstellung total positiver Zahlen durch Quadrate, M. Zeit. 11 (1921) 246–75; and in: "Ges. Abh.," vol. 1, 1966, 47–76.