Extension of Pólya's theorem to signomials with rational exponents

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Abstract

Pólya proved that if a real, homogeneous polynomial is positive on the nonnegative orthant (except at the origin), then it is the quotient of two homogeneous polynomials with no negative coefficients. We generalize this from polynomials to signomials with arbitrary rational exponents; we also show that Pólya's theorem does not generalize to arbitrary signomials (i.e., with irrational (real) exponents).

1 Introduction

Let $n \in \mathbb{N} := \{0, 1, 2, \dots\}$, let $X := (X_1, \dots, X_n)$ be indeterminates, let $\alpha := (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ be a multi-exponent, and let $\|\alpha\| = \alpha_1 + \cdots + \alpha_n$. Write $X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$. Let $x := (x_1, \dots, x_n) \in \mathbb{R}^n$. We write $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_{++} = (0, \infty)$, and we write $\Delta_n = \{ x \in \mathbb{R}_+^n \mid \sum_i x_i = 1 \}$ for the closed, $(n-1)$-dimensional simplex.

Suppose $F \in \mathbb{R}[X]$ is homogeneous of degree d. Suppose also $F > 0$ on Δ_n; since F is homogeneous, this assumption is equivalent to assuming that $F > 0$ on *all* of $\mathbb{R}_+^n \setminus \{(0, \dots, 0)\}$. We can now state—the conclusion of—Pólya's theorem [Pólya, 1928], in its weak and strong forms:

Theorem 1.1 (Weak form of Pólya's[1] theorem)**.** *Let F be as above. Then F is the quotient of two homogeneous polynomials $G, H \in \mathbb{R}[X]$ $(H \neq 0)$ with every*

[1]This theorem had been proved by Poincaré [1888] for $n = 2$, and by Meissner [1911] for $n = 3$. According to [Hardy, et al, 1934–91, p. 57], Meissner's method is applicable in principle for $n > 3$, though his method (unlike Pólya's) does not lead to the stronger conclusion in (1.2) below.

coefficient being strictly[2] *positive:* $HF = G$. $\square$

Theorem 1.2 (Strong form of Pólya's theorem)**.** *Let F be as above. Then for all sufficiently large $p \in \mathbb{N}$,*

$$(X_1 + \cdots + X_n)^p F(X_1, \ldots, X_n) = G(X_1, \ldots, X_n), \tag{1.2.1}$$

for some (*necessarily real, homogeneous polynomial*) *G with every coefficient being strictly positive* (*recall footnote* 2)*.* $\square$

These theorems are also proved in [Hardy, et al, 1934–91, pp. 57–60].

In this paper we consider the question of whether (1.1) and/or (1.2) admit reasonable extensions from polynomials to arbitrary signomials, which can be defined roughly as "real polynomials with arbitrary real exponents" (such as $3X_1^{\pi}X_2^{-1}X_3^{0} - 5X_1^{-4}X_2^{-2}X_3^{\sqrt{2}}$). A signomial with no negative coefficients is known as a *posynomial*; thus the G and H in (1.1) and (1.2) above are posynomials.

Our three main results are:

- that (1.2) does not extend to signomials with non-integer exponents (4.1);
- that (1.1) extends to signomials with rational exponents (5.1); and
- that (1.1) does not extend to arbitrary signomials (i.e., signomials with arbitrary irrational exponents) (6.1). Specifically, for every $\epsilon \in (-1, 1)$, let $F_\epsilon(X_1, X_2)$ be the homogeneous signomial $X_1^2 - X_1^{1+\epsilon}X_2^{1-\epsilon} + X_2^2$. Then we show that

    1. $F_\epsilon > 0$ on Δ_2, but
    2. F_ϵ is the quotient of two posynomials if and only if ϵ is rational.

It is remarkable that for a signomial F that is positive on Δ_n, the (semi-)*algebraic* condition that F be the quotient of two posynomials is closely related to the *arithmetic* condition that the exponents of F be rational.

In section 2 we begin with careful definitions of the vocabulary of (formal) signomials (as elements of the group ring $\mathbb{R}[\mathbb{R}^n]$). Then we consider various elementary algebraic properties of this ring: for example, it is an integral domain, but not a unique factorization domain or a Noetherian domain. In section 3, we consider the function $f : \mathbb{R}^n_{++} \to \mathbb{R}$ (denoted by $f(x)$) determined by a formal signomial $f(X)$. We show that different formal signomials $f(X)$ determine different signomial functions $f(x)$ (3.1), and that a signomial function is algebraic over $\mathbb{R}(X)$ (2.4) if and only if all its exponents are rational (3.3). We also investigate when the composition of signomial functions is a signomial function ((3.5) and (3.6)). In sections 4–6 we prove our three main results (already listed in the previous paragraph). Finally, in section 8, we summarize what has been known

[2] I.e., for every $\alpha \in \mathbb{N}^n$ with $\|\alpha\| = \deg G$, the coefficient of X^α in G is positive (and not merely nonnegative); and similarly for H. See also Remark 5.4 below.

(for a long time) about $\mu := \min_{x\in\Delta_n} F(x)$; this minimum is one of the parameters in upper bounds on the "Pólya exponent" $p_F := \min\{\, p \in \mathbb{N} \mid (1.2.1) \text{ holds}\,\}$ (see (5.3) below); our point here is that Pólya's original method of constructing p and G in (1.2) is effective, despite impressions (suggested by some) to the contrary.

Signomials have been largely neglected by real algebraic and real analytic geometers. Yet signomials have played a role in various branches of mathematics:

- Signomials in one variable arise as solutions of certain classical differential equations (see (3.2) below). They also appear in Müntz's theorem:

- Posynomials in more than one variable seem to have been first studied by Zener [1961], who showed how to use the arithmetic-geometric inequality to find the minimum of a posynomial f over $\mathbb{R}^n_{++}$ (at least when f has $n+1$ terms). [Duffin, et al, 1967], however, seems to be the first publication to have coined the word "posynomial" itself, as an abbreviation for "positive polynomial."[3]

  A posynomial function f need not be convex, so it is not clear, *a priori*, that a local minimum of f on a suitable domain is a global minimum. However, Rockafellar [1970] observed that $\ln f(e^{x_1}, \ldots, e^{x_n})$ (defined on all of $\mathbb{R}^n$) is convex. Thus every "posynomial program" is equivalent to a *convex program*, i.e., the problem of minimizing a convex function over a convex domain; and in such problems, every local minimizing point *is* a global minimizing point.

  It was not until [Passy, et al, 1967] that signomials (then called "generalized polynomials") in more than one variable were studied; the word "signomial," however (which seems to come from the words "signum" and "polynomial" used by Passy and Wilde) did not appear in print until [Duffin, et al, 1973], apparently.

  Signomials and especially posynomials are used in geometric programming [Duffin, et al, 1967], which has diverse applications to chemical engineering, management science, structural engineering, water-quality management, nuclear engineering, transportation planning, production planning, and regional economics. Recent applications have been made by electrical engineers modeling analog integrated circuits (see [Daems, et al, 2001] and [2002]).

  In most engineering and other applications, the signomials actually used have fixed exponents that are *rational*; cf., e.g., [Sherali, 1998]. So in case

[3] While [Duffin, et al, 1967] defines a "positive polynomial" to be a posynomial (i.e., with arbitrary real exponents, and with positive coefficients), real algebraists define (somewhat vaguely) a "positive polynomial" to be a polynomial (i.e., with exponents in $\mathbb{N}$) that takes only positive values on a prescribed subset of $\mathbb{R}^n$; cf., e.g., the titles of [Marshall, 2000] and [Prestel, et al, 2001]. Thus Pólya's theorem 1.1 above, and our theorem 5.1 below, may be viewed as establishing connections between these two definitions, when the exponents are in $\mathbb{N}$ or $\mathbb{Q}$, respectively.

Pólya's theorem, or our extension thereof (5.1), some day find engineering applications, the restriction to rational exponents in (5.1) should not be a problem.

- Miller [1994] considered a class of functions $f : \mathbb{R}^n \to \mathbb{R}$ that properly contains the class of (extensions by 0 to $\mathbb{R}^n$ of) signomial functions. Specifically, he considered terms built up (in a formal language) from variable symbols $x_1, x_2, \dots$ (an arbitrary one of which we denote by x) and from constants in $\mathbb{R}$ by the usual operation symbols $+$, $-$, and $\cdot$, together with the class of operation symbols $\{\, x^r \mid r \in \mathbb{R} \,\}$; the symbol x^r indicates the function $\mathbb{R} \to \mathbb{R}$ defined by

$$x \mapsto \begin{cases} x^r & \text{if } x > 0 \\ 0 & \text{if } x \le 0. \end{cases}$$

He considered the structure

$$\mathbb{R}_{\mathrm{an}}^{\mathbb{R}} := (\mathbb{R}, <, +, -, \cdot, 0, 1, (x^r)_{r \in \mathbb{R}}, (\tilde{f})_{f \in \mathbb{R}\{X,n\}, n \in \mathbb{N}}),$$

where $(\tilde{f})_{f \in \mathbb{R}\{X,n\}, n \in \mathbb{N}}$ denotes a certain class of functions $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ that are analytic on $[-1, 1]^n$. He proved that the theory of $\mathbb{R}_{\mathrm{an}}^{\mathbb{R}}$ admits quantifier-elimination and analytic cell-decomposition, and is universally axiomatizable, o-minimal, and polynomially bounded. Note that Miller's class of functions is, by its definition, closed under composition, while the class of signomial functions (even if we extend them, by 0, from their usual domain $\mathbb{R}_{++}^n$ to all of $\mathbb{R}^n$) is not: for example, $(x_1 + 1)^{1/2}$ is not a signomial function (3.5).

- [Wang, 2004] extended Descartes' rule of signs to arbitrary signomials in one variable. (Actually, he considered only signomials with *nonnegative* real exponents; but Wang's result is easily extended to arbitrary signomials f in one variable, as claimed, since we may apply his result to the "essential part" f_+ of f (defined in (2.3) below), in just the same way that we use f_+ in (5.1) below.)

In a future paper we shall give a version of the Fourier-Budan theorem (on counting the roots of $f \in \mathbb{R}[X_1]$ in an interval $(a, b]$) for 1-variable signomial functions f (here we must assume that $a \ge 0$).

In another paper, we shall investigate the extent to which Krivine's Positivstellensatz and Schmüdgen's Positivstellensatz [1991] (which can be derived from Pólya's theorem, as Schweighofer showed in [2002] and [submitted]), as well as a similar (1-variable) theorem of Pólya-Szegö, extend from polynomials to signomials.

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2 Generalities on formal signomials

We keep the same notations as in section 1, except that from now on we allow the multi-exponent α to be an arbitrary element of $\mathbb{R}^n$.

Definitions 2.1. A *formal signomial* (or simply a *signomial*, when no confusion can result) is an element of the group ring $\mathbb{R}[\mathbb{R}^n]$; the latter is defined to be the set of finite formal sums

$$f := f(X) := \sum_{i=1}^{m} c_i X^{\alpha_i}, \tag{2.1.1}$$

where $m \in \mathbb{N}$, the "coefficients" c_i of f are nonzero elements of (the ring) $\mathbb{R}$, and the (multi-)"exponents" $\alpha_i := (\alpha_{i,1}, \ldots, \alpha_{i,n})$ of f are distinct elements of (the additive group) $\mathbb{R}^n$; here X denotes n indeterminates as above (thus we may call f a signomial "in X," for clarity). We call a single term $c_i X^{\alpha_i}$ a *formal* (*gemeralized*) *monomial*; the X_j*-degree* of $c_i X^{\alpha_i}$ (assuming $c_i \neq 0$) is defined to be $\alpha_{i,j}$, while its *degree* is defined to be $\|\alpha_i\| = \alpha_{i,1} + \cdots + \alpha_{i,n}$. We call f *homogeneous* (of degree $d \in \mathbb{R}$) if each of its monomials has degee d. We define the *support* $\operatorname{supp} f$ of f to be $\{\, \alpha_i \mid c_i \neq 0 \,\}$.

Definition 2.2. A formal signomial each of whose coefficients is nonnegative is called a (*formal*) *posynomial*.

(Thus each signomial is the difference of two posynomials.)

We reserve the term "(real) polynomial" for a formal signomial f with $\operatorname{supp} f \subset \mathbb{N}^n \subset \mathbb{R}^n$. As usual, we continue to denote the ring of polynomials by $\mathbb{R}[X]$ (rather than by $\mathbb{R}[\mathbb{N}^n]$), and its field of fractions by $\mathbb{R}(X)$. The ring operations on $\mathbb{R}[\mathbb{R}^n]$ are straightforward generalizations of the operations on $\mathbb{R}[X]$.

[Gilmer, 1984] provides a comprehensive presentation of the algebraic properties of (semi)group rings at various levels of generality and abstraction. Below we mention some algebraic properties of our particular group ring, $\mathbb{R}[\mathbb{R}^n]$.

For $n > 0$, $\mathbb{R}[\mathbb{R}^n]$ is isomorphic to the iterated group ring $(\mathbb{R}[\mathbb{R}^{n-1}])[\mathbb{R}]$, by writing $X' := (X_1, \ldots, X_{n-1})$ (so that $X = (X'; X_n)$), and by writing $f(X_1, \ldots, X_n) \in \mathbb{R}[\mathbb{R}^n]$ as

$$f_1(X')X_n^{\beta_1} + \cdots + f_K(X')X_n^{\beta_K}, \tag{2.2.1}$$

for some $K \in \mathbb{N}$, some $\beta_1 < \cdots < \beta_K$ in $\mathbb{R}$, and some $f_k \in \mathbb{R}[\mathbb{R}^{n-1}] \setminus \{0\}$.

The units of the ring $\mathbb{R}[\mathbb{R}^n]$ are the nonzero (generalized) monomials cX^α, for $c \in \mathbb{R}^\times$ and $\alpha \in \mathbb{R}^n$. Indeed, the inverse of cX^α is $c^{-1}X^{-\alpha}$; conversely, for $f, g \in \mathbb{R}[\mathbb{R}^n]$, if fg is a nonzero monomial (such as 1), then both f and g are monomials. The latter fact can be seen by writing f as in (2.2.1) and writing g similarly as $g_1(X')X_n^{\gamma_1} + \cdots + g_L(X')X_n^{\gamma_L}$, and then considering the terms in fg of X_n-degree $\beta_1 + \gamma_1$, and the terms in fg of X_n-degree $\beta_K + \gamma_L$, to conclude that $K = L = 1$; then use induction on n to conclude that $f_1 g_1 \in \mathbb{R}[\mathbb{R}^{n-1}]$ is also a monomial.

By a similar argument, we also see that $\mathbb{R}[\mathbb{R}^n]$ is an integral domain, i.e., for $f, g \in \mathbb{R}[\mathbb{R}^n]$,

$$[f \neq 0 \text{ and } g \neq 0] \Rightarrow fg \neq 0.$$

We denote the field of fractions of $\mathbb{R}[\mathbb{R}^n]$ by $\mathbb{R}(\mathbb{R}^n)$; then $\mathbb{R}(X)$ is a subfield of $\mathbb{R}(\mathbb{R}^n)$.

For $n > 0$, $\mathbb{R}[\mathbb{R}^n]$ is not a unique factorization domain and it is not Noetherian, as seen, for example, by the (nontrivial) factorizations

$$\begin{aligned} X_1 - 1 &= (X_1^{1/2} + 1)(X_1^{1/2} - 1) \\ &= (X_1^{1/2} + 1)(X_1^{1/4} + 1)(X_1^{1/4} - 1) \\ &= \cdots . \end{aligned}$$

Definitions 2.3. First we define a partial order $\preceq$ on $\mathbb{R}^n$ as follows: for $\alpha, \beta \in \mathbb{R}^n$, we define $\alpha \preceq \beta$ to mean that $\alpha_1 \leq \beta_1, \ldots, \alpha_n \leq \beta_n$. Then, for $f \in \mathbb{R}[\mathbb{R}^n] \setminus \{0\}$ we define $\alpha_f \in \mathbb{R}^n$ to be the greatest lower bound (with respect to $\preceq$) of $\operatorname{supp} f$. Finally, we define the *essential part* $f_+(X)$ of $f(X)$ to be $X^{-\alpha_f} f(X)$.

Thus $f(X) = X^{\alpha_f} f_+(X)$, $\operatorname{supp} f = \alpha_f + \operatorname{supp} f_+$, and $\operatorname{supp} f_+ \subset \mathbb{R}^n_+$. For each $j \in \{1, \ldots, n\}$, there exists an $\alpha \in \operatorname{supp} f_+$ $(= (\operatorname{supp} f) - \alpha_f)$ with $\alpha_j = 0$. For example, (re)consider

$$f(X_1, X_2, X_3) := 3X_1^{\pi} X_2^{-1} X_3^0 - 5X_1^{-4} X_2^{-2} X_3^{\sqrt{2}}; \text{ then } \alpha_f = (-4, -2, 0) \text{ and}$$
$$f_+(X_1, X_2, X_3) = 3X_1^{4+\pi} X_2^1 X_3^0 - 5X_1^0 X_2^0 X_3^{\sqrt{2}}.$$

Definitions 2.4. For formal signomials f, g with $g \neq 0$, we say that the (formal) quotient $f/g \in \mathbb{R}(\mathbb{R}^n)$ is *algebraic* over $\mathbb{R}(X)$ (or simply *algebraic*) if there exists a polynomial $P \in \mathbb{R}(X_1, \ldots, X_n)[Y] \setminus \{0\}$ involving a single additional indeterminate Y such that

$$P\Big(X_1, \ldots, X_n; \frac{f(X_1, \ldots, X_n)}{g(X_1, \ldots, X_n)}\Big) = 0.$$

(In this case we may actually take $P \in \mathbb{R}[X_1, \ldots, X_n][Y] \setminus \{0\}$.) If f/g is not algebraic, then we call it *transcendental*. We call f *essentially algebraic* if f_+ is algebraic (recall (2.3)).

As usual, the relative algebraic closure of $\mathbb{R}(X)$ in $\mathbb{R}(\mathbb{R}^n)$ (i.e., the set of those quotients of signomials that are algebraic), forms a subfield of $\mathbb{R}(\mathbb{R}^n)$. In (3.3) below, we shall see that a signomial f is algebraic if and only if $\operatorname{supp} f \subset \mathbb{Q}^n$.

3 Generalities on signomial functions

Each formal signomial $f(X) = \sum_{i=1}^m c_i X^{\alpha_i}$ determines a function (also denoted by f) from $\mathbb{R}^n_{++}$ to $\mathbb{R}$, defined by $f(x) = \sum_i c_i x^{\alpha_i}$. Such a function is called a *signomial function*; and if $f(X)$ is a formal posynomial, then the function $f(x)$ determined by it is called a *posynomial function*. Writing the multi-exponent α_f of (2.3) as $(\alpha_{f,1}, \ldots, \alpha_{f,n})$, we extend $f(x)$ to those faces $\{x_j = 0\}$ of the boundary of $\mathbb{R}^n_{++}$ for which $\alpha_{f,j} \geq 0$; thus $f(x)$ can be defined at least[4] on

$$\mathbb{R}^n_+ \setminus \bigcup_{\alpha_{f,j}<0} \{\, x \in \mathbb{R}^n_+ \mid x_j = 0\},$$

which we call the *natural domain* of f. (Here, in case for some j, $\alpha_{f,j} \geq 0$, and for some i, $\alpha_{i,j} = 0$, we define $f(x)$ when $x_j = 0$ according to the convention that $0^0 = 1$.) In particular, since $\operatorname{supp} f_+ \subset \mathbb{R}^n_+$, $f_+(X)$ determines a real-valued function on all of $\mathbb{R}^n_+$, and for all $x \in \mathbb{R}^n_{++}$, $\operatorname{sgn} f(x) = \operatorname{sgn} f_+(x)$.

Every signomial function is real analytic on $\mathbb{R}^n_{++}$; and its extension to its natural domain ($\subseteq \mathbb{R}^n_+$) is continuous.

Lemma 3.1. *Suppose $f_1, \ldots, f_e$ are nonzero formal signomials. Then there exists an $x \in \mathbb{R}^n_{++}$ such that $f_1(x), \ldots, f_e(x)$ are all nonzero real numbers. More precisely, there exists $x' := (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}_{++}$ such that for all sufficiently large $x_n \in \mathbb{R}_{++}$, $f_1(x', x_n), \ldots, f_e(x'; x_n)$ are all nonzero real numbers.*

Proof. As in (2.2.1), for each $j \in \{1, \ldots, e\}$ write

$$f_j(X) = f_{j,1}(X')X_n^{\beta_{j,1}} + \cdots + f_{j,K_j}(X')X_n^{\beta_{j,K_j}},$$

for some $K_j \in \mathbb{N}$, some $\beta_{j,1} < \cdots < \beta_{j,K_j}$ in $\mathbb{R}$, and some $f_{j,k} \in \mathbb{R}[\mathbb{R}^{n-1}] \setminus \{0\}$.

For $n = 1$, note that each $f_{j,k}$ is just a nonzero constant. So for each j,

$$\begin{aligned} &\lim_{x_1 \to +\infty} \frac{f_j(x_1)}{x_1^{\beta_{j,K_j}}} \\ &= \lim_{x_1 \to +\infty} (f_{j,1}x_1^{\beta_{j,1}-\beta_{j,K_j}} + \cdots + f_{j,K_j-1}x_1^{\beta_{j,K_j-1}-\beta_{j,K_j}} + f_{j,K_j}) = f_{j,K_j} \neq 0, \end{aligned}$$

and we are done. For $n > 1$, apply induction to find $x' \in \mathbb{R}^{n-1}_{++}$ such that for each j, $f_{j,K_j}(x') \neq 0$. Then apply the 1-variable case to each $f_j(x'; X_n) \in \mathbb{R}[\mathbb{R}] \setminus \{0\}$. $\square$

In view of the case $e = 1$ of the lemma, we need no longer distinguish carefully, in every case, between $f_1(X)$ and $f_1(x)$.

Remark 3.2 (on differential equations). Every signomial function $y = f(x_1)$ in one variable is a solution of a suitable Cauchy-Euler differential equation, i.e., a d'th-order differential equation of the form

$$a_0 y + a_1 x_1 y' + a_2 x_1^2 y'' + \cdots + a_d x_1^d y^{(d)} = 0, \tag{3.2.1}$$

[4] In certain cases we may even extend f to points $x \in \mathbb{R}^n$ with $x_j < 0$ for certain j. Specifically, recall that if $x_j < 0$, then the complete set of values of $x_j^{\alpha_{i,j}}$ is $\{\, |x_j|^{\alpha_{i,j}} \cdot e^{\pi\sqrt{-1}(2k+1)\alpha_{i,j}} \mid k \in \mathbb{Z}\,\}$; this set contains a real number if and only if $(2k+1)\alpha_{i,j}$ is an integer, for some $k \in \mathbb{Z}$. Thus we may allow negative x_j in f if and only if for all i, $\alpha_{i,j}$ is a rational number with an odd denominator.

for $d \in \mathbb{N}$ and real constants a_i. To see this, suppose we are given a signomial function $y = f(x_1) = c_0 x_1^{\beta_0} + \cdots + c_d x_1^{\beta_d}$ with $\beta_0 < \cdots < \beta_d$. Recall that (3.2.1) can be solved either by substituting $x_1 = e^t$ (which transforms (3.2.1) into a differential equation with constant coefficients), or by substituting $y = x_1^r$, and solving the following polynomial equation (of degree d) for r:

$$a_0 + a_1 r + a_2 r(r-1) + \cdots + a_d r(r-1)(r-2)\cdots(r-d+1) = 0. \tag{3.2.2}$$

Since the polynomials

$$1,\ r,\ r(r-1),\ r(r-1)(r-2),\ \ldots,\ r(r-1)(r-2)\cdots(r-d+1)$$

form a basis for the real vector space of all real polynomials of degree $\leq d$ in r, there exist (unique) $a_0, \ldots, a_d \in \mathbb{R}$ such that (3.2.2) is $(r-\beta_0)(r-\beta_1)\cdots(r-\beta_d) = 0$. Then for each i, $y = x_1^{\beta_i}$ will satisfy (3.2.1), whence so will $y = f(x_1)$. The claim is proved.

The converse is false (except for $d = 1$). For example, to solve $y + x_1 y' + x_1^2 y'' = 0$, we try $y = x_1^r$, for some unknown r. We get $r(r-1)x_1^r + r x_1^r + x_1^r = 0$, whence $r^2 + 1 = 0$. This suggests the two complex solutions

$$\begin{aligned} x_1^{\sqrt{-1}} &= e^{\sqrt{-1}\ln x_1} = \cos\ln x_1 + \sqrt{-1}\sin\ln x_1 \text{ and} \\ x_1^{-\sqrt{-1}} &= e^{-\sqrt{-1}\ln x_1} = \cos\ln x_1 - \sqrt{-1}\sin\ln x_1, \end{aligned}$$

which lead to the two ($\mathbb{R}$-linearly independent) real solutions $y_1 = \cos\ln x_1$ and $y_2 = \sin\ln x_1$, neither of which is a signomial function (because, e.g., they both oscillate infinitely often as $x_1 \to +\infty$). □

As promised after (2.4) above, we now state and prove

Proposition 3.3. *A signomial $f \in \mathbb{R}[\mathbb{R}^n]$ is algebraic if and only if* $\operatorname{supp} f \subset \mathbb{Q}^n$.

Proof. Suppose first that $\operatorname{supp} f \subset \mathbb{Q}^n$. Any rational power $X_j^{p/q}$, for $p, q \in \mathbb{Z}$ with $q > 0$, is algebraic, since $P(X_j, X_j^{p/q}) = 0$, where $P(X_j, Y) = Y^q - X_j^p \in \mathbb{R}(X)[Y]$. Since f is an $\mathbb{R}$-linear combination of products of such powers, it, too, is algebraic.

For the converse, we continue to write $X' = (X_1, \ldots, X_{n-1})$. Write f as in (2.2.1). We need a lemma.

Lemma 3.4. *Suppose that $P(X; f(X)) = 0$, for some $P \in \mathbb{R}[X][Y] \setminus \{0\}$. Write*

$$P(X;Y) = p_0(X) + \cdots + p_j(X)Y^j + \cdots + p_d(X)Y^d \tag{3.4.1}$$

$$= P(X', X_n; Y) = \sum_{j=0}^{d} \big(p_{j,0}(X') + \cdots + p_{j,l}(X')X_n^l + \cdots + p_{j,e_j}(X')X_n^{e_j}\big)Y^j,$$

for some $d, e_j \in \mathbb{N}$ and some $p_j \in \mathbb{R}[X]$ and $p_{j,l} \in \mathbb{R}[X']$; for those j with $p_j \neq 0$, choose e_j so that $p_{j,e_j} \neq 0$. Let

$$D = \max_{\substack{0 \leq j \leq d \\ p_j \neq 0}} (e_j + j\beta_K),$$

where β_K is as in (2.2.1). *Let*

$$P_D(X', X_n; Y) = \sum_{\substack{p_j \neq 0 \\ e_j + j\beta_K = D}} p_{j,e_j}(X') X_n^{e_j} Y^j. \tag{3.4.2}$$

Then

(3.4.3) $P_D \neq 0$*;*

(3.4.4) $P_D(X', X_n; f_K(X')X_n^{\beta_K})$ *consists precisely of those terms in* $P(X', X_n; f(X', X_n))$ *with* X_n*-degree* D *(before any cancellation); and hence*

(3.4.5) $P_D(X', X_n; f_K(X')X_n^{\beta_K}) = 0$.

Proof of Lemma 3.4. (3.4.3) holds since there is some j_0 such that $p_{j_0} \neq 0$ and $e_{j_0} + j_0\beta_K = D$, and $p_{j_0,e_{j_0}} \neq 0$ by the choice of e_{j_0}. (3.4.4) follows from (3.4.2) and

$$P(X', X_n; f(X', X_n)) =$$
$$\sum_{j=0}^{d} \big(p_{j,0}(X') + \cdots + p_{j,e_j}(X')X_n^{e_j}\big)\big(f_1(X')X_n^{\beta_1} + \cdots + f_K(X')X_n^{\beta_K}\big)^j; \tag{3.4.6}$$

and (3.4.6), in turn, follows from (3.4.1) and (2.2.1). (3.4.5) follows from (3.4.4) and the hypothesis that $P(X, f(X)) = 0$. This proves the lemma. □

Returning to the proof of the converse of (3.3), suppose f is algebraic. We must show that each exponent occurring in f is rational. There is a P as in the hypothesis of (3.4). From (3.4.5) we conclude that $P_D(X', X_n; f_K(X')X_n^{\beta_K}) = 0$. For later reference we remark that since $P_D \neq 0$ (3.4.3), we have so far shown that

$$f_K(X')X_n^{\beta_K}, \text{ and hence } f_1(X')X_n^{\beta_1} + \cdots + f_{K-1}(X')X_n^{\beta_{K-1}}, \text{ is algebraic.} \tag{3.4.7}$$

Now we use induction on n. For $n = 1$, note that f_K is just a nonzero constant. The above paragraph shows that $P_D(X_1; f_K X_1^{\beta_K}) = 0$. And by (3.4.4), every term in the expansion of $P_D(X_1; f_K X_1^{\beta_K})$ (before cancellation, that is) has (X_1-)degree D. In order for cancellation of nonzero terms to occur, the number of such terms must be at least two; i.e., there exist $j_1 < j_2$ in the set $\{0, \ldots, d\}$ such that $e_{j_1} + j_1\beta_K = e_{j_2} + j_2\beta_K$ (and $p_{j_1}(X_1)p_{j_2}(X_1) \neq 0$). Therefore

$$\beta_K = \frac{e_{j_1} - e_{j_2}}{j_2 - j_1} \in \mathbb{Q},$$

as required. If $K = 1$, we are done for the case $n = 1$; if $K > 1$, we use a separate induction on K.

Now assume $n > 1$ and that every algebraic signomial in $\mathbb{R}[\mathbb{R}^{n-1}]$ has only rational exponents. Pick $x' := (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$ such that $f_K(x') \neq 0$

and $P_D(x', X_n; Y) \neq 0 \in \mathbb{R}[X_n; Y]$; this is possible by applying Lemma 3.1 (with n replaced by $n-1$) to f_K and the $(X_n; Y)$-coefficients $p_{j,e_j}(X') \in \mathbb{R}[X'] \subset \mathbb{R}[\mathbb{R}^{n-1}]$ of P_D, not all of which are zero in $\mathbb{R}[\mathbb{R}^{n-1}]$, by (3.4.3). Then $P_D(x', X_n; f(x', X_n)) = 0$. Now by the 1-variable case above, β_K is rational. Thus $X_n^{\beta_K}$ is algebraic, whence $f_K(X')$ is, too (using (3.4.7)). By the inductive hypothesis, every exponent occurring in $f_K(X')$ is rational. Thus, if $K=1$, we are done. If $K>1$, repeat the above argument for $f_1(X')X_n^{\beta_1} + \cdots + f_{K-1}(X')X_n^{\beta_{K-1}}$, considering (3.4.7). Thus all the exponents occurring in f are rational, and (3.3) is proved. $\square$

Thus we sometimes refer to a signomial all of whose exponents are rational as an "algebraic signomial"; the others can be called transcendental signomials.

Example 3.5. The class of signomial functions is not closed under composition. For example, the composition of $f(x_1) := x_1^{1/2}$ and $g(x_1) := x_1+1$ is $(f\circ g)(x_1) = f(g(x_1)) = (x_1+1)^{1/2}$, which is not a signomial function. (Proof: Suppose

$$(x_1+1)^{1/2} = h_1 x_1^{\beta_1} + \cdots + h_e x_1^{\beta_e} \in \mathbb{R}[\mathbb{R}],$$

with $\beta_1 < \cdots < \beta_e$ and $h_i \in \mathbb{R} \setminus \{0\}$. Then

$$x_1 + 1 = (h_1 x_1^{\beta_1} + \cdots + h_e x_1^{\beta_e})^2.$$

But the right hand side is either a monomial (in case $e=1$), or it contains at least three terms, of degrees $2\beta_1 < \beta_1 + \beta_2 < 2\beta_e$, in case $e>1$.)

Remark 3.6 (on composition). While the class of signomial functions is not closed under arbitrary composition, usually we may plug *monomials* into a signomial function, and still have a signomial function, as follows. Let $f(X)$ denote a formal signomial as in (2.1.1). Let Y denote new indeterminates $(Y_1, \ldots, Y_l)$ (some $l \in \mathbb{N}$), and let $b_1 Y^{\beta_1}, \ldots, b_n Y^{\beta_n}$ be n formal monomials in Y, where $b_j \in \mathbb{R}$ and $\beta_j := (\beta_{j,1}, \ldots, \beta_{j,l}) \in \mathbb{R}^l$. We then define

$$f(b_1 Y^{\beta_1}, \ldots, b_n Y^{\beta_n}) := \sum_{i=1}^{m} c_i \cdot b_1^{\alpha_{i,1}} \cdots b_n^{\alpha_{i,n}} Y_1^{\alpha_{i,1}\beta_{1,1} + \cdots + \alpha_{i,n}\beta_{n,1}} \cdots Y_l^{\alpha_{i,1}\beta_{1,l} + \cdots + \alpha_{i,n}\beta_{n,l}},$$

which will be a formal signomial in $\mathbb{R}[\mathbb{R}^l]$ provided that each power $b_j^{\alpha_{i,j}}$ is real (which will be the case if either $b_j > 0$, or $\alpha_{i,j}$ is a rational number with odd denominator, and where 0^0 is considered to be 1).

4 Impossibility of extending the strong form of Pólya's theorem to arbitrary signomials

We first note that already for $n=2$ (the first nontrivial case), the strong form of Pólya's theorem (1.2) does not extend to signomials with arbitrary, non-integer exponents. To state this precisely, first write (X, Y) instead of (X_1, X_2).

Theorem 4.1 (Falsity of (1.2) for arbitrary signomials). *For $n = 2$ there exists a homogeneous signomial $F \in \mathbb{R}[\mathbb{R}^2]$ such that $F > 0$ on Δ_2, but for every $p \in \mathbb{N}$, $(X+Y)^p F(X,Y)$ is* not *a posynomial—i.e., it has a negative coefficient.* (*We may even take such an F to have rational exponents.*)

Before beginning the proof, we need

Lemma 4.2. *For any $\epsilon \in \mathbb{R}$, let*

$$F_\epsilon(X,Y) = X^2 - X^{1+\epsilon}Y^{1-\epsilon} + Y^2. \tag{4.2.1}$$

If $|\epsilon| < 1$, then $F_\epsilon > 0$ on $\Delta_2 := \{\, (x,y) \in \mathbb{R}^2_+ \mid x + y = 1 \,\}$ (and hence on the rest of $\mathbb{R}^2_+ \setminus \{(0,0)\}$, since F is homogeneous).

Proof of Lemma 4.2. F_ϵ determines a continuous function $f(x,\epsilon) := F_\epsilon(x, 1-x)$ of two variables on $([0,1] \times [-1,1]) \setminus \{(0,-1),(1,1)\}$, since there we avoid the expression 0^0. It suffices to show that $f > 0$ on this region (which includes the region $[0,1] \times (-1,1)$).

For each $x \in (0,1)$,

$$\begin{aligned}\frac{\partial f}{\partial \epsilon} &= -x^{1+\epsilon}(\ln x)(1-x)^{1-\epsilon} + x^{1+\epsilon}(1-x)^{1-\epsilon}\ln(1-x)\\ &= x^{1+\epsilon}(1-x)^{1-\epsilon}(\ln(1-x) - \ln x).\end{aligned}$$

Therefore for all $\epsilon_0 \in [-1,1]$,

$$\frac{\partial f}{\partial \epsilon}(x,\epsilon_0) \begin{cases} > 0 & \text{for } 0 < x < \frac{1}{2}, \\ = 0 & \text{for } x = \frac{1}{2}, \text{ and} \\ < 0 & \text{for } \frac{1}{2} < x < 1. \end{cases}$$

So for $x \in (0, \frac{1}{2})$,

$$\min_{\epsilon\in[-1,1]} f(x,\epsilon) = f(x,-1) = x^2 - x^0(1-x)^2 + (1-x)^2 = x^2 > 0.$$

Similarly, for $x \in (\frac{1}{2}, 1)$,

$$\min_{\epsilon\in[-1,1]} f(x,\epsilon) = f(x,1) = x^2 - x^2(1-x)^0 + (1-x)^2 = (1-x)^2 > 0.$$

For $x = \frac{1}{2}$, we have, for all $\epsilon \in [-1,1]$,

$$f\Big(\frac{1}{2},\epsilon\Big) = \Big(\frac{1}{2}\Big)^2 - \Big(\frac{1}{2}\Big)^{1+\epsilon}\Big(\frac{1}{2}\Big)^{1-\epsilon} + \Big(\frac{1}{2}\Big)^2 = \Big(\frac{1}{2}\Big)^2 - \Big(\frac{1}{2}\Big)^2 + \Big(\frac{1}{2}\Big)^2 = \frac{1}{4} > 0.$$

Finally,

$$\begin{aligned}&\text{for all } \epsilon \in (-1,1], \quad f(0,\epsilon) = 0^2 - 0^{1+\epsilon}1^{1-\epsilon} + 1^2 = 1 > 0, \text{ and}\\ &\text{for all } \epsilon \in [-1,1), \quad f(1,\epsilon) = 1^2 - 1^{1+\epsilon}0^{1-\epsilon} + 0^2 = 1 > 0. \qquad \square\end{aligned}$$

Proof of (4.1). We shall show that we may take F in (4.1) to be the F_ϵ in (4.2.1), for any $\epsilon \in (-1,1)$ other than 0 (whether rational or not). The previous lemma shows that then $F > 0$ on Δ_2. It remains to show that for every $p \in \mathbb{N}$, $(X+Y)^p F(X,Y)$ has at least one negative coefficient. For this, observe first that

$$\begin{aligned}(X+Y)^p F(X,Y) &= (X+Y)^p (X^2 - X^{1+\epsilon}Y^{1-\epsilon} + Y^2) \\ &= (X+Y)^p (X^2+Y^2) - (X+Y)^p \cdot X^{1+\epsilon}Y^{1-\epsilon}. \qquad (4.2.2)\end{aligned}$$

Next observe that none of the terms contributed by $(X+Y)^p(X^2+Y^2)$ to (4.2.2) can offset any of the (negative) terms contributed by $-(X+Y)^p \cdot X^{1+\epsilon}Y^{1-\epsilon}$, since all the terms in the first product have integer exponents, while all the terms in the second product have non-integer exponents. $\square$

Having thus ruled out the possibility of extending the strong form (1.2) of Pólya's theorem to arbitrary signomials, we turn next, in sections 5 and 6 below, to the question of the extendability of the *weak* form (1.1) of Pólya's theorem.

5 Extension of the weak form of Pólya's theorem to essentially algebraic signomials

Theorem 5.1 (Weak form of Pólya for essentially algebraic signomials)**.** *Suppose F is a homogeneous signomial in $\mathbb{R}[\mathbb{R}^n]$. Suppose also that F is essentially algebraic* (2.4)*, i.e., that the essential part F_+ of F (recall* (2.3)*) is algebraic—equivalently, that* $\operatorname{supp} F_+ \subset \mathbb{Q}^n$ *(recall* (3.3)*). Finally, suppose that $F_+ > 0$ on Δ_n (and hence on the rest of $\mathbb{R}^n_+ \setminus \{(0,\ldots,0)\}$). Then F is the quotient of two homogeneous posynomials* (2.2) *G and H in $\mathbb{R}[\mathbb{R}^n] \setminus \{0\}$: $HF = G$.*

Proof. Let $q \in \mathbb{N} \setminus \{0\}$ be the least common denominator of the exponents occurring in F_+. Define a new signomial F^* as follows:

$$F^*(X_1,\ldots,X_n) = F_+(X_1^q,\ldots,X_n^q) \qquad (5.1.1)$$

(recall (3.6)). Then F^* is even a polynomial $\in \mathbb{R}[X]$, homogeneous of degree $q \cdot \deg F_+$. Also, $F^* > 0$ on $\mathbb{R}^n_+ \setminus \{(0,\ldots,0)\} \supset \Delta_n$, since for any $x \in \mathbb{R}^n_+ \setminus \{(0,\ldots,0)\}$, $F^*(x) = F_+(x_1^q,\ldots,x_n^q) > 0$. So there exist $G^*, H^* \in \mathbb{R}[X] \setminus \{0\}$, with all coefficients strictly positive (recall footnote 2 to Theorem 1.1), such that

$$H^*(X_1,\ldots,X_n)F^*(X_1,\ldots,X_n) = G^*(X_1,\ldots,X_n), \qquad (5.1.2)$$

by (1.1). (We may even take H^* to be $(X_1+\cdots+X_n)^p$ for sufficiently large $p \in \mathbb{N}$, by (1.2).) Replacing each X_j in (5.1.2) by $X_j^{1/q}$, we get

$$H^*(X_1^{1/q},\ldots,X_n^{1/q})F^*(X_1^{1/q},\ldots,X_n^{1/q}) = G^*(X_1^{1/q},\ldots,X_n^{1/q}). \qquad (5.1.3)$$

Defining

$$G'(X_1, \dots, X_n) := G^*(X_1^{1/q}, \dots, X_n^{1/q}) \text{ and} \tag{5.1.4}$$

$$H'(X_1, \dots, X_n) := H^*(X_1^{1/q}, \dots, X_n^{1/q}) \tag{5.1.5}$$

by (3.6), we see that G' and H' are nonzero posynomials.[5] Recalling (5.1.1) and (5.1.3)–(5.1.5), we get

$$H'(X_1, \dots, X_n)F_+(X_1, \dots, X_n) = G'(X_1, \dots, X_n).$$

Multiplying both sides by X^{α_F} (recalling (2.3)), we get

$$H'(X)(X^{\alpha_F} F_+(X)) = X^{\alpha_F} G'(X).$$

Finally, recalling that $F(X) = X^{\alpha_F} F_+(X)$, we get

$$H'(X)F(X) = X^{\alpha_F} G'(X).$$

Thus we may take $G(X) = X^{\alpha_F} G'(X)$ and $H(X) = H'(X)$ in (5.1). □

Remark 5.2 (on H). The above proof actually shows that we may take $H(X)$ in (5.1) to be

$$(X_1^{1/q} + \cdots + X_n^{1/q})^p \in \mathbb{R}[X_1^{1/q}, \dots, X_n^{1/q}] \subset \mathbb{R}[\mathbb{Q}^n] \subset \mathbb{R}[\mathbb{R}^n], \tag{5.2.1}$$

where $q \in \mathbb{N} \setminus \{0\}$ is as in the proof, and where $p \in \mathbb{N}$ is sufficiently large.

Remark 5.3 (on bounds). Given a particular F as in (1.2), let us call the minimum $p \in \mathbb{N}$ for which (1.2.1) holds, the *Pólya exponent* of F; we denote it by p_F. Then [de Loera, et al, 1996, 2001] showed that

$$p_F \leq \frac{nd^2 \max_i |c_i|}{\mu} + nd, \tag{5.3.1}$$

where, as before, $d = \deg F$, the c_i are as in (2.1.1), and

$$\mu := \mu(c) := \min_{x \in \Delta_n} F(x). \tag{5.3.2}$$

[Powers, et al, 2001] improved this, as follows. Letting

$$\overline{c} = \max_i \Big(\frac{\alpha_{i,1}!\, \alpha_{i,2}! \, \cdots \alpha_{i,n}!}{d!} \, |c_i| \Big),$$

they showed

$$p_F \leq \frac{d(d-1)}{2} \cdot \frac{\overline{c}}{\mu} - d + 1, \tag{5.3.3}$$

[5] In fact, for every $\alpha \in \frac{1}{q}\mathbb{N}^n \subset \mathbb{Q}^n$ with $\|\alpha\| = \deg G'$, the coefficient of X^α in G' is strictly positive, and similarly for H'. Recall footnote 2 to Theorem 1.1 above, and see Remark 5.4 below.

which is sharp at least for $d = 2$ and for certain values of μ. These two bounds obviously apply equally well to the minimum possible $p \in \mathbb{N}$ in (5.2.1), upon replacing μ by

$$\mu^* := \min_{x \in \Delta_n} F^*(x)$$

(recall (5.1.1)).

Remark 5.4 (on strictly positive coefficients). For $n = 2$, is "every" coefficient of, for example, the homogeneous polynomial $G := X_1^2 X_2 + X_2^3$ positive? I.e., should one ignore the "missing" monomials X_1^3 and $X_1 X_2^2$? In [Hardy, et al, 1934–91, p. 57], Pólya introduced his theorem(s) by noting that they "assert…that a positive form can be represented in a manner which renders its positive character intuitive." If we interpret Pólya's theorems as allowing G to omit some monomials of degree $\deg H + \deg F$ (or $p + \deg F$), then the conclusion of those theorems would be too weak to represent F in a manner making its positive character obvious. For example, taking $F = X_1^2 X_2 + X_2^3$ in (1.1) above, we would be able to take $H = 1$ and $G = F$; even though G has no negative coefficients, G is not positive at $(1, 0) \in \Delta_2$. (And in general, for any $n \geq 1$ and $e \geq 1$, a real form G of degree e in X with no negative coefficient is positive on Δ_n if and only if the coefficients in G of all the monomials $X_1^e, \ldots, X_n^e$ are strictly positive.) Thus Pólya's intention was to assert that for every $\alpha \in \mathbb{N}^n$ with $\|\alpha\| = \deg G$, the coefficient of X^α in G is positive (and not merely nonnegative); and similarly for H in (1.1). (In fact, [Hardy, et al, 1934–91] states this clarification explicitly in Appendix I, and notes its importance in the application of Pólya's theorem to proving Habicht's theorem on Hilbert's 17th problem.)

On the other hand, it is easy to prove (the correct interpretation of) Pólya's theorems from the loose interpretation thereof. Namely, let ϵ be any real number such that $0 < \epsilon < \mu$ (5.3.2), and apply the loose interpretation of (1.1) or (1.2) to $F(X_1, \ldots, X_n) - \epsilon(X_1 + \cdots + X_n)^d$ (whose minimum on Δ_n is $\mu - \epsilon > 0$, by the definition of Δ_n). We get

$$H(X_1, \ldots, X_n)(F(X_1, \ldots, X_n) - \epsilon(X_1 + \cdots + X_n)^d) = G(X),$$

for some G and H with no negative coefficients, whence

$$H(X_1, \ldots, X_n)F(X_1, \ldots, X_n) = \epsilon(X_1 + \cdots + X_n)^d H(X_1, \ldots, X_n) + G(X), \tag{5.4.1}$$

where the right hand side obviously has strictly positive coefficients on all monomials of degree $\deg G$, as required. For (1.2), we are done, since there $H = (X_1 + \cdots + X_n)^p$, which also has all possible coefficients positive. For (1.1), however, we are not quite done, since (5.4.1) does not seem to guarantee that the coefficient in H of *each* monomial of degree $p := \deg H$ is positive; but it does guarantee at least that the coefficient in H of each monomial $X_1^p, \ldots, X_n^p$ is positive, and this is enough to render the positive character of H on Δ_n obvious.

6 Impossibility of extending the weak form of Pólya's theorem to arbitrary (transcendental) signomials

Theorem 6.1 (Falsity of (1.1) for arbitrary signomials)**.** *For $n = 2$ there exists a homogeneous signomial $F \in \mathbb{R}[\mathbb{R}^2]$ such that $F > 0$ on Δ_2, but F is not the quotient of any two (nonzero) posynomials.*

Proof. Write (X, Y) instead of (X_1, X_2). Let

$$F_\epsilon(X, Y) = X^2 - X^{1+\epsilon}Y^{1-\epsilon} + Y^2. \tag{6.1.1}$$

Then for all $\epsilon \in (-1, 1)$, $F_\epsilon > 0$ on Δ_n, by (4.2). We shall show that for all *irrational* (real) ϵ, however, F_ϵ is not the quotient of any two nonzero posynomials; then we will be able to take F in (6.1) to be F_ϵ, for any $\epsilon \in (-1, 1) \setminus \mathbb{Q}$.

So suppose that

$$H(X, Y)(X^2 - X^{1+\epsilon}Y^{1-\epsilon} + Y^2) = G(X, Y), \tag{6.1.2}$$

for some $\epsilon \in \mathbb{R}$ and some posynomials G and H ($\neq 0$); we shall deduce from this that $\epsilon \in \mathbb{Q}$.

We may assume that G and H are homogeneous; otherwise, extract from G and H the homogeneous components of lowest degrees. Let $d = \deg H$ ($\in \mathbb{R}$). Write

$$\begin{aligned} G(X, Y) &= a_1X^{\alpha_1}Y^{2+d-\alpha_1} + a_2X^{\alpha_2}Y^{2+d-\alpha_2} + \cdots + a_nX^{\alpha_n}Y^{2+d-\alpha_n} \text{ and} \\ H(X, Y) &= b_1X^{\beta_1}Y^{d-\beta_1} + b_2X^{\beta_2}Y^{d-\beta_2} + \cdots + b_mX^{\beta_m}Y^{d-\beta_m}, \end{aligned} \tag{6.1.3}$$

where $n, m \in \mathbb{N} \setminus \{0\}$,

$$\begin{aligned} &\alpha_1 < \alpha_2 < \cdots < \alpha_n, \\ &\beta_1 < \beta_2 < \cdots < \beta_m \text{ (all in } \mathbb{R}), \end{aligned}$$

and each $a_i, b_i \in \mathbb{R}_{++}$. From (6.1.1) and (6.1.3) we get

$$\begin{aligned} &H(X, Y)F_\epsilon(X, Y) \\ &= (b_1X^{\beta_1}Y^{d-\beta_1} + b_2X^{\beta_2}Y^{d-\beta_2} + \cdots + b_mX^{\beta_m}Y^{d-\beta_m})(X^2 - X^{1+\epsilon}Y^{1-\epsilon} + Y^2) \\ &= b_1X^{\beta_1+2}Y^{d-\beta_1} + b_2X^{\beta_2+2}Y^{d-\beta_2} + \cdots + b_mX^{\beta_m+2}Y^{d-\beta_m} \\ &- b_1X^{\beta_1+1+\epsilon}Y^{d-\beta_1+1-\epsilon} - b_2X^{\beta_2+1+\epsilon}Y^{d-\beta_2+1-\epsilon} - \cdots - b_mX^{\beta_m+1+\epsilon}Y^{d-\beta_m+1-\epsilon} \\ &+ b_1X^{\beta_1}Y^{d-\beta_1+2} + b_2X^{\beta_2}Y^{d-\beta_2+2} + \cdots + b_mX^{\beta_m}Y^{d-\beta_m+2}. \end{aligned} \tag{6.1.4}$$

We need

Lemma 6.2. *Under the above assumptions, for each $i \in \{1, 2, \ldots, m\}$,*

$$\begin{aligned} &\textit{either } \beta_i + 1 + \epsilon = \beta_j + 2, \textit{ for some } j \in \{1, 2, \ldots, i-1\}, \\ &\textit{and/or } \beta_i + 1 + \epsilon = \beta_k, \qquad \textit{for some } k \in \{i+1, i+2, \ldots, m\}. \end{aligned}$$

Proof of Lemma 6.2. According to (6.1.1) and (6.1.2), we are assuming that HF_ϵ is a posynomial. Therefore, the negative term $-b_i X^{\beta_i+1+\epsilon} Y^{d-\beta_i+1-\epsilon}$ in (6.1.4) must be offset by some positive term(s) with the same exponent. Examining (6.1.4), we see that such a positive term can only be either $b_j X^{\beta_j+2} Y^{d-\beta_j}$ or $b_k X^{\beta_k} Y^{d-\beta_k+2}$, for some j or k as in (6.2). Now compare X-exponents. $\square$

Returning to the proof of (6.1), we have

$$\begin{aligned} \text{either } \beta_j &= \beta_i + (-1+\epsilon) \\ \text{or } \beta_k &= \beta_i + (1+\epsilon), \end{aligned} \tag{6.2.1}$$

by the lemma. Now form a sequence $i_1, i_2, \ldots, i_l, i_{l+1}, \ldots \in \{1, 2, \ldots, m\}$, as follows. Let $i_1 = 1$. For $l \geq 1$, once i_l has been determined, define i_{l+1} to be

that $j \in \{1, 2, \ldots, i_l - 1\}$ s.t. $\beta_j = \beta_{i_l} + (-1+\epsilon)$, if such a j exists; otherwise, that $k \in \{i_l + 1, \ldots, m\}$ s.t. $\beta_k = \beta_{i_l} + (1+\epsilon)$.

By (6.2.1), i_{l+1} is well defined by the above instructions.

Thus

$$\begin{aligned} \text{either } \beta_{i_{l+1}} &= \beta_{i_l} + (-1+\epsilon) \\ \text{or } \beta_{i_{l+1}} &= \beta_{i_l} + (1+\epsilon). \end{aligned}$$

Therefore for all $l \geq 1$,

$$\beta_{i_l} = \beta_{i_1} + m_l(-1+\epsilon) + m'_l(1+\epsilon), \tag{6.2.2}$$

for some $m_l, m'_l \in \mathbb{N}$ such that $m_1 = m'_1 = 0$ and for all $l \geq 1$,

$$\begin{aligned} \text{either } &(m_{l+1} = m_l + 1 \text{ and } m'_{l+1} = m'_l) \\ \text{or } &(m_{l+1} = m_l \text{ and } m'_{l+1} = m'_l + 1). \end{aligned}$$

(Therefore, for all $l \geq 1$,

$$m_l + m'_l = l - 1, \tag{6.2.3}$$

by induction on l.) Since $|\{1, 2, \ldots, m\}| = m$, after $\leq m+1$ steps, we find $i_{l_1} = i_{l_2}$, for some $l_1 < l_2$ $(l_1, l_2 \in \{1, 2, \ldots, m+1\})$. Then $\beta_{i_{l_1}} = \beta_{i_{l_2}}$, whence by (6.2.2),

$$\beta_{i_1} + m_{l_1}(-1+\epsilon) + m'_{l_1}(1+\epsilon) = \beta_{i_1} + m_{l_2}(-1+\epsilon) + m'_{l_2}(1+\epsilon).$$

Therefore

$$((m_{l_1} + m'_{l_1}) - (m_{l_2} + m'_{l_2}))\epsilon = -m_{l_2} + m'_{l_2} - (-m_{l_1} + m'_{l_1}),$$

whence by (6.2.3),

$$\epsilon = \frac{m_{l_1} - m'_{l_1} - m_{l_2} + m'_{l_2}}{(l_1 - 1) - (l_2 - 1)}.$$

Since $l_1 \neq l_2$, we conclude that $\epsilon \in \mathbb{Q}$. $\square$

7 Polya's theorem with "fewsignomials" G and H

Note: This section was still in preparation as of October 3, 2003.

As before, let F be a signomial that is homogenous of degree $d \in \mathbb{R}$ in $X := (X_1, \ldots, X_n)$. Suppose $F > 0$ on Δ_n, and all the exponents occurring in F are rational. Then we concluded in Theorem 5.1 above that $HF = G$, for some nonzero homogeneous posynomials G, H. Now suppose, in addition, that $q \in \mathbb{N}^+$ is a common denominator of the exponents of F. We saw in Remark 5.2 that we may then even take G and H to be in $\mathbb{R}[X_1^{1/q}, \ldots, X_n^{1/q}] \subset \mathbb{R}[\mathbb{Q}^n] \subset R[\mathbb{R}^n]$. As the exponents of F . . . .

Theorem 7.1. *. . . .*

. . . .

[Note: The above section was still in preparation as of October 3, 2003.]

8 The effectiveness of Pólya's original method

The bounds for the Pólya exponent p_F of F in (5.3) above are expressed in terms of the minimum μ of F on Δ_n (5.3.2). The question therefore arises: *what do we know about μ?* Over the years, I have heard a few lectures where it was asserted that we know "nothing" about μ, or at least that the problem of the determination of μ is so mysterious as to render Pólya's original method non-constructive, and therewith also Habicht's method [1940] for solving Hilbert's 17th problem in the case of a strictly definite homogeneous polynomial (Habicht relied on Pólya's theorem). On the other hand, in [1982] I had asserted, without proof, that Habicht's (and, implicitly, Pólya's) method is constructive. We shall now show that μ can, in principle, be computed from n, d, and c, in a strong sense made precise below.

First, fix n and d in $\mathbb{N}$, and let $f := f_{n,d} \in \mathbb{Z}[C; X]$ denote the general homogeneous polynomial of degree d in X with indeterminate coefficients $C := (C_1, \ldots, C_N)$, where $N = \binom{d+n-1}{n-1}$:

$$f(C; X) = \sum_{\substack{\alpha \in \mathbb{N}^n \\ \|\alpha\| = d}} C_{\nu(\alpha)} X^\alpha,$$

where ν is any fixed bijection of $\{\, \alpha \in \mathbb{N}^n \mid \|\alpha\| = d \,\}$ onto $\{1, 2, \ldots, N\}$. As in (5.3.2), for $c \in \mathbb{R}^N$, let

$$\mu(c) := \min_{x \in \Delta_n} f(c; x). \tag{8.0.1}$$

Actually, it is enough for most purposes to consider c on the unit sphere

$$S^{N-1} := \{\, c \in \mathbb{R}^N \mid c_1^2 + \cdots + c_N^2 = 1 \,\}.$$

The fact that $\mu(c)$ exists and is unique for all $c \in \mathbb{R}^N$ follows from the compactness of Δ_n, as does the fact that μ is continuous on $\mathbb{R}^N$ (and hence uniformly continuous on S^{N-1}). We shall return to these observations below.

As usual, a function $A \to B$ (where A and B are subsets of Euclidean spaces) is called *semialgebraic* if its graph (in the product space) is a semialgebraic set.

Proposition 8.1. *The function μ is semialgebraic on R^N.*

Proof. The definition (8.0.1) can be expressed by Φ, which is defined by

$$\begin{aligned}\Phi(c;\mu) \;:\leftrightarrow\; &\Phi_{n,d}(c_1,\ldots,c_N;\mu) \;:\leftrightarrow \\ &[(\forall x \in \Delta_n\,(f(c;x) \geq \mu)) \;\wedge\; (\forall \lambda > \mu,\, \exists x \in \Delta_n\,(f(c;x) < \lambda))];\end{aligned} \tag{8.1.1}$$

this is easily converted into a formula in the language of ordered rings (and below, we shall allow such informalities in other formulae). By [Tarski, 1930–67], we can construct (primitive recursively, from n and d alone) (1) a quantifier-free formula Ψ defined by

$$\Psi(c_1,\ldots,c_N;\mu) \;:\leftrightarrow\; \bigvee_i \Psi_i(c;\mu), \tag{8.1.2}$$

where each Ψ_i is defined by

$$\begin{aligned}&\Psi_i(c_1,\ldots,c_N;\mu) \;:\leftrightarrow \\ &\qquad\Big(p_i(c_1,\ldots,c_N;\mu) = 0 \;\wedge\; \bigwedge_j (q_{i,j}(c_1,\ldots,c_N;\mu) > 0)\Big)\end{aligned} \tag{8.1.3}$$

(for some p_i and $q_{i,j} \in \mathbb{Z}[C_1,\ldots,C_N;M]$, where M is also an indeterminate), and (2) a formal proof, from the axioms of real closed (ordered) fields, of

$$\forall c_1,\ldots,c_N,\mu\,(\Phi(c_1,\ldots,c_N;\mu) \leftrightarrow \Psi(c_1,\ldots,c_N;\mu)). \tag{8.1.4}$$

Thus (8.1.2)–(8.1.3) constitute a semialgebraic description of μ. ∎

The statement that μ is well defined on $\mathbb{R}^N$ can be expressed (semi)formally as:

$$\forall c\, \exists!\, \mu\, \Phi(c;\mu). \tag{8.1.5}$$

And the statement that μ is uniformly continuous on S^{N-1} can be expressed as:

$$\forall \epsilon > 0\; \exists \delta > 0\; \Omega(\epsilon,\delta), \tag{8.1.6}$$

where Ω is the formula defined by

$$\begin{aligned}&\Omega(\epsilon,\delta) \;:\leftrightarrow\; \forall c_1,\ldots,c_N\; \forall c'_1,\ldots,c'_N\; \forall \mu,\mu' \\ &\quad\Big[\Big(c_1^2+\cdots+c_N^2 = 1 \;\wedge\; {c'_1}^2+\cdots+{c'_N}^2 = 1 \;\wedge\; \bigwedge_i (|c'_i - c_i| < \delta)\Big) \\ &\qquad\qquad \to\; \big((\Phi(c;\mu) \wedge \Phi(c';\mu')) \;\to\; |\mu' - \mu| < \epsilon\big)\Big].\end{aligned} \tag{8.1.7}$$

(8.1.5) and (8.1.6) may be proved via easy "transcendental" methods (based on the Dedekind-completeness of—the order on—$\mathbb{R}$). For a second kind of proof, since the formulae (8.1.5) and (8.1.6) have no free variables, if we apply Tarski's theorem to them, we will get equivalent formulae (respectively) in the language of ordered rings that not only have no quantified variables, but also have no free variables; such formulae are just Boolean combinations of equations and inequalities involving terms built up from 0 and 1 by the ring operations $+$, $-$, and $\cdot$, and hence are trivially decidable. Since (8.1.5) and (8.1.6) hold over a model of the axioms for real closed fields (viz., $\mathbb{R}$), we can predict that the decision procedure will produce, primitive recursively, formal, "elementary" (messy) proofs of (8.1.5) and (8.1.6) (as opposed to proofs of their negations), purely from the axioms of real closed (ordered) fields (without using the completeness axiom). One could seek a third kind of proof (which, like the previous kind, does not use the completeness axiom) by a direct algebraic approach that exploits the particular properties of (8.1.1), (8.1.5), and (8.1.6), rather than relying on the general, all-purpose method provided by Tarski; this third approach would comply with the philosophical requirements of the program described in [Brumfiel, 1979, Introduction].

Anyway, the fact that μ is uniformly continuous and semialgebraic makes it theoretically computable in the following sense. First, for each i in (8.1.2), define the formula Θ_i by

$$\Theta_i(c_1, \ldots, c_N) \;:\leftrightarrow\; \exists\mu\, \Psi_i(c_1, \ldots, c_N, \mu). \tag{8.1.8}$$

Again by Tarski's theorem, we can construct, primitive recursively, a quantifier-free formula Υ_i defined by

$$\Upsilon_i(c_1, \ldots, c_N) \;:\leftrightarrow\; \bigvee_k \Big(s_{i,k}(c) = 0 \;\wedge\; \bigwedge_l (t_{i,k,l}(c) > 0) \Big) \tag{8.1.9}$$

(for some $s_{i,k}$ and $t_{i,k,l} \in \mathbb{Z}[C_1, \ldots, C_N]$), and a formal proof of

$$\forall c_1, \ldots, c_N\; (\Theta_i(c_1, \ldots, c_N) \leftrightarrow \Upsilon_i(c_1, \ldots, c_N)). \tag{8.1.10}$$

Write $S_i = \{c \in S^{N-1} \mid \Upsilon_i(c)\}$. Then $S^{N-1} = \bigcup_i S_i$, by (8.1.2)–(8.1.5) and (8.1.8)–(8.1.10).

If the coefficients $c \in S^{N-1}$ happen to be rational (or even real algebraic), then we may compute $s_{i,k}(c)$ and $t_{i,k,l}(c)$ exactly, and determine with certainty those i for which $c \in S_i$. For any such i, there must be exactly one $\mu \in \mathbb{R}$ such that $\Psi_i(c, \mu)$, by (8.1.2)–(8.1.5) and (8.1.8)–(8.1.10). We could take $\Psi_i(c, \mu)$ as a reasonable description (obtained primitive recursively from n, d, and c) of (the real algebraic number) μ. Alternatively, we could describe μ by means of primitive recursive functions $\rho_1, \rho_2 : \mathbb{N} \to \mathbb{Q}$ and $\rho_3 : \mathbb{N} \to \mathbb{N}$ (constructed from n, d, and c using Tarski's theorem, combined with an interval-bisection search for a real root of p_i) such that for all $l \in \mathbb{N}$, $\rho_3(l)$ is the code for a formal proof (from the axioms of real closed ordered fields) of the statement Ξ_l defined by

$$\Xi_l \;:\leftrightarrow\; \exists\mu \Big(\Psi_i(c; \mu) \;\wedge\; \rho_1(l) < \mu < \rho_2(l) \;\wedge\; \rho_2(l) - \rho_1(l) \leq \frac{1}{2^l} \Big).$$

If, on the other hand, some of the $c_1, \ldots, c_N$ are irrational, then we proceed as follows. Define a function $g : \mathbb{R}_{++} \to (0,1]$ by

$$g(\epsilon) = \sup\{\, \delta \in (0,1] \mid \Omega(\epsilon,\delta) \,\}.$$

By arguments similar to those above, g is well defined and semialgebraic, and $\Omega(\epsilon, g(\epsilon))$ holds for all $\epsilon > 0$. Moreover, we may construct a primitive recursive function $\rho_4 : \mathbb{N}\setminus\{0\} \to \mathbb{N}\setminus\{0\}$ such that for all $l \in \mathbb{N}\setminus\{0\}$,

$$0 < \frac{1}{\rho_4(l)} < g\Big(\frac{1}{l}\Big), \quad \text{whence} \quad \Omega\Big(\frac{1}{l}, \frac{1}{\rho_4(l)}\Big).$$

Now, if we want to approximate $\mu(c)$ to accuracy $1/l$ (for some $l \in \mathbb{N}\setminus\{0\}$), then for each i for which c_i is irrational (or at least non-real-algebraic), we find a rational number c_i' such that $|c_i' - c_i| < 1/(2\rho_4(l))$. (Presumably the irrational real number c_i is presented as some kind of limit of rational numbers, whether by a Cauchy sequence, a Dedekind cut, or an infinite decimal, etc.) Then successive rational approximations c_i' and c_i'' to c_i will differ from each other by

$$< \frac{1}{2\rho_4(l)} + \frac{1}{2\rho_4(l)} = \frac{1}{\rho_4(l)} < g\Big(\frac{1}{l}\Big).$$

Then we compute $\mu(c) := \mu(c_1', \ldots, c_N')$ and $\mu(c'')$ as in the previous paragraph. Even though successive approximations c' and c'' may oscillate, say, between S_{i_1} and S_{i_2} (for two (or more) distinct i_1 and i_2), we know that the successive values of $\mu(c')$ and $\mu(c'')$ converge to $\mu(c)$, since μ is continuous; i.e., $|\mu(c'') - \mu(c')| < 1/l$, by (8.1.6)–(8.1.7), as desired.

The paragraphs above show that μ can, in principle, be effectively computed from F, using Tarski's theorem. We note, also, that approximations to μ can be found very efficiently by modern methods of optimization, such as that of Lasserre [2001] (simplified by [Schweighofer, submitted]).

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