RECENT PROGRESS ON THE WHITE NOISE APPROACH TO THE LÉVY LAPLACIAN

HUI-HSIUNG KUO

Department of Mathematics
Louisiana State University
Baton Rouge, LA 70808, USA
E-mail: kuo@math.lsu.edu

Let ϕ be a function defined on $L^2(0,1)$. In 1922 P. Lévy introduced the Laplacian $\Delta_L \phi$ of ϕ for harmonic analysis on the space $L^2(0,1)$ by

$$\Delta\phi(x) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle \phi''(x)e_n, e_n\rangle,$$

where $\{e_n; n \geq 1\}$ is an orthonormal basis for $L^2(0,1)$. This Laplacian, called the Lévy Laplacian, has no finite dimensional analogue. It is intrinsically infinite dimensional and possesses peculiar properties. In 1975 T. Hida introduced white noise theory which can be used to study Lévy's functional analysis. In this paper we study recent progress on some properties of the Lévy Laplacian from the white noise viewpoint, e.g., its relationships to the Lévy group, the spherical mean, Wiener processes, Gross Laplacian, and the Fourier transform. More significantly, we discuss K. Saitô's novel idea on the diagonalization of the Lévy Laplacian operator to construct a domain of this operator and to find semigroups and stochastic processes generated by the Lévy Laplacian. We present some of recent results due to Saitô and his collaborators.

1. Introduction

Let H be an infinite dimensional real separable Hilbert space. Suppose a real-valued function f defined on H is twice Fréchet differentiable such that $f''(x)$ is a trace class operator of H. Gross[30] introduced in 1967 the Laplacian $\Delta_G f$ of f defined by

$$\Delta_G f(x) = \mathrm{trace}_H\, f''(x).$$

Thus for any orthonormal basis $\{e_n; n \geq 1\}$ for H, we have

$$\Delta_G f(x) = \sum_{n=1}^{\infty} \langle f''(x)e_n, e_n\rangle. \tag{1}$$

Obviously, this Laplacian operator Δ_G is an infinite dimensional analogue of the finite dimensional Laplacian operator. However, the topology of H is too strong for this Laplacian operator. For example, consider the function $f(x) = |x|^2$, where $|\cdot|$ is the norm on H . The second Fréchet derivative of f is given by $f''(x) = 2I$ with I being the identity operator of H. Hence $\Delta_G f$ is not defined. This was one of the motivations for Gross[29] to introduce the concept of abstract Wiener space in 1965. For example, let K be a Hilbert-Schmidt operator of H and consider the function

$$g(x) = |Kx|^2, \quad x \in H. \tag{2}$$

Then $g''(x) = 2K^*K$ and so $\Delta_G g$ is given by

$$\Delta_G g(x) = 2\,\mathrm{trace}_H\, K^*K.$$

Note that $\Delta_G g(x)$ being defined is reflected by the fact that $|Kx|$ is a measurable semi-norm, see Gross[29] and Kuo[58]. Let $\|\cdot\|$ be a measurable norm on H and let B be the completion of H with respect to $\|\cdot\|$. The pair (H, B) is called an *abstract Wiener space*. Let μ be the standard Gaussian measure on B and define

$$p_t(x, A) = \mu\Big(\frac{A - x}{\sqrt{t}}\Big). \tag{3}$$

Then for any bounded Lipschitz function f on B, the function $u(t, x) = \int_B f(y)\, p_t(x, dy)$ satisfies the heat equation:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta_G u, \quad u(0, x) = f(x). \tag{4}$$

Thus $p_t(x, \cdot)$ is a fundamental solution of the heat equation on the abstract Wiener space (H, B). Moreover, the measures $\{p_t(x, \cdot);\, t > 0, x \in B\}$ are the transition probabilities of a Wiener process on B.

On the other hand, let H be the real Hilbert space $L^2(0, 1)$. Take $h \in L^1(0, 1)$. The function

$$\phi(x) = \int_0^1 h(t)x(t)^2\, dt, \quad x \in L^2(0, 1), \tag{5}$$

can be regarded as the infinite dimensional analogue of the function $f(x_1, \dots, x_n) = \sum_{k=1}^n a_k x_k^2$ on $\mathbb{R}^n$. Since $\Delta f(x_1, \dots, x_n) = 2\sum_{k=1}^n a_k$, the corresponding infinite dimensional Laplacian of φ is given by

$$\Delta_L \phi(x) = 2\int_0^1 h(t)\, dt. \tag{6}$$

Obviously, this Laplacian is different from the Gross Laplacian. In fact, we have the second derivative of ϕ given by

$$\langle \phi''(x)\xi, \eta\rangle = \int_0^1 h(t)\xi(t)\eta(t)\,dt, \quad \xi, \eta \in L^2(0,1).$$

If an orthonormal basis $\{e_n; n \geq 1\}$ for $L^2(0,1)$ is uniformly bounded and equally dense (see the next section for definition,) then it is a fact that

$$\Delta_L \phi(x) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle \phi''(x)e_n, e_n\rangle. \tag{7}$$

The Laplacian as defined in Eq. (7) was introduced by Lévy[72] in 1922. Observe that if the Gross Laplacian $\Delta_G\phi(x)$ exists, then the Lévy Laplacian $\Delta_L\phi(x)$ vanishes. Moreover, let $h(\cdot) = 1$, then the function $\phi(x)$ in Eq. (5) is given by $\phi(x) = |x|^2$. In this case, the Gross Laplacian $\Delta_G\phi(x)$ does not exist, while we have the Lévy Laplacian $\Delta_L\phi(x) = 2$.

Recall that the expression $\Delta_L\phi(x)$ in Eq. (6) is motivated by the finite dimensional function $f(x_1, \ldots, x_n) = \sum_{k=1}^n a_k x_k^2$ on $\mathbb{R}^n$, which has the Laplacian $\Delta f(x_1, \ldots, x_n) = 2\sum_{k=1}^n a_k$. However, the Lévy Laplacian as defined by Eq. (7) has no finite dimensional analogue. In this sense, the Lévy Laplacian is an intrinsically infinite dimensional object.

In 1975 Hida[31] introduced the white noise theory. One of the main motivations for this theory was to interpret Lévy's functional analysis by using the white noise language. Within this theory, we can clearly see that the Lévy Laplacian vanishes on ordinary white noise functions and is a rather interesting operator acting on generalized white noise functions.

Another approach to study the Lévy Laplacian has been carried out by M. N. Feller[12−28] in a series of papers. In particular, he has studied the harmonic analysis associated with the Lévy Laplacian. See also the book by Polishchuk[88].

The Lévy Laplacian has been discovered to be closely related to many mathematical objects, e.g., the infinite dimensional rotation group in the work of Hida[41,42] and Obata[78,82], and Yang-Mills equations in the papers by Accardi[1] and his collaborators[2−8] and by Léandre and Volovich[71]. The differential equations involving the Lévy Laplacian have been studied by M. N. Feller as mentioned above, by Chung-Ji-Saitô[10], and by Lávička[70]. The semigroups generated by the Lévy Laplacian and associated stochastic processes have been constructed and studied by Saitô[91,93,94,98−104] and his collaborators[11,69,68,76,77,105−107].

In this paper we will briefly explain the white noise approach to the Lévy Laplacian in Section 2. Some important results will be reviewed in Section 3. In Section 4 we will make some observations which lead to the novel discovery of Saitô on the diagonalization and domains of the Lévy Laplacian in Section 5. Finally we will discuss the semigroups and stochastic processes associated with the Lévy Laplacian in Section 6.

2. White Noise Approach

Let $\mathcal{S}$ denote the Schwartz space of real-valued rapidly decreasing functions on $\mathbb{R}$. The space $\mathcal{S}$ is a nuclear space with the family $\{|\cdot|_p;\, p \geq 0\}$ of norms defined by

$$|f|_p = |A^p f|_0, \quad f \in \mathcal{S}, \tag{8}$$

where $A = -d^2/dx^2 + x^2 + 1$ and $|\cdot|_0$ is the $L^2(\mathbb{R})$-norm. Thus we have a Gel'fand triple

$$\mathcal{S} \subset L^2(\mathbb{R}) \subset \mathcal{S}', \tag{9}$$

where $\mathcal{S}'$ is the dual space of $\mathcal{S}$. Let μ be the standard Gaussian measure on $\mathcal{S}'$. The probability space $(\mathcal{S}', \mu)$ is called a *white noise space*.

Let (L^2) denote the complex Hilbert space $L^2(\mathcal{S}', \mu)$. By the Wiener-Itô theorem, each $\phi \in (L^2)$ has a unique decomposition

$$\phi(x) = \sum_{n=0}^{\infty} I_n(f_n)(x) = \sum_{n=0}^{\infty} \langle :x^{\otimes n}:, f_n \rangle, \quad f_n \in \widehat{L}^2(\mathbb{R}^n), \tag{10}$$

where I_n is the multiple Wiener integral of order n and $:x^{\otimes n}:$ is the n-th order Wick tensor of $x \in \mathcal{S}'$ (see page 33 in the book by Kuo[64].) Moreover, the (L^2)-norm of ϕ is given by

$$\|\phi\|_0 = \left(\sum_{n=0}^{\infty} n!\, |f_n|^2_{L^2(\mathbb{R}^n)} \right)^{1/2}. \tag{11}$$

Note that the Hilbert space (L^2) is unitarily equivalent to the Fock space $\Gamma\big(L^2(\mathbb{R})\big)$ of $L^2(\mathbb{R})$ by the correspondence

$$(f_0, f_1, \ldots, f_n, \ldots) \longleftrightarrow \sum_{n=0}^{\infty} I_n(f_n).$$

For each $p \geq 0$ and ϕ as given in Eq. (10), define $\|\phi\|_p$ by

$$\|\phi\|_p = \left(\sum_{n=0}^{\infty} n!\, |(A^p)^{\otimes n} f_n|^2_{L^2(\mathbb{R}^n)} \right)^{1/2},$$

Note that $\|\phi\|_p = \|\Gamma(A^p)\phi\|_0$ with $\Gamma(A^p)$ being the second quantization of the operator A^p. Let $(\mathcal{S})_p = \{\phi;\, \|\phi\|_p < \infty\}$. Then $(\mathcal{S})_p$ is a complex Hilbert space with norm $\|\cdot\|_p$.

Let $(\mathcal{S}) = \cap_{p\geq 0}\,(\mathcal{S})_p$ with topology given by the family $\{\|\cdot\|_p;\, p \geq 0\}$ of norms. It can be checked that $(\mathcal{S})$ is a nuclear space and $(\mathcal{S}) \subset (L^2)$. Let $(\mathcal{S})^*$ be the dual space of $(\mathcal{S})$. Then we have a Gel'fand triple

$$(\mathcal{S}) \subset (L^2) \subset (\mathcal{S})^*. \tag{12}$$

Here the identification of (L^2) with its dual space is given by the conjugate, i.e., the bilinear pairing $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ of $(\mathcal{S})^*$ and $(\mathcal{S})$ and the inner product $(\cdot, \cdot)$ of (L^2) are related by the equality

$$\langle\!\langle \phi, \psi \rangle\!\rangle = (\phi, \bar{\psi}), \quad \phi \in (L^2),\ \psi \in (\mathcal{S}).$$

It turns out that $(\mathcal{S})^* = \cup_{p\geq 0}\,(\mathcal{S})_{-p}$. Here $(\mathcal{S})_{-p}$ being the completion of (L^2) with respect to the norm

$$\|\phi\|_{-p} = \left(\sum_{n=0}^{\infty} n!\, |(A^{-p})^{\otimes n} f_n|^2_{L^2(\mathbb{R}^n)} \right)^{1/2}. \tag{13}$$

The nuclear space $(\mathcal{S})$ serves as the space of test functions on the white noise space $\mathcal{S}'$. Its dual space $(\mathcal{S})^*$ is the corresponding space of generalized functions on $\mathcal{S}'$.

Let $\mathcal{S}_c$ denote the complexification of $\mathcal{S}$. For each $\xi \in \mathcal{S}_c$, we define the renormalization of $e^{\langle \cdot, \xi \rangle}$ to be the function $:e^{\langle \cdot, \xi \rangle}: = e^{\langle \cdot, \xi \rangle - \frac{1}{2}\langle \xi, \xi \rangle}$. The function $:e^{\langle \cdot, \xi \rangle}:$ is a test function and has the Wiener-Itô decomposition

$$:e^{\langle x, \xi \rangle}: = \sum_{n=0}^{\infty} \frac{1}{n!} \langle :x^{\otimes n}:, \xi^{\otimes n} \rangle, \quad x \in \mathcal{S}'. \tag{14}$$

A generalized function Φ in $(\mathcal{S})^*$ can be represented by

$$\Phi = \sum_{n=0}^{\infty} \langle :\cdot^{\otimes n}:, F_n \rangle, \quad F_n \in \mathcal{S}'(\mathbb{R}^n), \tag{15}$$

and there exists a positive number p depending on Φ such that $\|\Phi\|_{-p}$ as defined in Eq. (13) is a finite number.

An important tool in white noise theory is the S-transform introduced by Kubo and Takenaka[53−56]. The S-*transform* of a generalized function Φ in $(\mathcal{S})^*$ is defined to be the following function on $\mathcal{S}_c$,

$$S\Phi(\xi) = \langle\!\langle \Phi, :e^{\langle \cdot, \xi \rangle}: \rangle\!\rangle = e^{-\langle \xi, \xi \rangle / 2} \langle\!\langle \Phi, e^{\langle \cdot, \xi \rangle} \rangle\!\rangle, \quad \xi \in \mathcal{S}_c.$$

If Φ is given by Eq. (15), then in view of Eq. (14) we have

$$S\Phi(\xi) = \sum_{n=0}^{\infty} n! \Big\langle F_n,\, \frac{1}{n!}\xi^{\otimes n} \Big\rangle = \sum_{n=0}^{\infty} \langle F_n,\, \xi^{\otimes n}\rangle, \quad \xi \in \mathcal{S}_c. \tag{16}$$

The S-transform uniquely specifies a generalized function, i.e., $S\Phi = S\Psi$ implies that $\Phi = \Psi$. The next fundamental theorem due to Potthoff and Streit[89] gives a characterization of generalized functions in the space $(\mathcal{S})^*$ in terms of their S-transforms. This theorem has been extended to other spaces of generalized functions. See the survey paper by Kuo[66] and references therein.

Theorem 2.1. *A complex-valued function F on $\mathcal{S}_c$ is the S-transform of a generalized function in the space $(\mathcal{S})^*$ if and only if it satisfies the following two conditions:*

(a) For any $\xi, \eta \in \mathcal{S}_c$, the function $F(\xi + z\eta)$ is an entire function of $z \in \mathbb{C}$.
(b) There exist constants $K, a, p \geq 0$ such that

$$|F(\xi)| \leq K \exp\Big[a|\xi|_p^2\Big], \quad \forall\, \xi \in \mathcal{S}_c.$$

The stochastic process $B(t,x) = \langle x,\, 1_{[0,t)}\rangle$ for $x \in \mathcal{S}'$ and $t \geq 0$ and $= -\langle x,\, 1_{[t,0)}\rangle$ for $x \in \mathcal{S}'$ and $t < 0$ is a Brownian motion. Thus an element x in the white noise space $(\mathcal{S}', \mu)$ can be regarded as $\dot{B}$. Moreover, for each $t \in \mathbb{R}$, $\dot{B}(t)$ is a generalized function in the space $(\mathcal{S})^*$ and its S-transform is given by

$$S(\dot{B}(t))(\xi) = \xi(t), \quad \xi \in \mathcal{S}_c.$$

In white noise theory the family $\{\dot{B}(t);\, t \in \mathbb{R}\}$ is taken as a continuum system of coordinates. The renormalization $:\dot{B}(t)^2:$ of $\dot{B}(t)^2$, defined as the limit of $\big(\epsilon^{-1}(B(t+\epsilon) - B(t))\big)^2 - \epsilon^{-1}$ in the space $(\mathcal{S})^*$ as $\epsilon \to 0$, is a generalized function and has the S-transform $S(:\dot{B}(t)^2:)(\xi) = \xi(t)^2$, $\xi \in \mathcal{S}_c$. For any $h \in L^1(\mathbb{R})$, the function

$$\Phi = \int_{\mathbb{R}} h(t) :\dot{B}(t)^2:\, dt$$

is a generalized function with the S-transform

$$S(\Phi)(\xi) = \int_{\mathbb{R}} h(t)\xi(t)^2\, dt,$$

which is the white noise setup of the function in Eq. (5). On the other hand, the white noise setup of the function in Eq. (2) is given by

$$g(\xi) = \int_{\mathbb{R}^2} k(s,t)\xi(s)\xi(t)\,dsdt, \quad \xi \in \mathcal{S}_c,$$

where $k \in L^2(\mathbb{R}^2)$. This function g is the S-transform of the function

$$\phi = \int_{\mathbb{R}^2} k(s,t)\,dB(s)dB(t) = \int_{\mathbb{R}^2} k(s,t) :\dot{B}(s)\dot{B}(t):\, dsdt,$$

which belongs to the space (L^2) of ordinary Brownian functions.

Now, let $F = S\Phi$ be the S-transform of a generalized function $\Phi \in (\mathcal{S})^*$. By condition (b) in Theorem 2.1 F has functional derivatives of all order. Suppose the second functional derivative is given by

$$\langle F''(\xi)\eta, \zeta\rangle = \int_{\mathbb{R}} F_s''(\xi;t)\eta(t)\zeta(t)\,dt + \int_{\mathbb{R}^2} F_r''(\xi;s,t)\eta(s)\zeta(t)\,dsdt,$$

where $F_s''(\xi;\cdot) \in L^1(\mathbb{R})$ and $F_r''(\xi;\cdot,\cdot) \in L^1(\mathbb{R}^2)$. The functions F_s'' and F_r'' are the *singular* and *regular parts* of F'', respectively.

Let T be a finite interval on $\mathbb{R}$. If the function $\int_T F_s''(\xi;t)\,dt$, $\xi \in \mathcal{S}_c$, satisfies the conditions in Theorem 2.1, then we define the Lévy Laplacians of F and Φ by

$$\Delta_L F(\xi) = \frac{1}{|T|}\int_T F_s''(\xi;t)\,dt, \quad \xi \in \mathcal{S}_c, \tag{17}$$

$$\Delta_L \Phi = S^{-1}\Big(\frac{1}{|T|}\int_T F_s''(\cdot;t)\,dt\Big).$$

Furthermore, if the regular part $F_r''(\xi;s,t)$ is the kernel function of a trace class operator $F_r''(\xi)$ of $L^2(\mathbb{R})$ such that the function $\xi \mapsto \mathrm{trace}_{L^2(\mathbb{R})}\,F_r''(\xi)$ satisfies the conditions in Theorem 2.1, then we define the Volterra Laplacians of F and Φ by

$$\Delta_V F(\xi) = \mathrm{trace}_{L^2(\mathbb{R})}\,F_r''(\xi) \quad \xi \in \mathcal{S}_c,$$

$$\Delta_V \Phi = S^{-1}\Big(\mathrm{trace}_{L^2(\mathbb{R})}\,F_r''(\cdot)\Big).$$

Note that the Volterra Laplacian is an extension of the Gross Laplacian to generalized functions. In this paper we will not discuss the Volterra Laplacian and simply refer to the book by Kuo[64] for more information.

The definition of the Lévy Laplacian in Eq. (17) is related to the one given in Eq. (7) as follows. Suppose an orthonormal basis $\{e_n; n \geq 1\}$ for

$L^2(T)$ is equally dense, i.e.,

$$\frac{1}{N}\sum_{n=1}^{N} e_n(t)^2 \longrightarrow \frac{1}{|T|} \quad \text{in } L^2(T),$$

and uniformly bounded, i.e., $\sup_{n\geq 1}\sup_{t\in T} |e_n(t)| < \infty$. Then for any function F satisfying the above conditions we have

$$\Delta_L F(\xi) = \frac{1}{|T|}\int_T F_s''(\xi;t)\,dt = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\langle F''(x)e_n, e_n\rangle, \quad \xi \in \mathcal{S}_c.$$

For the proof of this fact, see page 212 in the book by Kuo[64].

3. Properties of the Lévy Laplacian

In this section we will explain some important properties concerning the Lévy Laplacian. In particular, we will state several theorems which show that the Lévy Laplacian is closely related to other objects.

3.1. *The Lévy group*

Let $\mathbb{N}$ denote the set of natural numbers. A permutation σ of $\mathbb{N}$ is called a *Lévy permutation* if it satisfies the condition:

$$\lim_{N\to\infty}\frac{1}{N}\Big|\big\{n \leq N;\ \sigma(n) > N\big\}\Big| = 0,$$

where $|A|$ is the cardinality of A. Note that for any two Lévy permutations σ_1 and σ_2, we have

$$\Big|\big\{n \leq N;\ \sigma_1\sigma_2(n) > N\big\}\Big| \leq \Big|\big\{n \leq N;\ \sigma_1(n) > N\big\}\Big| + \Big|\big\{n \leq N;\ \sigma_2(n) > N\big\}\Big|.$$

Thus the set $\mathcal{P}_L(\mathbb{N})$ of all Lévy permutations forms a group, which we call the *Lévy permutation group*. Examples of non-Lévy permutations are easy to construct. For example, consider the following disjoint partition of $\mathbb{N}$,

$$\mathbb{N} = A_1 \cup B_1 \cup A_2 \cup B_2 \cup \cdots \cup A_n \cup B_n \cup \cdots,$$

where A_n's and B_n's are ordered sets given by

$$\begin{aligned} A_1 &= \{1,2\}, & B_1 &= \{3,4\},\\ A_2 &= \{5,6,7,8\}, & B_2 &= \{9,10,11,12\},\\ A_3 &= \{13,14,\dots,24\}, & B_3 &= \{25,26,\dots,36\},\\ &\cdots\cdots, & &\cdots\cdots. \end{aligned}$$

Here $|A_n| = |B_n|$ for all $n \geq 1$ and $|A_n| = 2\big(|A_1| + |A_2| + \cdots + |A_{n-1}|\big)$ for $n \geq 2$. Thus the cardinality of A_n is given by

$$|A_1| = 2, \quad |A_n| = 4 \cdot 3^{n-2}, \quad n \geq 2.$$

Let σ be the permutation taking A_n to B_n and B_n to A_n for each n while the order is kept. Then obviously we have

$$\limsup_{N\to\infty} \frac{1}{N} \Big| \big\{ n \leq N;\ \sigma(n) > N \big\} \Big| = \frac{1}{2}.$$

Thus σ is not a Lévy permutation.

Recall the Gel'fand triple $\mathcal{S} \subset L^2(\mathbb{R}) \subset \mathcal{S}'$ in Eq. (9). Let $\mathcal{O}\big(\mathcal{S}, L^2(\mathbb{R})\big)$ denote the collection of all unitary operators U of $L^2(\mathbb{R})$ such that the restriction of U to $\mathcal{S}$ is a homeomorphism of $\mathcal{S}$. The set $\mathcal{O}\big(\mathcal{S}, L^2(\mathbb{R})\big)$ is a group under the composition of operators. This group is an infinite dimensional rotation group introduced by Yoshizawa[108]. Many interesting subgroups of $\mathcal{O}\big(\mathcal{S}, L^2(\mathbb{R})\big)$ have been discovered by Hida[31,34–37,41,42]. One of them is the Lévy group.

Let $\{e_n;\ n \geq 1\} \subset \mathcal{S}$ be an orthonormal basis for $L^2(\mathbb{R})$. For each $\sigma \in \mathcal{P}_L(\mathbb{N})$, let g_σ denote the linear operator of $L^2(\mathbb{R})$ defined by

$$x = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n \ \longmapsto\ g_\sigma x = \sum_{n=1}^{\infty} \langle x, e_{\sigma^{-1}(n)} \rangle e_n.$$

Obviously, g_σ is a unitary operator of $L^2(\mathbb{R})$ for each $\sigma \in \mathcal{P}_L(\mathbb{N})$. The *Lévy group* is defined to be

$$\mathcal{G}_L = \Big\{ g_\sigma\,;\ \sigma \in \mathcal{P}_L(\mathbb{N}) \text{ and } \ g_\sigma : \mathcal{S} \to \mathcal{S} \text{ is a homeomorphism} \Big\}.$$

It is a subgroup of the infinite dimensional rotation group $\mathcal{O}\big(\mathcal{S}, L^2(\mathbb{R})\big)$. For each $g \in \mathcal{G}_L$, its restriction to $\mathcal{S}$ can be extended to the complexification $\mathcal{S}_c$ of $\mathcal{S}$. Thus we can define an operator U_g acting on functions F defined on the space $\mathcal{S}_c$ by

$$U_g F(\xi) = F(g^{-1}\xi), \quad \xi \in \mathcal{S}_c.$$

Theorem 3.1. (Hida and Saitô[50]) *The Lévy Laplacian Δ_L commutes with U_g for all g in the Lévy group $\mathcal{G}_L$, i.e., if F is a function so that $\Delta_L F$ exists, then the Lévy Laplacian of $U_g F$ also exists and the equality holds*

$$\Delta_L U_g F = U_g \Delta_L F.$$

For further properties of the Lévy permutations and the Lévy group, see the papers by Obata[78–80,82].

3.2. *Spherical mean and the Lévy Laplacian*

The concept of spherical mean for functionals defined on the space $L^2(0,1)$ was introduced by P. Lévy[74]. It is closely related to the Lévy Laplacian. We will explain this relationship from the papers by Obata[81], Kuo-Obata-Saitô[67], and the book by Kuo[64]. Functionals defined on $L^2(0,1)$ will be replaced by those functions defined on $\mathcal{S}_c$ which are the S-transforms of generalized functions in $(\mathcal{S})^*$.

Let $\{e_n; n \geq 1\} \subset \mathcal{S}$ be an orthonormal basis for $L^2(\mathbb{R})$. For each $n \geq 2$, let $\lambda^{(n-1)}$ denote the uniform probability measure on the unit sphere S^{n-1} of $\mathbb{R}^n$ and define a mapping θ_n from S^{n-1} into the space $\mathcal{S}$ by

$$\theta_n(x_1, x_2, \ldots, x_n) = x_1 e_1 + x_2 e_2 + x_n e_n.$$

Let F be a function defined on $\mathcal{S}_c$. The *spherical mean* of F at $\xi \in \mathcal{S}_c$ with radius ρ is defined by

$$MF(\xi;\rho) = \lim_{n\to\infty} \int_{S^{n-1}} F(\xi + \rho\theta_n(x))\, d\lambda^{(n-1)}(x),$$

provided that the limit exists. The next theorem has appeared in the papers by Obata[78,81] and Kuo-Obata-Saitô [67]. For the proof, see Theorem 12.27 in the book by Kuo[64].

Theorem 3.2. *Suppose F is a function defined on $\mathcal{S}_c$ and satisfies the conditions:*

(a) $F(\xi+\eta) = F(\xi) + \langle F'(\xi), \eta\rangle + \frac{1}{2}\langle F''(\xi)\eta, \eta\rangle + o(|\eta|_0^2)$. *(Here $|\cdot|_0$ is the $L^2(\mathbb{R})$-norm.)*

(b) *F has the spherical mean $MF(\xi;\rho)$ for all small $\rho > 0$.*

Then we have

$$\Delta_L F(\xi) = 2 \lim_{\rho\to 0} \frac{MF(\xi;\rho) - F(\xi)}{\rho^2}$$

in the sense that if one side exists, then the other side also exists and the equality holds.

A function F defined on $\mathcal{S}_c$ is said to satisfy the *mean value property* if for any $\xi \in \mathcal{S}_c$ there exists ρ_0 such that

$$MF(\xi;\rho) = F(\xi) \quad \text{for all } \ 0 < \rho < \rho_0.$$

It is shown in Obata[81] that all S-transforms $F = S\phi$ for $\phi \in (L^2)$ satisfy the mean value property. On the other hand, suppose F satisfies the mean value property. Then by the above theorem we have $\Delta_L F = 0$, namely, the function F is Δ_L-harmonic. In particular, all functions in (L^2) and their S-transforms are Δ_L-harmonic. This is an interesting property of the Lévy Laplacian. For comparison, the Gross Laplacian is a densely defined operator on the space (L^2).

3.3. *Associated stochastic processes*

Let μ be the white noise measure on $\mathcal{S}'$. Define $p_t(x, A) = \mu\big((A - x)/\sqrt{t}\,\big)$ as in Eq. (3) for an abstract Wiener space. Then the probability measures $\{p_t(x, \cdot);\, t > 0, x \in \mathcal{S}'\}$ are the transition probabilities of an $\mathcal{S}'$-valued standard Wiener process $W(t)$.

For an orthonormal basis $\{e_n; n \geq 1\} \subset \mathcal{S}$ for $L^2(\mathbb{R})$, the Wiener process $W(t)$ can be represented by

$$W(t) = \sum_{n=1}^{\infty} \langle W(t), e_n \rangle e_n.$$

For fixed t, it follows from Theorem 3.1 in Itô[52] that this series converges almost surely as an $\mathcal{S}'$-valued random variable. For each n, the stochastic process $\langle W(t), e_n \rangle$ is an one-dimensional Brownian motion.

Let $|\cdot|_{-p}$ be the dual norm of $|\cdot|_p$ in Eq. (8), i.e., $|f|_{-p} = |A^{-p} f|_0$ and let $\mathcal{S}_{-p}$ be the completion of $L^2(\mathbb{R})$ with respect to the norm $|\cdot|_{-p}$. Then $\big(L^2(\mathbb{R}), \mathcal{S}_{-p}\big)$ is an abstract Wiener space for any $p > 1/2$. Hence the Wiener process $W(t)$ is actually supported in $\mathcal{S}_{-p}$ for any $p > 1/2$. By Eq. (4) the Gross Laplacian Δ_G is the infinitesimal generator of the Wiener process $W(t)$.

Note that for each N, the stochastic process $\sum_{n=1}^{N} \langle W(t), e_n \rangle e_n$ is a Brownian motion in the space spanned by $\{e_1, e_2, \ldots, e_N\}$. Consider the weighted Brownian motion

$$\widetilde{W}_N(t) = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \langle W(t), e_n \rangle e_n. \tag{18}$$

The stochastic process $\widetilde{W}_N(t)$ is related to the Lévy Laplacian by the next theorem from Theorem 12.28 in the book by Kuo[64].

Theorem 3.3. *Let F be the S-transform of a generalized function in $(\mathcal{S})^*$. Then*

$$\Delta_L F(\xi) = 2 \lim_{N \to \infty} \lim_{t \to 0} \frac{E\big(F(\xi + \widetilde{W}_N(t)) - F(\xi)\big)}{t}$$

in the sense that if one side exists, then the other side also exists and the equality holds.

In fact, it is easy to check that for any N we have

$$2\lim_{t\to 0}\frac{E\big(F(\xi+\widetilde{W}_N(t)\big)-F(\xi)}{t}=\frac{1}{N}\sum_{n=1}^{N}\langle F''(\xi)e_n,e_n\rangle$$

and so the equality in the above theorem follows from Eq. (7).

The sequence $\{\widetilde{W}_N(t);\, N\geq 1\}$ of weighted Brownian motions has rather peculiar properties as $N\to\infty$. If we regard $\widetilde{W}_N(t), N\geq 1$, as $L^2(\mathbb{R})$-valued stochastic processes, then by the strong law of large numbers we have

$$|\widetilde{W}_N(t)|_0^2=\frac{1}{N}\sum_{n=1}^{N}\langle W(t),e_n\rangle^2\longrightarrow t\quad\text{almost surely},$$

and

$$\begin{aligned}&|\widetilde{W}_{2N}(t)-\widetilde{W}_N(t)|_0^2\\&=\frac{1}{2N}\sum_{n=N+1}^{2N}\langle W(t),e_n\rangle^2+\Big(\frac{1}{\sqrt{2N}}-\frac{1}{\sqrt{N}}\Big)^2\sum_{n=1}^{N}\langle W(t),e_n\rangle^2\\&=\frac{1}{2}\frac{1}{N}\sum_{n=N+1}^{2N}\langle W(t),e_n\rangle^2+\Big(\frac{1}{\sqrt{2}}-1\Big)^2\frac{1}{N}\sum_{n=1}^{N}\langle W(t),e_n\rangle^2\\&\longrightarrow(2-\sqrt{2})t\quad\text{almost surely.}\end{aligned}$$

Therefore, the sequence $\{\widetilde{W}_N(t);\, N\geq 1\}$ does not converge almost surely in $L^2(\mathbb{R})$ as $N\to\infty$.

On the other hand, we can also regard $\widetilde{W}_N(t), N\geq 1$, as $\mathcal{S}_{-p}$-valued stochastic processes. Take e_n to be the Hermite function of order $n-1$, then $Ae_n=2n\,e_n$, $n\geq 1$. Hence for $p>1/2$ we have

$$\begin{aligned}|\widetilde{W}_N(t)|_{-p}^2&=\frac{1}{N}\sum_{n=1}^{N}\frac{1}{(2n)^{2p}}\langle W(t),e_n\rangle^2\\&\longrightarrow 0\quad\text{almost surely.}\end{aligned}$$

Hence $\widetilde{W}_N(t)$ converges to 0 almost surely in $\mathcal{S}_{-p}$ as $N\to\infty$ for any $p>1/2$.

3.4. *Relationship to the Gross Laplacian*

A relationship between the Gross and Lévy Laplacian was discovered by Kuo-Obata-Saitô[67] in 1990. Their idea has been pursued further by Paycha[86,87] and Zhang[109]. In §12.6 of the book by Kuo[64] a more general setup has been discussed, namely, consider the question as to whether it is possible to find operators J_ϵ and positive numbers $c(\epsilon)$ such that

$$\Delta_L F = \lim_{\epsilon \longrightarrow 0} c(\epsilon) \Delta_G (F \circ J_\epsilon) \ ? \tag{19}$$

Note that the question is to express the Lévy Laplacian in terms of a renormalization of the Gross Laplacian. Consider the function

$$F(\xi) = \int_T f(t) \xi(t)^2 \, dt + \int_{T^2} g(s,t) \xi(s) \xi(t) \, ds dt,$$

where $f \in L^1(T)$ and $g \in L^1(T^2)$. If the equality in Eq. (19) holds for this particular function F, then as derived in §12.6 of the book by Kuo[64] the following two equalities must hold

$$\lim_{\epsilon \longrightarrow 0} c(\epsilon) \sum_{n=1}^{\infty} \int_T f(t) |(J_\epsilon e_n)(t)|^2 \, dt = \frac{1}{|T|} \int_T f(t) \, dt \tag{20}$$

$$\lim_{\epsilon \longrightarrow 0} c(\epsilon) \sum_{n=1}^{\infty} \int_{T^2} g(s,t) (J_\epsilon e_n)(s) (J_\epsilon e_n)(t) \, ds dt = 0. \tag{21}$$

It can be easily shown that the equality in Eq. (20) is satisfied if $\{e_n; \, n \geq 1\}$ is uniformly bounded and

$$c(\epsilon) \sum_{n=1}^{\infty} (J_\epsilon e_n)(t)^2 \longrightarrow \frac{1}{|T|} \quad \text{in } L^2(T). \tag{22}$$

In particular, we can integrate t over T to get

$$\lim_{\epsilon \longrightarrow 0} c(\epsilon) \|J_\epsilon\|_{HS}^2 = \lim_{\epsilon \longrightarrow 0} c(\epsilon) \sum_{n=1}^{\infty} |J_\epsilon e_n|_0^2 = 1, \tag{23}$$

where $\| \cdot \|_{HS}$ denotes the Hilbert-Schmidt operator norm. On the other hand, the equality in Eq. (21) is satisfied if

$$\lim_{\epsilon \longrightarrow 0} c(\epsilon) \|J_\epsilon^* J_\epsilon\|_{HS} = 0. \tag{24}$$

We may take $c(\epsilon) = \|J_\epsilon\|_{HS}^{-2}$ so that Eq. (23) is automatically satisfied and Eq. (24) becomes

$$\lim_{\epsilon \longrightarrow 0} \|J_\epsilon\|_{HS}^{-2} \|J_\epsilon^* J_\epsilon\|_{HS} = 0. \tag{25}$$

In fact, the conditions in Eqs. (22) and (25) together with the condition that $J_\epsilon \to I$ strongly on $L^2(\mathbb{R})$ are also sufficient for the equality in Eq. (19) to hold. This is the next theorem from §12.6 of the book by Kuo[64].

Theorem 3.4. *Let $\{J_\epsilon;\, \epsilon > 0\}$ be a family of continuous linear operators from $\mathcal{S}'$ into $\mathcal{S}$ satisfying the following conditions:*

(a) $J_\epsilon \to I$ *strongly on* $L^2(\mathbb{R})$ *as* $\epsilon \to 0$.

(b) $\lim_{\epsilon\to 0} \|J_\epsilon\|_{HS}^{-2} \|J_\epsilon^* J_\epsilon\|_{HS} = 0$.

(c) There exists a uniformly bounded orthonormal basis $\{e_n;\, n \geq 1\}$ for $L^2(T)$ such that

$$\lim_{\epsilon\to 0} \|J_\epsilon\|_{HS}^{-2} \sum_{n=1}^{\infty} (J_\epsilon e_n)(t)^2 = \frac{1}{|T|} \quad \text{in } L^2(T).$$

Then for any S-transform F with $\Delta_L F$ being defined as in Eq. (17) we have the equality

$$\Delta_L F = \lim_{\epsilon\to 0} \|J_\epsilon\|_{HS}^{-2} \Delta_G (F \circ J_\epsilon).$$

3.5. *Fourier transform and commutators*

The Fourier transform $\mathcal{F}\Phi$ of a generalized function Φ in $(\mathcal{S})^*$ is defined to be the generalized function with S-transform given by

$$(S\mathcal{F}\Phi)(\xi) = (S\Phi)(-i\xi) e^{-\frac{1}{2}\langle \xi, \xi\rangle}, \quad \xi \in \mathcal{S}_c. \tag{26}$$

The Fourier transform $\mathcal{F}$ is a continuous linear operator from $(\mathcal{S})^*$ into itself. It is proved in Hida-Kuo-Obata[47] that $\mathcal{F}$ is the unique (up to a constant multiple) continuous linear operator from $(\mathcal{S})^*$ into itself such that for any $\xi \in \mathcal{S}$ the following equalities hold:

$$\mathcal{F}\widetilde{D}_\xi = i\widetilde{Q}_\xi \mathcal{F}, \quad \mathcal{F}\widetilde{Q}_\xi = i\widetilde{D}_\xi \mathcal{F}, \tag{27}$$

where $\widetilde{D}_\xi$ is the continuous extension to $(\mathcal{S})^*$ of the differential operator D_ξ, in the direction of ξ, acting on $(\mathcal{S})$ and $\widetilde{Q}_\xi$ is the continuous extension to $(\mathcal{S})^*$ of the multiplication operator Q_ξ by $\langle \cdot, \xi\rangle$ acting on $(\mathcal{S})$.

The Dirac delta function δ_x at $x \in \mathcal{S}'$ was introduced in white noise theory by Kubo and Yokoi[57], namely,

$$\langle\!\langle \delta_x, \phi \rangle\!\rangle = \phi(x), \quad \phi \in (\mathcal{S}),$$

where the unique continuous version of ϕ is taken. The S-transform of δ_x is given by

$$(S\delta_x)(\xi) = e^{\langle x, \xi\rangle - \frac{1}{2}\langle \xi, \xi\rangle}, \quad \xi \in \mathcal{S}_c. \tag{28}$$

In particular, when $x = 0$, we see from Eqs. (26) and (28) that $\mathcal{F}1 = \delta_0$. Thus $\mathcal{F}$ is the unique linear operator from $(\mathcal{S})^*$ into itself satisfying the equalities in Eq. (27) and $\mathcal{F}1 = \delta_0$.

The next theorem is an interesting relationship between the Fourier transform and the Lévy Laplacian discovered by Hida and Saitô[50].

Theorem 3.5. *The Lévy Laplacian and the Fourier transform are related by the following equality:*

$$\mathcal{F}\big(\Delta_L + \tfrac{1}{2}\big) = -\big(\Delta_L + \tfrac{1}{2}\big)\mathcal{F}.$$

The Lévy Laplacian is related to the adjoint Δ_G^* of the Gross Laplacian and the number operator N by the equalities:

$$\big[\Delta_L, \Delta_G^*\big] = 2I, \quad \big[\Delta_L, N\big] = 2\Delta_L,$$

where $[A, B] = AB - BA$ is the commutator of A and B, and N is the number operator defined by

$$N\Big(\sum_{n=0}^{\infty} \langle :x^{\otimes n}:, f_n\rangle\Big) = \sum_{n=1}^{\infty} n\,\langle :x^{\otimes n}:, f_n\rangle$$

The number operator is continuous as an operator from $(\mathcal{S})$ into itself and from $(\mathcal{S})^*$ into itself. For more commutator identities, see the books by Hida-Kuo-Potthoff-Streit[48] and by Kuo[64].

4. Special elementary functions

The Lévy Laplacian is a very peculiar operator. For example, although it is defined from the second derivative, it has the derivation property, namely, for any F and G with $\Delta_L F$ and $\Delta_L G$ being defined, we have

$$\Delta_L(FG) = (\Delta_L F)G + F(\Delta_L G).$$

In terms of generalized functions, let $F = S\Phi$ and $G = S\Psi$. Then

$$\Delta_L(\Phi \diamond \Psi) = (\Delta_L \Phi) \diamond \Psi + \Phi \diamond (\Delta_L \Psi),$$

where $\diamond$ denotes the Wick product, i.e., $S(\Phi \diamond \Psi) = (S\Phi)(S\Psi)$.

There are several classes of special elementary functions on which the Lévy Laplacian operates in particular ways. The third class below is of great importance since it can be used to define various domains of the Lévy Laplacian with nice properties.

4.1. *Normal functionals*

Lévy considered the *normal functionals*, i.e., functions of the form

$$F(\xi) = \int_{\mathbb{R}^n} f(u_1, \ldots, u_n)\xi(u_1)^{j_1} \cdots \xi(u_n)^{j_n}\, du_1 \cdots du_n,$$

where $f \in L^1(\mathbb{R}^n)$ and $j_1, \ldots, j_n \geq 0$ are integers. This function is the S-transform of the generalized function in $(\mathcal{S})^*$

$$\Phi = \int_{\mathbb{R}^n} f(u_1, \ldots, u_n) :\dot{B}(u_1)^{j_1} \cdots \dot{B}(u_n)^{j_n}:\, du_1 \cdots du_n,$$

where $: \cdot :$ denotes the renormalization. The Lévy Laplacian $\Delta_L F$ exists, but the expression is somewhat complicated to be manageable.

4.2. *Gaussian functionals*

For a complex number $c \neq 1/2$, consider the Gaussian functional Φ_c which is the multiplicative renormalization

$$\Phi_c = \mathcal{N}\exp\left[c \int_{\mathbb{R}} \dot{B}(t)^2\, dt\right]. \tag{29}$$

The S-transform of Φ_c is given by

$$F_c = \exp\left[\frac{c}{1-2c} \int_{\mathbb{R}} \xi(t)^2\, dt\right], \quad \xi \in \mathcal{S}_c. \tag{30}$$

In 1984 we observed in the lecture notes (later published in 1993 as Nagoya University Lecture Notes[62]) a fact about the Lévy Laplacian

$$\Delta_L F_c = \frac{2c}{1-2c} F_c.$$

For any complex number $z \neq -1$, we can choose $c = z/2(1+z)$ so that $\Delta_L F_c = zF_c$. Thus F_c is an eigenfunction of Δ_L with eigenvalue z. On the other hand, as $c \to \infty$ in Eq. (30) we get the function

$$F_\infty = \exp\left[-\frac{1}{2} \int_{\mathbb{R}} \xi(t)^2\, dt\right], \quad \xi \in \mathcal{S}_c.$$

By Theorem 2.1 F_∞ is the S-transform of a generalized function, denoted by Φ_∞, in $(\mathcal{S})^*$. We can check that $\Delta_L F_\infty = -F_\infty$ and so F_∞ is an eigenfunction with eigenvalue -1. Hence every complex number z is an eigenvalue of the Lévy Laplacian. The corresponding eigenfunctions are the Gaussian functionals.

4.3. *Exponential functions*

In recent years Saitô has come up with a novel idea to start from the elementary exponential functions to construct a domain space on which the Lévy Laplacian can operate any number of times. This will produce many interesting objects related to the Lévy Laplacian, e.g., a semigroup whose infinitesimal generator is the L'evy Laplacian.

An *elementary exponential function* is a function of the form

$$F(\xi) = \int_{T^n} f(u_1, \ldots, u_n) e^{ia_1\xi(u_1)+\cdots+ia_n\xi(u_n)}\, du_1 \cdots du_n,$$

where $f \in L^1(T^n)$ and $a_1, \ldots, a_n \in \mathbb{C}$. It is easy to verify that F satisfies the conditions in Theorem 2.1 and so is the S-transform of a generalized function Φ in $(\mathcal{S})^*$ which can be denoted by

$$\Phi = \int_{T^n} f(u_1, \ldots, u_n) :e^{ia_1\dot{B}(u_1)+\cdots+ia_n\dot{B}(u_n)}:\, du_1 \cdots du_n. \tag{31}$$

A direct computation shows that the singular part F_s'' of F'' is given by

$$\begin{aligned} F''(\xi; t) = &-a_1^2 \int_{T^{n-1}} f(t, u_2, \ldots, u_n) e^{ia_1\xi(t)+ia_2\xi(u_2)+\cdots+ia_n\xi(u_n)}\, du_2 \cdots du_n \\ &- \quad \cdots\cdots\cdots\cdots \\ &- a_n^2 \int_{T^{n-1}} f(u_1, \ldots, u_{n-1}, t) e^{ia_1\xi(u_1)+\cdots+ia_{n-1}\xi(u_{n-1})+ia_n\xi(t)}\, du_1 \cdots du_{n-1}. \end{aligned}$$

Integrating the above equation over the t-variable to get

$$\begin{aligned} \Delta_L F(\xi) &= \frac{1}{|T|} \int_T F_s''(\xi; t)\, dt \\ &= \frac{1}{|T|} \big(- a_1^2 F(\xi) - \cdots - a_n^2 F(\xi) \big) \\ &= -\frac{a_1^2 + \cdots + a_n^2}{|T|} F(\xi). \end{aligned}$$

Therefore, the Lévy Laplacians of F and Φ are given by

$$\begin{aligned} \Delta_L F &= -\frac{a_1^2 + \cdots + a_n^2}{|T|} F, \\ \Delta_L \Phi &= -\frac{a_1^2 + \cdots + a_n^2}{|T|} \Phi. \end{aligned} \tag{32}$$

This equality, obtained by Saitô and Tsoi[105], is the starting point of Saitô's idea to construct various domains for the Lévy Laplacian, which we will explain in the next section.

5. Diagonalization and domains of the Lévy Laplacian

In order to study the Lévy Laplacian Δ_L we need to construct a space which can serve as a domain for Δ_L such that the powers Δ_L^k and the exponential $e^{t\Delta_L}$ are all continuous operators on this space. In this section we will explain Saitô's novel idea for constructing various domains for the Lévy Laplacian.

From now on, we fix a *Lévy diagonalization function* h, i.e., a continuous function $h : \mathbb{R} \to \mathbb{C}$ such that there exists a stochastic process $X(t)$ with

$$Ee^{irX(t)} = e^{th(r)}, \quad r \in \mathbb{R}. \tag{33}$$

For example, $h(r) = -|r|^\nu$, $1 \le \nu \le 2$, is such a function. In this case, we have the ν-stable process $X(t)$.

The following table shows a comparison of the construction for one- and infinite-dimensional cases. The Schwartz space $\mathcal{S}$ is constructed from the space $L^2(\mathbb{R})$ and the operator $A = -d^2/dx^2 + x^2 + 1$, while the space $\mathbb{D}_{-p,\infty}$ (see Step 4 below) is constructed from the space $(\mathcal{S})_{-p}$ and the Lévy Laplacian Δ_L.

Table 1. Comparison of one- and infinite-dimensional cases.

	space	operator	eigenvalues	powers	domain
one-dim	$L^2(\mathbb{R})$	A	$2k+2$	A^n, $(2k+2)^n$	$\mathcal{S}$
∞-dim	$(\mathcal{S})_{-p}$	Δ_L	$h(\lambda)$	Δ_L^n, $h(\lambda)^n$	$\mathbb{D}_{-p,\infty}$

Step 1. $\mathbb{E}_\lambda$: *The space of elementary exponential functions*

For each $\lambda \in \mathbb{R}$, let $\mathbb{E}_\lambda$ be the linear space spanned by generalized functions of the form in Eq. (31), i.e,

$$\Phi = \int_{T^n} f(u_1, \ldots, u_n) :e^{ia_1\dot{B}(u_1)+\cdots+ia_n\dot{B}(u_n)}: \, du_1 \cdots du_n,$$

where $n \ge 1$, $f \in L^1(T^n)$ and $a_1, \ldots, a_n \in \mathbb{C}$ satisfy the conditions:

(a) $a_1 + \cdots + a_n = \sqrt{|T|}\,\lambda$.

(b) $a_1^2 + \cdots + a_n^2 = -|T|h(\lambda)$.

Here condition (b) is motivated by Eq. (32) so that $h(\lambda)$ is an eigenvalue for these functions, while condition (a) is required for the uniqueness of the representation in Lemma 5.3 below.

Lemma 5.1. *For each $\lambda \in \mathbb{R}$, we have*

(1) $\mathbb{E}_\lambda \subset (\mathcal{S})_{-p}$ *for any* $p > 5/12$.
(2) $\mathbb{E}_\lambda \neq \{0\}$.
(3) $\Delta_L \Phi = h(\lambda)\Phi$ *for all* $\Phi \in \mathbb{E}_\lambda$.

Step 2. $\mathbb{E}_{\lambda,-p}$: *The completion of* $\mathbb{E}_\lambda$ *with respect to* $\|\cdot\|_{-p}$

Let $p > 5/12$ be fixed from now on. In view of the fact (1) in Lemma 5.1, we can complete the space $\mathbb{E}_\lambda$ with respect to the $(\mathcal{S})_{-p}$-norm $\|\cdot\|_{-p}$ defined in Eq. (13). Let $\mathbb{E}_{\lambda,-p}$ denote the completion.

Lemma 5.2. *For each $\lambda \in \mathbb{R}$, we have*

(1) $\mathbb{E}_{\lambda,-p} \subset (\mathcal{S})_{-p}$ *for any* $p > 5/12$.
(2) $\mathbb{E}_{\lambda,-p}$ *is a Hilbert space.*
(3) $\Delta_L \Phi = h(\lambda)\Phi$ *for all* $\Phi \in \mathbb{E}_{\lambda,-p}$.

Step 3. $\mathbb{D}_{-p,N}$: *The space of Pettis integrals of* $\mathbb{E}_{\lambda,-p}$ *over* λ

Suppose a weakly measurable function $\Phi : \mathbb{R} \to (\mathcal{S})^*$ is Pettis integrable (see page 243 in the book[64].) Then the Pettis integral $\int_{\mathbb{R}} \Phi(\lambda)\, d\lambda$ exists and is a generalized function in $(\mathcal{S})^*$. The next lemma is from the papers by Kuo-Obata-Saitô[68,69]. The condition (a) in Step 1 is needed for the proof of this lemma (see the papers[68,69].)

Lemma 5.3. *Let* $\Phi, \Psi : \mathbb{R} \to (\mathcal{S})^*$ *be Pettis integrable. Assume that* $\Phi(\lambda), \Psi(\lambda) \in \mathbb{E}_{\lambda,-p}$ *for almost all* $\lambda \in \mathbb{R}$ *and* $\int_{\mathbb{R}} \Phi(\lambda)\, d\lambda = \int_{\mathbb{R}} \Psi(\lambda)\, d\lambda$. *Then* $\Phi(\lambda) = \Psi(\lambda)$ *for almost all* $\lambda \in \mathbb{R}$.

Let $N \geq 0$ be an integer. Consider a Pettis integral Φ given by

$$\Phi = \int_{\mathbb{R}} \Phi(\lambda)\, d\lambda, \quad \Phi(\lambda) \in \mathbb{E}_{\lambda,-p} \quad \text{a.e. } (\lambda). \tag{34}$$

For such a generalized function Φ, define

$$[\![\Phi]\!]_{-p,N} = \left[\int_{\mathbb{R}} \|\Phi(\lambda)\|_{-p}^2 \left(\sum_{n=0}^{N} |h(\lambda)|^{2n} \right) d\lambda \right]^{1/2}. \tag{35}$$

Obviously, $0 \leq [\![\Phi]\!]_{-p,N} \leq \infty$. Observe that $[\![\Phi]\!]_{-p,N}$ is well-defined by Lemma 5.3. Moreover, see the infinite-dimensional case in Table 1 for the motivation of the definition for $[\![\Phi]\!]_{-p,N}$.

Now, let $\mathbb{D}_{-p,N}$ denote the space of all Pettis integrals $\Phi = \int_{\mathbb{R}} \Phi(\lambda)\, d\lambda$ with $\Phi(\lambda) \in \mathbb{E}_{\lambda,-p}$, λ-a.e., and $[\![\Phi]\!]_{-p,N} < \infty$. Observe that by Lemma 5.3 each element Φ in $\mathbb{D}_{-p,N}$ can be uniquely represented as a Pettis integral as in Eq. (34).

Lemma 5.4. *For any integer $N \geq 0$, we have*

(1) $\mathbb{D}_{-p,N}$ *is a Hilbert space with norm* $[\![\,\cdot\,]\!]_{-p,N}$.
(2) $\mathbb{D}_{-p,N+1} \subset \mathbb{D}_{-p,N}$.
(3) $\Delta_L\big(\int_{\mathbb{R}} \Phi(\lambda)\, d\lambda\big) = \int_{\mathbb{R}} h(\lambda)\Phi(\lambda)\, d\lambda$.
(4) $\Delta_L : \mathbb{D}_{-p,N+1} \longrightarrow \mathbb{D}_{-p,N}$ *is a continuous linear operator.*

Note that the assertion (3) in this lemma gives the diagonalization of the Lévy Laplacian with h being the diagonalization function. The next theorem is from the papers by Kuo-Obata-Saitô[68,69].

Theorem 5.1. *The Lévy Laplacian Δ_L, when restricted to $\mathbb{D}_{-p,N+1}$, is a densely defined self-adjoint operator on $\mathbb{D}_{-p,N}$ for any $p > 5/12$ and any integer $N \geq 0$.*

We point out something related to Eq. (35) and Table 1. For Φ given as in Eq. (34), define

$$[\![\Phi]\!]_{-p} = \left(\int_{\mathbb{R}} \|\Phi(\lambda)\|_{-p}^2 \, d\lambda \right)^{1/2}.$$

Then we can use the fact (3) in Lemma 5.4 to express the norm $[\![\Phi]\!]_{-p,N}$ in Eq. (35) as follows:

$$[\![\Phi]\!]_{-p,N} = \left(\sum_{n=0}^{N} \|\Delta_L^n \Phi\|_{-p}^2 \right)^{1/2}.$$

Step 4. $\mathbb{D}_{-p,\infty}$: *The projective limit of* $\{\mathbb{D}_{-p,N};\, N \geq 0\}$

From Step 3 we have the following chain with continuous inclusions:

$$\cdots\cdots\ \mathbb{D}_{-p,N+1} \subset \mathbb{D}_{-p,N} \subset \cdots \subset \mathbb{D}_{-p,1} \subset \mathbb{D}_{-p,0} \subset (\mathcal{S})_{-p}.$$

Finally, we define $\mathbb{D}_{-p,\infty}$ to be the projective limit of $\{\mathbb{D}_{-p,N};\, N \geq 0\}$. Note that $\mathbb{D}_{-p,\infty} = \cap_{N=0}^{\infty} \mathbb{D}_{-p,N}$ and the topology is given by the family $\{[\![\,\cdot\,]\!]_{-p,N};\, N \geq 0\}$ of norms.

The space $\mathbb{D}_{-p,\infty}$ is a domain of the Lévy Laplacian that we mentioned in the beginning of this section. It should be noticed that $\mathbb{D}_{-p,\infty}$ depends not only on $p > 5/12$, but also on the diagonalization function $h(\lambda)$.

Theorem 5.2. *The Lévy Laplacian Δ_L is a continuous linear operator from $\mathbb{D}_{-p,\infty}$ into itself. (Hence all powers Δ_L^n of Δ_L are also continuous linear operators from $\mathbb{D}_{-p,\infty}$ into itself.)*

We summary the above discussion regarding to various spaces and the Lévy Laplacian Δ_L as follows:

(1) We have the spaces $\mathbb{E}_\lambda$, $\mathbb{E}_{\lambda,-p}$, $\mathbb{D}_{-p,N}$, $\mathbb{D}_{-p,\infty}$.
(2) The Lévy Laplacian Δ_L, when regarded as an operator on $\mathbb{E}_\lambda$ and $\mathbb{E}_{\lambda,-p}$, is a multiplication operator by $h(\lambda)$.
(3) The Lévy Laplacian Δ_L as an operator on $\mathbb{D}_{-p,N}$ and $\mathbb{D}_{-p,\infty}$ is a diagonal operator.
(4) On the space $\mathbb{D}_{-p,\infty}$ we can apply the Lévy Laplacian any number of times. Moreover, we can construct a semigroup on $\mathbb{D}_{-p,\infty}$ for Δ_L, which we will do in the next section.

6. Associated Semigroups and Stochastic Processes

Let (H, B) be an abstract Wiener space and let $p_t(x, \cdot)$ be the Gaussian measure defined in Eq. (3). For a bounded continuous function f on B and for $t > 0$, define

$$P_t f(x) = \int_B f(y)\, p_t(x, dy). \tag{36}$$

Let $P_0 = I$. Then the family $\{P_t;\, t \geq 0\}$ is a semigroup and its infinitesimal generator is the Gross Laplacian Δ_G. On the other hand, let W_t be a standard Wiener process with state space B. Then

$$P_t f(x) = E\big[f(W_t)\,\big|\,W_0 = x\big]. \tag{37}$$

Note that $\big(L^2(\mathbb{R}), (\mathcal{S})_{-p}\big)$ is an abstract Wiener space for any $p > 1/2$. Hence we can easily formulate the above two equations in the white noise space for the Gross Laplacian Δ_G with domain being the space $(\mathcal{S})$ of test functions in Eq. (12).

Question: For the Lévy Laplacian, what are the corresponding objects of the semigroup $\{P_t;\, t \geq 0\}$ in Eq. (36), the stochastic process W_t in Eq. (37), and the domain space $(\mathcal{S})$?

First of all, we take $\mathbb{D}_{-p,\infty}$ to be the analogue of $(\mathcal{S})$ for the Lévy Laplacian. As we have seen in the previous section, the Lévy Laplacian Δ_L is a continuous linear operator from $\mathbb{D}_{-p,\infty}$ into itself. Thus we can repeatedly apply Δ_L on $\mathbb{D}_{-p,\infty}$.

6.1. *Semigroups generated by the Lévy Laplacian*

To construct a semigroup $\{G_t;\, t \geq 0\}$ for the Lévy Laplacian, note that G_t is informally given by $G_t = e^{t\Delta_L}$. Since Δ_L is diagonalized along the $\mathbb{E}_{\lambda,-p}$ spaces with eigenvalues $h(\lambda)$, $\lambda \in \mathbb{R}$, we can define G_t as follows. Each $\Phi \in \mathbb{D}_{-p,\infty}$ can be uniquely represented by

$$\Phi = \int_{\mathbb{R}} \Phi(\lambda)\, d\lambda, \quad \Phi(\lambda) \in \mathbb{E}_{\lambda,-p}.$$

For this Φ, we define

$$G_t\Phi = \int_{\mathbb{R}} e^{th(\lambda)}\Phi(\lambda)\, d\lambda.$$

Observe that by Eq. (33) we have $\Re h(\lambda) \leq 0$. Hence by Eq. (35),

$$\begin{aligned}
[\![G_t\Phi]\!]_{-p,N}^2 &= \int_{\mathbb{R}} \|e^{th(\lambda)}\Phi(\lambda)\|_{-p}^2 \left(\sum_{n=0}^{N} |h(\lambda)|^{2n} \right) d\lambda \\
&= \int_{\mathbb{R}} e^{2t\Re h(\lambda)} \|\Phi(\lambda)\|_{-p}^2 \left(\sum_{n=0}^{N} |h(\lambda)|^{2n} \right) d\lambda \\
&\leq \int_{\mathbb{R}} \|\Phi(\lambda)\|_{-p}^2 \left(\sum_{n=0}^{N} |h(\lambda)|^{2n} \right) d\lambda \\
&= [\![\Phi]\!]_{-p,N}^2.
\end{aligned}$$

Thus the family $\{G_t;\, t \geq 0\}$ of linear operators from $\mathbb{D}_{-p,\infty}$ into itself is equi-continuous. In fact, we have the next theorem from the papers by Kuo-Obata-Saitô[68,69].

Theorem 6.1. *The family $\{G_t;\, t \geq 0\}$ is an equi-continuous semigroup of class (C_0) generated by the Lévy Laplacian Δ_L as a continuous linear operator defined on the space $\mathbb{D}_{-p,\infty}$.*

We should mention that the semigroup $\{G_t;\, t \geq 0\}$ depends on the choice of the diagonalization function $h(\lambda)$ in Eq. (33). For a different choice of the function $h(\lambda)$, we would get a different semigroup. On the other hand, the domain $\mathbb{D}_{-p,\infty}$ of the Lévy Laplacian and the semigroup depends on p. But this dependence is not so essential.

6.2. *Associated stochastic processes*

Unlike the Gross Laplacian, we cannot expect the analogue of Eq. (37) in the following form

$$G_t\Phi(x) = E\big[\Phi(W_t)\,\big|\,W_0 = x\big].$$

This is obviously because $\Phi \in \mathbb{D}_{-p,\infty}$ is a generalized function and so is not defined pointwise. However, we can consider the Lévy Laplacian as an operator acting on S-transforms of generalized functions in $\Phi \in \mathbb{D}_{-p,\infty}$ as defined in Eq. (17). In that case, we will be able to obtain an analogue of Eq. (37) for the Lévy Laplacian.

Let $\widetilde{\mathbb{D}}_{-p,\infty} = S\mathbb{D}_{-p,\infty}$ and endow the topology induced from $\mathbb{D}_{-p,\infty}$ by the S-transform. To avoid confusion, we now use $\widetilde{\Delta}_L$ to denote the Lévy Laplacian acting on the space $\widetilde{\mathbb{D}}_{-p,\infty}$, namely,

$$\widetilde{\Delta}_L = S\Delta_L S^{-1} :\ \widetilde{\mathbb{D}}_{-p,\infty} \longrightarrow\ \widetilde{\mathbb{D}}_{-p,\infty}.$$

Thus $\widetilde{\Delta}_L$ is a continuous linear operator from $\widetilde{\mathbb{D}}_{-p,\infty}$ into itself. Define

$$\widetilde{G}_t = SG_tS^{-1} :\ \widetilde{\mathbb{D}}_{-p,\infty} \longrightarrow\ \widetilde{\mathbb{D}}_{-p,\infty}.$$

By the above Theorem 6.1, the family $\{\widetilde{G}_t;\, t \geq 0\}$ is an equi-continuous semigroup of class (C_0) generated by the Lévy Laplacian $\widetilde{\Delta}_L$ as a continuous linear operator defined on the space $\widetilde{\mathbb{D}}_{-p,\infty}$.

Recall the stochastic process $X(t)$ in Eq. (33) which has characteristic function $e^{th(r)}$. Take a function $\eta_T \in \mathcal{S}$ such that $\eta_T = 1/\sqrt{T}$ on T. By using $X(t)$ and η_T, we define an $\mathcal{S}$-valued stochastic process by

$$\mathbb{X}_t = \xi + iX(t)\,\eta_T, \quad \xi \in \mathcal{S}.$$

The next theorem is from the papers by Kuo-Obata-Saitô[68,69].

Theorem 6.2. *For all $F \in \widetilde{\mathbb{D}}_{-p,\infty}$, the following equality holds*

$$\widetilde{G}_tF(\xi) = E\big[F(\mathbb{X}_t)\,\big|\,\mathbb{X}_0 = \xi\big].$$

This theorem is the analogue of Eq. (37) for the Lévy Laplacian. We can interpret $\mathbb{X}_t$ as an $\mathcal{S}$-valued stochastic process generated by the Lévy Laplacian $\widetilde{\Delta}_L$. Note that $\widetilde{\Delta}_L$ and $\mathbb{X}_t$ are intrinsically linked together by the diagonalization function $h(r)$.

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