

# ON SERIES-PARALLEL EXTENSIONS OF UNIFORM MATROIDS

BRAHIM CHAOURAR AND JAMES OXLEY

ABSTRACT. This paper gives an excluded-minor characterization of the class of matroids that are series-parallel extensions of uniform matroids.

## 1. INTRODUCTION

The terminology used here will follow Oxley [6]. A matroid  $M$  is a *series-parallel extension* of a matroid  $N$  if  $M$  can be obtained from  $N$  by a sequence of operations each consisting of a series or parallel extension, where the last two operations involve the addition of an element in series or in parallel, respectively, to an existing element. Let  $\mathcal{M}$  be the class of all series-parallel extensions of uniform matroids together with all minors of such matroids. Clearly  $\mathcal{M}$  is closed under the taking of duals. It is not difficult to show that, to obtain all the members of  $\mathcal{M}$  from uniform matroids, one must allow, along with the operations of series and parallel extension, the addition of loops or coloops. The purpose of this note is to characterize  $\mathcal{M}$  by excluded minors. There are exactly five 3-connected matroids of rank 3 on a 6-element set. These matroids can be obtained from  $M(K_4)$  by relaxing zero, one, two, three, or four circuit-hyperplanes. The matroids are, respectively,  $M(K_4)$ , the rank-3 whirl  $\mathcal{W}^3$ ,  $Q_6$ ,  $P_6$ , and the uniform matroid  $U_{3,6}$  (see [6, p. 295]). All but the last of these five matroids is an excluded minor for  $\mathcal{M}$ . There are two further excluded minors,  $U_{2,4} \oplus_2 U_{2,4}$ , which consists of two disjoint 3-point lines in the plane, and  $U_{2,4} \oplus U_{2,4}$ .

**Theorem 1.1.** *A matroid  $M$  is a minor of a series-parallel extension of a uniform matroid if and only if  $M$  has no minor isomorphic to any of  $M(K_4)$ ,  $\mathcal{W}^3$ ,  $P_6$ ,  $Q_6$ ,  $U_{2,4} \oplus_2 U_{2,4}$ , or  $U_{2,4} \oplus U_{2,4}$ .*

## 2. THE PROOF

The proof of Theorem 1.1 will use the following three lemmas. The first is a well-known extension (see, for example, [8, Theorem 14.2.2] or [6, Corollary 11.2.15]) of a graph result of Dirac [3], Ádám [1], and Duffin [4]; the second is a result of Bixby [2]; and the third was proved by Walton [7] (see also [5]).

**Lemma 2.1.** *A connected matroid with at least one element is a series-parallel network if and only if it has no minor isomorphic to  $U_{2,4}$  or  $M(K_4)$ .*

**Lemma 2.2.** *Let  $M$  be a connected non-binary matroid. If  $e \in E(M)$ , then  $M$  has a  $U_{2,4}$ -minor using  $e$ .*

---

Date: March 10, 2003.

1991 Mathematics Subject Classification. 05B35.

**Lemma 2.3.** *Let  $M$  be a 3-connected matroid having no minor isomorphic to any of  $M(K_4)$ ,  $\mathcal{W}^3$ ,  $P_6$ , or  $Q_6$ . Then  $M$  is uniform.*

*Proof of Theorem 1.1.* It is straightforward to check that each of  $M(K_4)$ ,  $\mathcal{W}^3$ ,  $P_6$ ,  $Q_6$ ,  $U_{2,4} \oplus_2 U_{2,4}$ , and  $U_{2,4} \oplus U_{2,4}$  is an excluded minor for  $\mathcal{M}$ . Now let  $N$  be an excluded minor that is not in this list. If  $N$  is disconnected, then each component of  $N$  is in  $\mathcal{M}$ . No component of  $N$  can be a series-parallel network so, by Lemma 2.1, each component has a minor isomorphic to  $M(K_4)$  or  $U_{2,4}$ . As  $N$  has no  $M(K_4)$ -minor, it follows that  $N$  has  $U_{2,4} \oplus U_{2,4}$  as a minor; a contradiction. We conclude that  $N$  is connected. If  $N$  is 3-connected, then, by Lemma 2.3,  $N$  is uniform; a contradiction. We deduce that  $N$  is not 3-connected. Thus  $N$  is a 2-sum with basepoints  $p_1$  and  $p_2$  of two connected matroids  $N_1$  and  $N_2$  each of which has at least three elements. Both  $N_1$  and  $N_2$  are minors of  $N$  so neither has  $M(K_4)$  as a minor. If  $N_i$  has no  $U_{2,4}$ -minor, then it is a series-parallel network and so  $N$  is a series-parallel extension of a member of  $\mathcal{M}$ ; a contradiction. Therefore both  $N_1$  and  $N_2$  have minors isomorphic to  $U_{2,4}$ . Thus, by Lemma 2.2, each  $N_i$  has a  $U_{2,4}$ -minor using  $p_i$ . Hence  $N$  has  $U_{2,4} \oplus_2 U_{2,4}$  as a minor; a contradiction.  $\square$

Since  $\mathcal{M}$  can be obtained from the class of uniform matroids by a sequence of series extensions, parallel extensions, or direct sums with loops or coloops, it is natural to consider the class of matroids that can be derived from the class of uniform matroids by series extensions, parallel extensions, and direct sums. This class is easily seen to be minor-closed and all its excluded minors are connected. The next result is obtained by making the obvious modifications to the last proof.

**Corollary 2.4.** *The excluded minors for the class of matroids that can be constructed from uniform matroids by a sequence of series extensions, parallel extensions, or direct sums are  $M(K_4)$ ,  $\mathcal{W}^3$ ,  $P_6$ ,  $Q_6$ , and  $U_{2,4} \oplus_2 U_{2,4}$ .*

#### ACKNOWLEDGEMENTS

The second author's research was partially supported by a grant from the National Security Agency.

#### REFERENCES

- [1] A. Ádám, Über zweipolige elektrische Netze. I, *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **2** (1957), 211–218.
- [2] R.E. Bixby,  $l$ -matrices and a characterization of non-binary matroids, *Discrete Math.* **8** (1974), 139–145.
- [3] G.A. Dirac, A property of 4-chromatic graphs and some remarks on critical graphs, *J. London Math. Soc.* **27** (1952), 85–92.
- [4] R.J. Duffin, Topology of series-parallel networks, *J. Math. Anal. Appl.* **10** (1965), 303–318.
- [5] J.G. Oxley, A characterization of certain excluded-minor classes of matroids, *Europ. J. Combinatorics* **10** (1989), 275–279.
- [6] J.G. Oxley, *Matroid Theory*, Oxford University Press, New York, 1992.
- [7] P.N. Walton, *Some Topics in Combinatorial Theory*, D.Phil. thesis, University of Oxford, 1981.
- [8] D.J.A. Welsh, *Matroid Theory*, Academic Press, London, 1976.

RIYADH COLLEGE OF TECHNOLOGY, P.O. BOX 42826, RIYADH 11551, SAUDI ARABIA  
*E-mail address:* bchaourar@hotmail.com

DEPARTMENT OF MATHEMATICS, LOUISIANA STATE UNIVERSITY, BATON ROUGE, LOUISIANA  
 70803–4918, USA  
*E-mail address:* oxley@math.lsu.edu