

# PALEY-WIENER THEOREMS FOR THE $\Theta$ -SPHERICAL TRANSFORM: AN OVERVIEW

GESTUR ÓLAFSSON AND ANGELA PASQUALE

## 1. INTRODUCTION

Harmonic analysis has its origin in the work of Fourier on the heat equation, which led him to consider the expansion of an “arbitrary”  $2\pi$ -periodic function into superposition of trigonometric functions:

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

with

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} dt.$$

One can interpret this expansion either as the spectral decomposition of the differential operators with constant coefficients, or as decomposition of  $L^2([0, 2\pi])$  into irreducible representations of the compact Lie group  $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ . There is no “definition” of harmonic analysis that includes all its different aspects, but the basic idea is *to study functions, or function spaces, in terms of decomposition into simpler, basic functions*. This includes spectral decomposition of differential operators, theory of special functions, integral transforms related to special functions, atomic decomposition of function spaces, and study of functions defined on a Lie group (or a homogeneous space) by decomposing them into pieces associated with the unitary irreducible representations of the group. The meaning of “simpler” or “basic” functions depends then on the context in which we are working.

The basic example of noncompact Lie group is the real line  $\mathbb{R}$  considered as an additive group. In this case all the unitary irreducible representations are one dimensional and given by the exponential functions  $t \mapsto e^{i\lambda t}$  with  $\lambda \in \mathbb{R}$ . To each (sufficiently regular) function  $f : \mathbb{R} \rightarrow \mathbb{C}$  we associate its Fourier transform

$$(\mathcal{F}f)(\lambda) \sim \frac{1}{2\pi} \int_{\mathbb{R}} f(x) e^{-i\lambda x} dx.$$

Here the Fourier inversion

$$f(x) \sim \int_{\mathbb{R}} (\mathcal{F}f)(\lambda) e^{i\lambda x} d\lambda$$

provides the required decomposition with respect to the irreducible representations or the spectral decomposition of the differential operators on  $\mathbb{R}$  with constant coefficients. Harmonic analysis also studies in which sense this decomposition has to be considered. For instance, the symbol “ $\sim$ ”

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means convergence in  $L^2$  norm if  $f \in L^2(\mathbb{R})$ , and even pointwise convergence if  $f$  is smooth and compactly supported.

One of the fundamental questions in harmonic analysis is to determine the image of different function spaces under the Fourier transform. For smooth, compactly supported functions, the answer is given by the classical theorem of Paley and Wiener, which characterizes this image in terms of holomorphic extendibility and growth conditions.

The classical one-dimensional Fourier analysis on  $\mathbb{R}$  has several generalizations. The real line can be replaced by a higher-dimensional Euclidean space, by a locally compact Hausdorff topological group, by a Lie group, or by a homogeneous space. Among the homogeneous spaces, the symmetric spaces play an important role for their numerous applications to other branches of mathematics and to physics.

The aim of this paper is to give an overview of several types of Paley-Wiener theorems, leading up to the Paley-Wiener theorem for the  $\Theta$ -spherical Fourier transform. The theory of  $\Theta$ -spherical functions is relatively new, and originates from the interplay of the harmonic analysis on symmetric spaces and the theory of special functions associated with root systems. As of now, only some basic theorems, many of them only for special cases, have been proven, leaving most of the theory to be developed. The basic functions, the  $\Theta$ -spherical functions, have singular behaviour as we approach the boundary of the domain where they live. Therefore the space of compactly supported functions is easier to handle than the  $L^p$ -spaces and many other natural function spaces, and it is therefore natural to look for a Paley-Wiener-type theorem. Up to now such a theorem has been proven only in very special cases, and still, its formulation and proof are very technical. In this paper we shall not go into details of the proofs, but present an overview which explains the different examples which have inspired and motivated the theory of  $\Theta$ -spherical functions. We refer to [ÓP01], [ÓP02], [Pa02a], [Pa02b] and [ÓP03] for details.

Starting from the Euclidean case, we move to Harish-Chandra's theory of spherical functions on Riemannian symmetric spaces of noncompact type. The latter provided the geometric background to Heckman-Opdam's theory of hypergeometric functions associated with root systems, which was developed from the late 80ies. The theory of Heckman and Opdam not only yielded a natural class of geometrically motivated special functions, but also provided new methods and tools for understanding the geometric theory.

A generalization of Harish-Chandra's theory to a different class of symmetric spaces found its roots in the theoretical physics. In fact, the studies of the causal structures of space-time underlined the physical relevance of non-Riemannian symmetric spaces which are endowed with a  $G$ -invariant ordering. A special class of ordered symmetric spaces where a theory of spherical functions could be developed are the noncompactly causal (NCC) symmetric spaces. This theory started in the early 90ies with the article [FHÓ94].

It turned out that the Heckman-Opdam theory was the ideal tool to extend the results of [FHÓ94]. At this point it seemed therefore natural to enclose all these different but related theories in a single one, namely the theory of  $\Theta$ -spherical functions.

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## 2. THE EUCLIDEAN PALEY-WIENER THEOREM

Let  $\mathfrak{a}$  be an  $r$ -dimensional Euclidean real vector space, and let  $\mathfrak{a}_{\mathbb{C}}$ ,  $\mathfrak{a}^*$  and  $\mathfrak{a}_{\mathbb{C}}^*$  respectively denote the complexification, the real dual and the complex dual of the vector space  $\mathfrak{a}$ . The Euclidean Fourier

transform of a sufficiently regular function  $f : \mathfrak{a} \rightarrow \mathbb{C}$  is the function  $\mathcal{F}f : \mathfrak{a}_\mathbb{C}^* \rightarrow \mathbb{C}$  defined by

$$\mathcal{F}f(\lambda) := \int_{\mathfrak{a}} f(H) e^{\lambda(H)} dH,$$

where  $dH$  denotes the Lebesgue measure on  $\mathfrak{a}$ . The integral converges for instance when  $f$  is smooth and compactly supported.

For a compact subset  $E$  of  $\mathfrak{a}$  let  $\text{conv}(E)$  denote the closed convex hull of  $E$ , i.e. the intersection of all closed half-spaces in  $\mathfrak{a}$  containing  $E$ . The support function of  $E$  is the function  $q_E : \mathfrak{a}^* \rightarrow \mathbb{R}$  defined by

$$q_E(\lambda) := \sup_{H \in E} \lambda(H) = \sup_{H \in \text{conv}(E)} \lambda(H). \quad (1)$$

Let  $C$  be a compact convex subset of  $\mathfrak{a}$ , and let  $C_c^\infty(C)$  denote the space of smooth functions on  $\mathfrak{a}$  with support contained in  $C$ . The Paley-Wiener space  $\text{PW}(C)$  is the space of the entire functions  $g : \mathfrak{a}_\mathbb{C}^* \rightarrow \mathbb{C}$  which are of exponential type  $C$  and rapidly decreasing, i.e., for every  $N \in \mathbb{N}$  there is a constant  $C_N \geq 0$  such that

$$|g(\lambda)| \leq C_N (1 + |\lambda|)^{-N} e^{q_C(\text{Re } \lambda)}$$

for all  $\lambda \in \mathfrak{a}_\mathbb{C}^*$ .

The classical theorem Paley and Wiener characterizes  $\text{PW}(C)$  as the image of  $C_c^\infty(C)$  under the Euclidean Fourier transform (see e.g. [Hoer90], Theorem 7.3.1, or [JL01], Theorem 8.3 and Proposition 8.6).

**Theorem 2.1** (Paley-Wiener). *Let  $C$  be a compact convex subset of  $\mathfrak{a}$ . Then the Euclidean Fourier transform maps  $C_c^\infty(C)$  bijectively onto  $\text{PW}(C)$ . Moreover, if  $C$  is stable under the action of a finite group  $W$  of linear automorphisms of  $\mathfrak{a}$ , then the Fourier transform maps the subspace  $C_c^\infty(C)^W$  of  $W$ -invariant elements in  $C_c^\infty(C)$  onto the subspace  $\text{PW}(C)^W$  of  $W$ -invariant elements in  $\text{PW}(C)$ .*

Suppose that  $A$  is a connected simply connected abelian Lie group with Lie algebra  $\mathfrak{a}$ . Then  $\exp : \mathfrak{a} \rightarrow A$  is a diffeomorphism. Denote by  $\log$  the inverse of  $\exp$ . The Euclidean Fourier transform of a sufficiently regular functions  $f : A \rightarrow \mathbb{C}$  is the function  $\mathcal{F}_A f : \mathfrak{a}_\mathbb{C}^* \rightarrow \mathbb{C}$  defined by

$$\mathcal{F}_A f(\lambda) := \int_A f(a) e^{\lambda(\log a)} da, \quad (2)$$

where the Haar measure  $da$  on  $A$  is the pullback under the exponential map of the Haar measure  $dH$  on  $\mathfrak{a}$ . Let  $W$  be a finite group acting on  $\mathfrak{a}$  by linear automorphism. Then we define an action by  $W$  on  $A$  by  $w(\exp H) = \exp w(H)$ . Denote by  $C_c^\infty(C)$  the space of smooth functions on  $A$  with support in the compact set  $\exp C$ , and let  $C_c^\infty(C)^W$  be the subspace of  $W$ -invariant elements. Composition with  $\exp$  and Theorem 2.1 prove that the Euclidean Fourier transform  $\mathcal{F}_A$  is a bijection of  $C_c^\infty(C)$  onto  $\text{PW}(C)$  and restricts to a bijection of  $C_c^\infty(C)^W$  onto  $\text{PW}(C)^W$ .

### 3. HARMONIC ANALYSIS ON RIEMANNIAN SYMMETRIC SPACES

In this section we present the theory of spherical functions on Riemannian symmetric spaces as developed by Harish-Chandra, Helgason, Gangolli and others, and state Helgason-Gangolli's Paley-Wiener theorem for the associated spherical Fourier transform. The main references are [Hel84] and [GV88].

Let  $G$  be a connected noncompact semisimple Lie group with finite center, and let  $K$  be a maximal compact subgroup of  $G$ . Then the quotient manifold  $G/K$  can be endowed with the structure of Riemannian symmetric space of the noncompact type. Harish-Chandra's theory of spherical functions on  $G/K$  is the harmonic analysis of the  $K$ -invariant functions on  $G/K$ . (For technical reasons, the theory was in fact developed in the wider setting of  $G$  of the Harish-Chandra class. See [GV88], Chapter 2.)

There are several equivalent ways to define *spherical functions* on  $G/K$ . See e.g. [Hel84], Ch. IV, §§ 2–3. For this, let  $dk$  be a normalized Haar measure on  $K$  and let  $\mathbb{D}(G/K)$  denote the (commutative) algebra of invariant differential operators on  $G/K$ . We identify functions on  $G/K$  with right  $K$ -invariant functions on  $G$ .

*The integral equations:* Let  $\psi$  be a complex-valued continuous function on  $G$ , not identically zero. Then  $\psi$  is spherical if

$$\int_K \psi(xky) dk = \psi(x)\psi(y) \quad (3)$$

for all  $x, y \in G$ .

Notice that the integral equations (3) ensure that  $\psi$  is  $K$ -biinvariant and that  $\psi(e) = 1$ , where  $e$  denotes the unit element of  $G$ .

*The differential equations:* A left  $K$ -invariant complex-valued smooth function  $\psi$  on  $G/K$  is spherical if  $\psi(eK) = 1$  and if there exists a character  $\chi : \mathbb{D}(G/K) \rightarrow \mathbb{C}$  such that for all  $D \in \mathbb{D}(G/K)$  we have

$$D\psi = \chi(D)\psi. \quad (4)$$

Thus the spherical functions are the normalized joint eigenfunctions of  $\mathbb{D}(G/K)$ . Notice that, since  $\mathbb{D}(G/K)$  contains the Laplace operator, which is elliptic, all spherical functions are indeed real analytic functions on  $G$ .

*Character characterization:* A left  $K$ -biinvariant complex-valued function  $\psi$  on  $G$  is spherical if the map

$$f \mapsto \hat{f}(\psi) := \int_{G/K} f(x)\psi(x) dx \quad (5)$$

is a homomorphism of the (commutative) convolution algebra  $C_c^\infty(G/K)^K$  of  $K$ -biinvariant complex-valued functions on  $G$ .

In short, the spherical functions play for the harmonic analysis of radial (that is,  $K$ -invariant) functions on  $G/K$  the same role as the exponential functions for the harmonic analysis on the real line.

Let  $\theta : G \rightarrow G$  be the Cartan involution on  $G$  corresponding to  $K$ , i.e.,  $K = \{k \in G \mid \theta(k) = k\}$ . Denote by the same letter the derived involution  $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$ . Then  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  where

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid \theta(X) = X\}$$

is the Lie algebra of  $K$  and

$$\mathfrak{p} = \{X \in \mathfrak{g} \mid \theta X = -X\}.$$

Let  $\mathfrak{a}$  be a maximal abelian subspace of  $\mathfrak{p}$ . Let  $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$  be the set of roots of  $\mathfrak{a}$  in  $\mathfrak{g}$ . For  $\alpha \in \Sigma$  let  $\mathfrak{g}_\alpha := \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{a}\}$  be the corresponding root space, and set  $m_\alpha := \dim \mathfrak{g}_\alpha$ . Let  $\mathfrak{m} = \{X \in \mathfrak{k} \mid [\mathfrak{a}, X] = \{0\}\}$  be the centralizer of  $\mathfrak{a}$  in  $\mathfrak{k}$ . Then

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha.$$

The set  $\{H \in \mathfrak{a} \mid \alpha(H) = 0 \text{ for some } \alpha \in \Sigma\}$  is a finite union of hyperplanes. We can therefore choose an  $X \in \mathfrak{a}$  such that  $\alpha(X) \neq 0$  for all  $\alpha \in \Sigma$ . Let  $\Sigma^+ := \{\alpha \in \Sigma \mid \alpha(X) > 0\}$ . Then  $\Sigma^+$  is a *positive system* of roots. Let  $\Pi$  be the corresponding fundamental system of *simple roots*. Let

$$\mathfrak{a}^+ := \{H \in \mathfrak{a} \mid \alpha(H) > 0 \text{ for all } \alpha \in \Sigma^+\}$$

and

$$\mathfrak{n} := \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha.$$

Then  $\mathfrak{n}$  is a nilpotent Lie algebra. We set  $A := \exp \mathfrak{a}$ ,  $A^+ := \exp \mathfrak{a}^+$  and  $N := \exp(\mathfrak{n})$ . The centralizer and the normalizer of  $A$  in  $K$  are respectively  $M = Z_K(A)$  and  $M' = N_K(A)$ .

**Theorem 3.1.** *The map*

$$K \times A \times N \ni (k, a, n) \mapsto kan \in G \quad (6)$$

*is an analytic diffeomorphism.*

The decomposition of Theorem 3.1 is known as *Iwasawa decomposition* of  $G$ . Define  $a_K : G \rightarrow A$  by

$$x \in Ka_K(x)N \quad (7)$$

**Theorem 3.2.** *We have  $G = KAK$  and the map  $K/M \times A^+ \ni (kM, a) \mapsto ka \in G/K$  is an analytic diffeomorphism onto an open dense subset of  $G/K$ .*

The Weyl group  $W := M'/M$  is a finite reflection group acting on  $\mathfrak{a}$  and – by duality – on  $\mathfrak{a}^*$ . Fix a  $W$  invariant inner product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{a}$ . For  $\alpha \in \Sigma$  define  $H_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$  by  $\alpha(H_\alpha) = 2$ . Let  $r_\alpha : \mathfrak{a}^* \rightarrow \mathfrak{a}^*$  be the reflection  $r_\alpha(\lambda) = \lambda - \lambda(H_\alpha)\alpha$ . Then  $W$  is generated by  $\{r_\alpha \mid \alpha \in \Pi\}$ . For  $a = \exp(H) \in A$  and  $\lambda \in \mathfrak{a}_\mathbb{C}^*$  let

$$a^\lambda := e^{\lambda(H)}$$

and notice that the homomorphism  $A \rightarrow \mathbb{C}^*$  are exactly the maps  $a \mapsto a^\lambda$ .

The complexification and complex dual of  $\mathfrak{a}$  are respectively denoted by  $\mathfrak{a}_\mathbb{C}$  and  $\mathfrak{a}_\mathbb{C}^*$ . We extend the inner product of  $\mathfrak{a}$  to  $\mathfrak{a}^*$  by duality and to  $\mathfrak{a}_\mathbb{C}$  and  $\mathfrak{a}_\mathbb{C}^*$  by  $\mathbb{C}$ -bilinearity. Finally, we set

$$\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha. \quad (8)$$

We have now set up all the notations to describe Harish-Chandra's results on spherical functions.

**Theorem 3.3** (Harish-Chandra). *Let  $\lambda \in \mathfrak{a}_\mathbb{C}^*$ . Then the function  $\varphi_\lambda : G \rightarrow \mathbb{C}$  given by*

$$\varphi_\lambda(x) := \int_K a(xk)^{\lambda - \rho} dk \quad (9)$$

*is a spherical function. It is a real analytic function of  $x \in G$  and a holomorphic function of  $\lambda \in \mathfrak{a}_\mathbb{C}^*$ . Every spherical function on  $G$  is of the form  $\varphi_\lambda$  for some  $\lambda \in \mathfrak{a}_\mathbb{C}^*$ . Furthermore  $\varphi_\lambda = \varphi_\mu$  if and only if there exists  $w \in W$  such that  $\lambda = w\mu$ .*

Notice that, since  $\mathfrak{a}^+$  is the interior of a fundamental domain of  $W$ , Theorem 3.2 implies that a continuous  $K$ -biinvariant functions  $f$  on  $G$  is uniquely determined by its restriction  $\text{Res}_{A^+}(f) = f|_{A^+}$  to  $A^+$ . Moreover, we can normalize the invariant measures  $dx$  on  $G/K$  and  $da$  on  $A$  so, that for  $f \in C_c(G/K)^K$  we have

$$\int_{G/K} f(x) dx = \int_{A^+} f(a)\delta(a) da = \frac{1}{|W|} \int_A f(a)\delta(a) da,$$

where  $|W|$  denotes the cardinality of the Weyl group and

$$\delta(a) := \prod_{\alpha \in \Sigma^+} |a^\alpha - a^{-\alpha}|^{m_\alpha}. \quad (10)$$

Furthermore, a  $K$ -biinvariant function  $\varphi$  is an eigenfunction for  $\mathbb{D}(G/K)$  if and only if  $\text{Res}_{A^+}(f)$  is an eigenfunction for the system of equation on  $A^+$  given by the radial components of operators from  $\mathbb{D}(G/K)$ . Thus harmonic analysis of  $K$ -invariant functions on  $G/K$  becomes the study of  $W$ -invariant functions on  $A$  or even of functions on the set  $A^+$ . Let  $\mathbb{D}_{G/K}(A)$  denote the commutative algebra of radial components along  $A^+$  of the differential operators in  $\mathbb{D}(G/K)$ . With each  $\lambda \in \mathfrak{a}_\mathbb{C}^*$

is associated a character  $\chi_\lambda$  of  $\mathbb{D}_{G/K}(A)$ , and the spherical function  $\varphi_\lambda$  is determined on  $A^+$  by the system of partial differential equations

$$D\varphi = \chi_\lambda(D)\varphi, \quad D \in \mathbb{D}_{G/K}(A). \quad (11)$$

Observe that  $\chi_\lambda = \chi_{w\lambda}$  for all  $w \in W$ .

Let  $\{H_i\}_{i=1}^r$  be a fixed orthonormal basis of  $\mathfrak{a}$ . For each  $H \in \mathfrak{a}$  let  $\partial(H)$  denote the corresponding directional derivative in  $\mathfrak{a}$ . Then the partial differential equation (11) on  $A^+$  corresponding to the Laplace operator can be written explicitly as

$$L\varphi = (\langle \lambda, \lambda \rangle - \langle \rho, \rho \rangle)\varphi \quad (12)$$

where

$$L := \sum_{i=1}^r \partial(H_i)^2 + \sum_{\alpha \in \Sigma^+} m_\alpha \frac{1 + e^{-2\alpha}}{1 - e^{-2\alpha}} \partial(A_\alpha) \quad (13)$$

denotes the radial component on  $A^+$  of the Laplace operator. A multidimensional variant of the classical method of Frobenius for determining local solutions of differential equations with regular singularities led Harish-Chandra to look for solutions of (12) of the form

$$\Phi_\lambda(m; a) = a^{\lambda - \rho} \sum_{\mu \in \Lambda} \Gamma_\mu(m; \lambda) a^{-\mu}, \quad a \in A^+. \quad (14)$$

Here  $\Lambda := \left\{ \sum_{j=1}^r n_j \alpha_j \mid n_j \in \mathbb{N}_0 \right\}$ , where  $\alpha_1, \dots, \alpha_r$  is for some enumeration of  $\Pi$ . Moreover  $m := \{m_\alpha \mid \alpha \in \Sigma\}$  denotes the set of multiplicities. With the initial condition  $\Gamma_0(m; \lambda) = 1$ , the coefficients  $\Gamma_\mu(m; \lambda)$  with  $\mu \in \Lambda \setminus \{0\}$  are uniquely determined by means of the recurrence relations

$$\langle \mu, \mu - 2\lambda \rangle \Gamma_\mu(m; \lambda) = 2 \sum_{\alpha \in \Sigma^+} m_\alpha \sum_{\substack{k \in \mathbb{N} \\ \mu - 2k\alpha \in \Lambda}} \Gamma_{\mu - 2k\alpha}(m; \lambda) \langle \mu + \rho - 2k\alpha - \lambda, \alpha \rangle \quad (15)$$

provided  $\lambda \in \mathfrak{a}_\mathbb{C}^*$  satisfies  $\langle \mu, \mu - 2\lambda \rangle \neq 0$  for all  $\mu \in \Lambda \setminus \{0\}$ . On this subset of  $\mathfrak{a}_\mathbb{C}^*$  the series on the right-hand side of (14) converges to a meromorphic function of  $\lambda$  and real analytic function of  $a \in A^+$  – in fact, there is tubular neighborhood  $U^+$  of  $A^+$  in the complexification of  $A$  on which this series converges to a holomorphic function –. It is remarkable that the function  $\Phi_\lambda$  turns out to solve the entire system (11) and allows to construct a basis for the smooth solutions of (11) on  $A^+$ .

**Theorem 3.4** (Harish-Chandra). *Let the notation be as above. Then the following properties hold for generic spectral parameter  $\lambda \in \mathfrak{a}_\mathbb{C}^*$ .*

- (1) *Let  $D \in \mathbb{D}_{G/K}(A)$ . Then  $D\Phi_\lambda = \chi_\lambda(D)\Phi_\lambda$ .*
- (2) *The functions  $\{\Phi_{w\lambda} \mid w \in W\}$  form a basis for the space of smooth solutions to the system (11) of differential equations on  $A^+$ .*
- (3) *There is a meromorphic function  $c$  on  $\mathfrak{a}_\mathbb{C}^*$  (depending only on  $\lambda$  and on the root structure) so that the spherical function  $\varphi_\lambda$  admits on  $A^+$  the expansion*

$$\varphi_\lambda(a) = \sum_{w \in W} c(w\lambda) \Phi_{w\lambda}(m; a). \quad (16)$$

The function  $c$  occurring in (16) is the so-called Harish-Chandra's  $c$ -function. It governs the asymptotic behavior of  $\varphi_\lambda$  on  $A^+$ . More precisely, let us write  $A^+ \ni a \rightarrow \infty$  to indicate that  $a \in A^+$  and  $\lim a^{-\alpha} = 0$  for all  $\alpha \in \Sigma^+$ . If  $\operatorname{Re}\langle \lambda + \rho, \alpha \rangle < 0$  for all  $\alpha \in \Sigma^+$ , then

$$\lim_{A^+ \ni a \rightarrow \infty} a^{\rho - \lambda} \varphi_\lambda(a) = c(\lambda).$$

The  $c$ -function is given by

$$c(\lambda) = \int_{\bar{N}} e^{-(\lambda+\rho)(H(\bar{n}))} d\bar{n},$$

where  $\bar{N} := \theta(N)$  and the Haar measure  $d\bar{n}$  on  $\bar{N}$  is normalized by the condition

$$c(\rho) := \int_{\bar{N}} e^{-\rho(H(\bar{n}))} d\bar{n} = 1. \quad (17)$$

Let  $\Sigma_1^+$  denote the set of indivisible positive roots. Hence  $\alpha \in \Sigma_1^+$  provided  $\alpha \in \Sigma^+$  but  $\alpha/2 \notin \Sigma^+$ . For  $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$  set

$$\lambda_{\alpha} := \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}. \quad (18)$$

Then Gindikin and Karpelevic proved the following explicit product formula for Harish-Chandra's  $c$ -function:

$$c(\lambda) = \kappa_0 \prod_{\alpha \in \Sigma_1^+} \frac{2^{-\lambda_{\alpha}} \Gamma(\lambda_{\alpha})}{\Gamma(\frac{1}{2}(\lambda_{\alpha} + m_{\alpha}/2 + 1)) \Gamma(\frac{1}{2}(\lambda_{\alpha} + m_{\alpha}/2 + m_{2\alpha}))}, \quad (19)$$

where the constant  $\kappa_0$  is chosen so that (17) holds. See e.g. [GV88], Theorem 4.7.5, or [Hel84], Ch. IV, Theorem 6.14.

**Example 3.5** (The real rank-one case). The real rank-one case corresponds to Riemannian symmetric spaces of the noncompact type for which  $\mathfrak{a}$  is one dimensional. The set  $\Sigma^+$  consists at most of two elements:  $\alpha$  and, possibly,  $2\alpha$ . By setting  $H_{\alpha}/2 \equiv 1$  and  $\alpha \equiv 1$ , we identify  $\mathfrak{a}$  and  $\mathfrak{a}^*$  with  $\mathbb{R}$ , and their complexifications  $\mathfrak{a}_{\mathbb{C}}$  and  $\mathfrak{a}_{\mathbb{C}}^*$  with  $\mathbb{C}$ . The Weyl chamber  $\mathfrak{a}^+$  coincides with the half-line  $(0, +\infty)$ . The Weyl group  $W$  reduces to  $\{-1, 1\}$  acting on  $\mathbb{R}$  and  $\mathbb{C}$  by multiplication. Moreover  $\rho(m) \equiv m_{\alpha}/2 + m_{2\alpha}$ . We normalize the inner product so that  $\langle \alpha, \alpha \rangle = 1$ . The algebra  $\mathbb{D}(G/K)$  is generated by the Laplace operator, and the system of differential equation (12) with spectral parameter  $\lambda \in \mathbb{C}$  is equivalent to the single Jacobi differential equation

$$\frac{d^2\varphi}{dt^2} + (m_{\alpha} \coth t + m_{2\alpha} \coth(2t)) \frac{d\varphi}{dt} = (\lambda^2 - \rho^2)\varphi, \quad t \in (0, +\infty). \quad (20)$$

The solution to (20) that behaves asymptotically as  $e^{(\lambda-\rho)t}$  for  $t \rightarrow +\infty$  is

$$\Phi_{\lambda}(m; t) = (2 \sinh t)^{\lambda-\rho} {}_2F_1 \left( \frac{\rho - \lambda}{2}, \frac{-m_{\alpha}/2 + 1 - \lambda}{2}; 1 - \lambda; -\sinh^{-2} t \right),$$

where  ${}_2F_1$  denotes the Gaussian hypergeometric function. The function  $\Phi_{\lambda}(m; t)$  coincides with the Jacobi function of second kind  $\Phi_{\nu}^{(a,b)}(t)$  with parameters  $a = (m_{\alpha} + m_{2\alpha} - 1)/2$ ,  $b = (m_{2\alpha} - 1)/2$  and  $\nu = -i\lambda$ . See e.g. [Koo84], Section 2. Harish-Chandra's spherical functions are Jacobi functions of first kind

$$\varphi_{\lambda}(t) = {}_2F_1 \left( \frac{m_{\alpha} + 2m_{2\alpha} + 2\lambda}{4}, \frac{m_{\alpha} + 2m_{2\alpha} - 2\lambda}{4}; \frac{m_{\alpha} + 2m_{2\alpha} + 1}{2}; -\sinh^2 t \right),$$

and the expansion (16) reduces to one of Kummer's relations between solutions of the hypergeometric equations. See e.g. [Er53], 2.9 (34).

**Example 3.6** (The complex case). The complex case corresponds to Riemannian symmetric spaces  $G/K$  for which the Lie algebra  $\mathfrak{g}$  of  $G$  is endowed with a complex structure. It is characterized by the fact that the root system  $\Sigma$  is reduced with all multiplicities  $m_{\alpha} = 2$ . It follows, in particular, that  $\rho = \sum_{\alpha \in \Sigma^+} \alpha$ . The Harish-Chandra series are given in  $A^+$  by  $\Phi_{\lambda}(2; a) := \Delta(a)^{-1} a^{\lambda}$  where

$$\Delta(a) := \prod_{\alpha \in \Sigma^+} (a^{\alpha} - a^{-\alpha}) \quad (21)$$

is Weyl's denominator. Set

$$\pi(\lambda) := \prod_{\alpha \in \Sigma^+} \lambda_\alpha. \quad (22)$$

An explicit formula for the spherical functions on  $G/K$  was determined by Harish-Chandra, namely

$$\varphi_\lambda(a) = \frac{\pi(\rho)}{\pi(\lambda)} \frac{\sum_{w \in W} (\det w) a^{w\lambda}}{\Delta(a)} \quad (23)$$

(see e.g. [Hel78], Ch. IV, Theorem 5.7). Notice that  $c(\lambda) = \pi(\rho)/\pi(\lambda)$ .

The *spherical Fourier transform*  $\mathcal{F}f$  of  $f \in C_c(G/K)^K$  is the  $W$ -invariant function on  $\mathfrak{a}_c^*$  defined by

$$(\mathcal{F}f)(\lambda) := \int_G f(x) \varphi_\lambda(x) dx = \int_{A^+} f(a) \varphi_\lambda(a) \delta(a) da = \frac{1}{|W|} \int_A f(a) \varphi_\lambda(a) \delta(a) da, \quad (24)$$

where  $dx$  and  $da$  are respectively the fixed normalizations of the Haar measures on  $G$  and  $A$ , and  $\delta$  is given by (10).

**Theorem 3.7** (Plancherel Theorem). *The spherical Fourier transform extends to an isometric isomorphism*

$$\mathcal{F} : L^2(G/K)^K \simeq L^2(A, \delta(a) da)^W \longrightarrow L^2(\mathfrak{ia}, |c(\lambda)|^{-2} d\lambda)^W$$

with pointwise inversion on  $C_c^\infty(G/K)^K$  given by

$$f(a) = \int_{\mathfrak{ia}^*} \mathcal{F}(f)(\lambda) \varphi_{-\lambda}(a) |c(\lambda)|^{-2} d\lambda, \quad a \in A. \quad (25)$$

In terms of representation theory, this corresponds to the decomposition of  $L^2(G/K)$  into direct integral of spherical principal series representations.

Recall from Section 2 the definition of Paley-Wiener space  $\text{PW}(C)^W$  associated with a compact convex  $W$ -invariant subset  $C$  of a Euclidean space. Here and in the following sections,  $W$  always denotes the Weyl group.

The characterization of the image of  $C_c^\infty(G/K)^K$  is the content of the Paley-Wiener theorem for the spherical Fourier transform. As in the Euclidean case, this theorem gives a finer characterization by describing, for every compact convex  $W$ -invariant subset  $C$  of  $\mathfrak{a}$ , the image of the space  $C_c^\infty(C)^W$  of elements  $f \in C_c^\infty(G/K)^K$  for which the support  $\text{supp } f|_A$  of their  $W$ -invariant restriction to  $A$  is contained in  $\exp(C)$ . The Paley-Wiener theorem was stated by Helgason in 1966 in [Hel66], where it was proven for the real rank-one and complex cases. The extension to arbitrary Riemannian symmetric spaces of the noncompact type was completed by Gangolli [Gan71] in 1971. The proof was later simplified by Rosenberg [Ros77].

**Theorem 3.8** (Helgason-Gangolli-Rosenberg). *Let  $C$  be a  $W$ -invariant compact convex subset of  $\mathfrak{a}$ . Then the Euclidean Fourier transform maps  $C_c^\infty(C)^W$  bijectively onto  $\text{PW}(C)^W$ .*

The original formulation of Theorem 3.8 considers only the case in which  $C$  is some Euclidean ball  $B_R := \{H \in \mathfrak{a} \mid |H| := \langle H, H \rangle^{1/2} \leq R\}$  with  $R > 0$ . Its extension to arbitrary  $W$ -invariant convex compact subsets of  $\mathfrak{a}$  is elementary. See e.g. [HS94] or [ÓP03]. Nevertheless, to simplify our exposition, here we only outline the proving method for the case of  $C = B_R$ .

The fact that the spherical Fourier transform maps  $C_c^\infty(B_R)^W$  into  $\text{PW}(B_R)^W$  is obtained by writing this transform as composition  $\mathcal{F} = \mathcal{F}_A \circ \mathcal{A}$  of the so-called Abel transform and of the Euclidean Fourier transform (2). Since the Abel transform can be easily seen to map  $C_c^\infty(B_R)^W$  into itself, the required inclusion follows then from the Euclidean Paley-Wiener theorem (Theorem 2.1). The thrust of the Paley-Wiener theorem for  $\mathcal{F}$  is to get the surjectivity, namely, that given

a holomorphic function  $h \in \text{PW}(B_R)^W$ , there is an  $f \in C_c^\infty(B_R)^W$  so that  $\mathcal{F}f = h$ . Motivated by the inversion formula (25), one defines  $f$  to be a suitable constant multiple of the wave packet

$$\mathcal{I}h(x) := \int_{i\mathfrak{a}^*} h(\lambda)\varphi_{-\lambda}(x) |c(\lambda)|^{-2} d\lambda, \quad x \in G. \quad (26)$$

Then  $\mathcal{I}h$  is well-defined, smooth and  $K$ -biinvariant. The crucial step is to show that  $\mathcal{I}h(\exp H) = 0$  for  $|H| > R$ . For this one uses the expansion (16) and the  $W$ -invariance of  $h$  to write

$$\mathcal{I}h(\exp H) = \int_{i\mathfrak{a}^*} h(-\lambda)c(-\lambda)^{-1}\Phi_\lambda(m; a) d\lambda.$$

At this step one would like to replace  $\Phi_\lambda(m; a)$  by its definition (14) and interchange summation and integration to get

$$\mathcal{I}h(\exp H) = \sum_{\mu \in \Lambda} e^{-(\mu+\rho)(H)} B_\mu(m; H) \quad (27)$$

with

$$B_\mu(m; H) := \int_{i\mathfrak{a}^*} h(-\lambda)\Gamma_\mu(m; \lambda)c(-\lambda)^{-1}e^{\lambda(H)} d\lambda. \quad (28)$$

Now the recursion relations (15) and the product formula (19) for  $c(\lambda)$  show that both  $\Gamma_\mu(m; \lambda)$  and  $c(-\lambda)^{-1}$  are holomorphic in  $\lambda$  provided  $\text{Re}\langle \lambda, \alpha \rangle < 0$  for all  $\alpha \in \Sigma^+$ . This, together with suitable estimates for  $\Gamma_\mu(m; \lambda)$ , allows one to shift the contour of integration in (28) and to prove, as in the Euclidean Paley-Wiener theorem, that  $B_\mu(m; H) = 0$  for  $|H| > R$ . The estimates that have been essential to justify (27) were established in [Gan71].

#### 4. THE HECKMAN-OPDAM THEORY

As noticed in Example 3.5, the spherical functions on rank-one Riemannian symmetric spaces are special instances of Jacobi functions of first kind, hence of hypergeometric functions. The specialization occurs with the choice of the multiplicities  $m_\alpha$  and  $m_{2\alpha}$  as certain nonnegative integers fixed by the geometry. Also in the higher-rank case, the geometry constraints the root multiplicities  $m_\alpha$  to assume certain specific values. The spherical functions as well are determined by the geometry, since the system of differential equations (11) originates from the algebra of  $G$ -invariant differential operators on  $G/K$ . Nevertheless, the differential equation (12) makes perfectly sense without the geometrical restrictions on  $m$ . This observation was the starting point of the theory of hypergeometric functions associated with root systems, which has been developed by Heckman and Opdam. Their goal was to reconstruct, for arbitrary complex values of multiplicities, the systems of differential equations (11). As analytic continuations (in the multiplicity parameters) of Harish-Chandra's spherical functions, the common eigenfunctions of these new systems would have provided a class of geometrically motivated multivariable hypergeometric functions. Heckman and Opdam could realize their program in a series of papers from 1988 to 1995 ([HO87],[Hec87], [Opd88a],[Opd88b], [Heck91], [Opd93], [Opd95]). The generalized spherical functions they constructed are nowadays known as *hypergeometric functions associated with root systems*. Our overview below is based mainly on [HS94], [Heck97] and [Opd95], to which we refer for details.

The Riemannian symmetric spaces of Harish-Chandra's theory are replaced in the theory of Heckman and Opdam by triples  $(\mathfrak{a}, \Sigma, m)$ , where  $\mathfrak{a}$  is an  $r$ -dimensional real Euclidean vector space,  $\Sigma$  is a root system in  $\mathfrak{a}^*$ , and  $m$  is a multiplicity function on  $\Sigma$ , that is a function  $m : \Sigma \rightarrow \mathbb{C}$  which is invariant under the Weyl group  $W$  of  $\Sigma$ . Setting  $m_\alpha := m(\alpha)$  for  $\alpha \in \Sigma$ , we therefore have  $m_{w\alpha} = m_\alpha$  for all  $w \in W$ . Because of our interest in Paley-Wiener theorems, we shall restrict ourselves here to the case in which all  $m_\alpha$  are nonnegative reals. Harmonic analysis results for some negative values of  $m_\alpha$  can be found in [Opd00].

We say that the triple is *geometric* if there is a Riemannian symmetric space of noncompact type  $G/K$  with restricted root system  $\Sigma$  for the corresponding pair  $(\mathfrak{g}, \mathfrak{a})$  so that  $m_\alpha$  is the multiplicity of the root  $\alpha$  for all  $\alpha \in \Sigma$ . Notice that we adopt the multiplicity notation commonly used in the theory of symmetric spaces. It differs from the notation employed by Heckman and Opdam in the following ways. The root system  $R$  used by Heckman and Opdam is related to our root system  $\Sigma$  by the relation  $R = \{2\alpha \mid \alpha \in \Sigma\}$ ; the multiplicity function  $k$  in Heckman-Opdam's work is related to our  $m$  by  $k_{2\alpha} = m_\alpha/2$ .

In the following our fixed inner product in  $\mathfrak{a}$  is denoted by  $\langle \cdot, \cdot \rangle$ . The dimension  $r$  of  $\mathfrak{a}$  is called the (real) rank of the triple  $(\mathfrak{a}, \Sigma, m)$ . The symbols  $\mathfrak{a}_\mathbb{C}$ ,  $\mathfrak{a}_\mathbb{C}^*$ ,  $H_\lambda$ ,  $\mathfrak{a}^+$ ,  $\Sigma^+$ ,  $\Pi$ ,  $r_\alpha$ ,  $\lambda_\alpha$  shall have the same meaning as in Section 3. To construct an analog of the Cartan subgroup, we first consider the complex torus  $A_\mathbb{C} := \mathfrak{a}_\mathbb{C}/\mathbb{Z}\{i\pi H_\alpha : \alpha \in \Sigma\}$  with Lie algebra  $\mathfrak{a}_\mathbb{C}$ . Let  $\exp : \mathfrak{a}_\mathbb{C} \rightarrow A_\mathbb{C}$  denote the canonical projection. Then  $A := \exp \mathfrak{a}$  is an abelian group so that  $\exp : \mathfrak{a} \rightarrow A$  is a diffeomorphism. We write  $\log$  for the inverse of  $\exp$ , and set  $A^+ := \exp \mathfrak{a}^+$ .

The restricted weight lattice of  $\Sigma$  is the set  $P$  of all  $\lambda \in \mathfrak{a}^*$  for which  $\lambda_\alpha \in \mathbb{Z}$  for all  $\alpha \in \Sigma$ . Notice that  $2\alpha \in P$  for all  $\alpha \in \Sigma$  and that  $P$  consists exactly of the elements  $\lambda \in \mathfrak{a}_\mathbb{C}^*$  for which the exponential  $e^\lambda(a) := e^{\lambda(\log a)} = a^\lambda$  is single valued on  $A_\mathbb{C}$ . We denote by  $\mathbb{C}[A_\mathbb{C}]$  the  $\mathbb{C}$ -linear span of the  $e^\lambda$  with  $\lambda \in P$ .

Let  $S(\mathfrak{a}_\mathbb{C})$  denote the symmetric algebra over  $\mathfrak{a}_\mathbb{C}$  considered as the space of polynomial functions on  $\mathfrak{a}_\mathbb{C}^*$ , and let  $S(\mathfrak{a}_\mathbb{C})^W$  be the subalgebra of  $W$ -invariant elements. For  $p \in S(\mathfrak{a}_\mathbb{C})$  write  $\partial(p)$  for the corresponding constant-coefficient differential operator on  $A_\mathbb{C}$  (or on  $\mathfrak{a}_\mathbb{C}$ ).

An algebra of differential operators playing the role of the algebra  $\mathbb{D}_{G/K}(A)$  was obtained by Heckman and Opdam as follows. Let  $\mathcal{R}$  be the subalgebra of the quotient field of  $\mathbb{C}[A_\mathbb{C}]$  generated the constant function 1 and by  $1/(1 - e^{-2\alpha})$  with  $\alpha \in \Sigma^+$ . Let  $\mathbb{D}\mathcal{R}^W$  denote the algebra of differential operators on  $A_\mathbb{C}$  with coefficients in  $\mathcal{R}$  which are invariant under the Weyl group  $W$ . If  $L(m)$  denotes the differential operator defined (for arbitrary multiplicity functions  $m$ ) by the right-hand side of (13), then  $L(m) \in \mathbb{D}\mathcal{R}^W$ . The elements in  $\mathbb{D}\mathcal{R}^W$  which commute with  $L(m)$  form therefore an algebra  $\mathbb{D}(\mathfrak{a}, \Sigma, m)$  which agrees with  $\mathbb{D}_{G/K}(A)$  when  $(\mathfrak{a}, \Sigma, m)$  is the geometric triple which corresponds to the Riemannian symmetric space of noncompact type  $G/K$ . Heckman and Opdam proved that  $\mathbb{D}(\mathfrak{a}, \Sigma, m)$  is indeed commutative and parameterized by the elements of  $S(\mathfrak{a}_\mathbb{C})^W$ . It follows, in particular, that  $\mathbb{D}(\mathfrak{a}, \Sigma, m)$  is generated by  $r (= \dim \mathfrak{a})$  elements. It is important to point out that the elements of  $\mathbb{D}(\mathfrak{a}, \Sigma, m)$  can be constructed algebraically. Indeed Cherednik determined an algebraic algorithm for constructing the operator  $D(m; p) \in \mathbb{D}(\mathfrak{a}, \Sigma, m)$  corresponding to the element  $p \in S(\mathfrak{a}_\mathbb{C})^W$  directly from  $p$  and the data  $(\mathfrak{a}, \Sigma, m)$ . His main tool are the so-called Dunkl-Cherednik differential-reflection operators. The operators  $D(m; p)$  are known as the *hypergeometric differential operators*.

Let  $\lambda \in \mathfrak{a}_\mathbb{C}^*$  be arbitrarily fixed. Then the system of differential equations

$$D(m; p)\varphi = p(\lambda)\varphi, \quad p \in S(\mathfrak{a}_\mathbb{C})^W \tag{29}$$

is called the *hypergeometric system of differential equations with spectral parameter*  $\lambda$  associated with the data  $(\mathfrak{a}, \Sigma, m)$ . For geometric triples it agrees with the system of partial differential equations (11) defining Harish-Chandra's spherical function of spectral parameter  $\lambda$ .

Formula (14) can still be employed to define the Harish-Chandra series. We extend the definition of Harish-Chandra's  $c$ -function to arbitrary multiplicity functions  $m$  by means of the Gindikin-Karpelevic formula (19). To underline its dependence on  $m$ , we replace the notation  $c(\lambda)$  by  $c(m; \lambda)$ . Similarly, we write  $\delta(m; a)$  instead of  $\delta(a)$  in (10), and  $\rho(m)$  instead of  $\rho$  in (8).

Notice that  $c(m; \lambda)$  is a meromorphic function of  $\lambda \in \mathfrak{a}_\mathbb{C}^*$ . Also the Harish-Chandra series  $\Phi_\lambda(m; a)$  are meromorphic in  $\lambda \in \mathfrak{a}_\mathbb{C}^*$  and singular on the walls of  $A^+$ . So a priori the sum (16) is only well-defined on  $A^+$  and meromorphic in  $\lambda$ . This singular behavior appears of course also in the Riemannian case, but, there, the right-hand side of (16) is known to be regular by Theorem 3.3.

Heckman and Opdam, who used the right-hand side of (16) to *define* their spherical functions, needed to develop tools to understand how the cancellation of singularities takes place. They proved the following fundamental result (cf. e.g. Theorem 4.4.2 in [HS94]).

**Theorem 4.1** (Heckman and Opdam). *For a fixed nonnegative multiplicity function  $m$  there is a  $W$ -invariant tubular neighborhood  $U$  of  $A$  in  $A_{\mathbb{C}}$  so that the function*

$$\varphi_{\lambda}(m; a) := \sum_{w \in W} c(m; w\lambda) \Phi_{w\lambda}(m; a), \quad a \in A^+,$$

*extends to a  $W$ -invariant holomorphic function of  $(\lambda, a) \in \mathfrak{a}_{\mathbb{C}}^* \times U$ .*

The functions  $\varphi_{\lambda}(m; a)$  are the so-called *hypergeometric functions associated with the triple  $(\mathfrak{a}, \Sigma, m)$*  (or, simply, *with root system  $\Sigma$* ). By construction, they reduce to Harish-Chandra's spherical functions in the geometric case.

Let  $C_c^{\infty}(A)^W$  denote the space of  $W$ -invariant  $C^{\infty}$  functions on  $A$  with compact support. The *Opdam transform* of  $f \in C_c^{\infty}(A)^W$  associated with the triple  $(\mathfrak{a}, \Sigma, m)$  is the  $W$ -invariant function  $\mathcal{F}f(m)$  on  $\mathfrak{a}_{\mathbb{C}}^*$  defined by

$$\mathcal{F}f(m; \lambda) := \frac{1}{|W|} \int_A f(a) \varphi_{\lambda}(m; a) \delta(m; a) da = \int_{A^+} f(a) \varphi_{\lambda}(m; a) \delta(m; a) da.$$

Notice that the assumption  $m_{\alpha} \geq 0$  for all  $\alpha \in \Sigma$  ensures that  $\delta(m; a)$  is a continuous  $W$ -invariant function of  $a \in A$ . The inversion formula for the Opdam transform is given by

$$f(a) = \kappa \int_{i\mathfrak{a}^*} \mathcal{F}f(m; \lambda) \varphi_{-\lambda}(m; a) |c(m; \lambda)|^{-2} d\lambda \quad (30)$$

where  $d\lambda$  is a suitable normalization of the Lebesgue measure on  $i\mathfrak{a}^*$  and  $\kappa > 0$  is a constant depending only on the normalization of the measures.

Recall from Section 2 the notation  $C_c^{\infty}(C)^W$  for the  $W$ -invariant  $C^{\infty}$  functions  $f : A \rightarrow \mathbb{C}$  with support contained in  $\exp C$ , where  $C$  is a compact, convex and  $W$ -invariant subset of  $\mathfrak{a}$ . The Paley-Wiener theorem for the Opdam transform have been proven in [Opd95], Theorems 8.6 and 9.13(4); see also [Opd00], p. 49.

**Theorem 4.2** (Opdam). *Let  $m$  be a fixed multiplicity function with  $m_{\alpha} \geq 0$  for all  $\alpha \in \Sigma$ . Suppose  $C$  is a compact, convex and  $W$ -invariant subset of  $\mathfrak{a}$ . Then  $\mathcal{F}(m)$  maps  $C_c^{\infty}(C)^W$  bijectively onto  $\text{PW}(C)^W$ .*

In its original form, Theorem 4.2 was stated for  $C = \text{conv } W(H)$ , where  $W(H)$  is the Weyl group orbit of an element  $H \in \mathfrak{a}$  and  $\text{conv}$  denotes the convex hull. The above formulation can be easily deduced from it.

It is important to observe that the proof of the surjectivity of Opdam transform follows Helgason's method. But the fact that  $\mathcal{F}(m)$  maps  $C_c^{\infty}(C)^W$  into  $\text{PW}(C)^W$  presents new difficulties. The crucial tool used in the Riemannian case was the decomposition  $\mathcal{F} = \mathcal{F}_A \circ \mathcal{A}$ . In the context of hypergeometric functions associated with root systems, the Abel transform  $\mathcal{A}$  does not have a direct extension – one could of course generalize the Abel transform as  $\mathcal{F}_A^{-1} \circ \mathcal{F}(m)$ , but this definition does not help to determine the support of  $\mathcal{F}(m)f$  –. Opdam's method is based on estimates for the hypergeometric functions  $\varphi_{\lambda}$ . Compared to the Riemannian case, the difficulty for estimating the hypergeometric functions relies essentially in the fact that the hypergeometric functions associated with root systems are generally not representable by integral formulas. Opdam's estimates have been obtained by introducing new regular nonsymmetric eigenfunctions of the Dunkl-Cherednik operators with Weyl groups symmetrization equal to the  $\varphi_{\lambda}$ . The important feature is that the Dunkl-Cherednik operators associated with elements in  $\mathfrak{a}_{\mathbb{C}}$  are first order differential-reflection operators. Indeed Opdam was able to extend to this setting a clever method, due to de Jeu [dJ93],

which uses the first order directional derivatives occurring in the eigenfunction problems to evaluate the growth of the eigenfunctions themselves.

## 5. SPHERICAL FUNCTIONS ON NONCOMPACTLY CAUSAL SYMMETRIC SPACES

The theory of spherical functions on Riemannian symmetric spaces depends on at least three closely related facts. First, the subgroup  $K$  is a maximal compact subgroup, i.e.,  $\theta$  is a Cartan involution. Secondly, the algebra  $\mathbb{D}(G/K)$  contains an elliptic differential operator, and hence all the joint eigenfunctions – or distributions – are real analytic. Finally we have the decompositions  $G = KAN = KAK$ . If we replace  $\theta$  by an arbitrary involution  $\tau : G \rightarrow G$  and set  $H = \{h \in G \mid \tau(h) = h\}$ , then  $H$  is no longer compact, on  $G/H$  there are in general no elliptic invariant differential operators,  $G \neq HAN$  (i.e., a corresponding generalized Iwasawa decomposition does not hold in general), and  $G \neq HAH$ . But it was shown in [FHÓ94] that there is a natural class of symmetric spaces, the *noncompactly causal symmetric spaces*, where an analogue theory of spherical functions defined on open  $H$ -invariant conal subset of  $G/H$  can be developed. For simplicity we shall assume in the following that  $G$  is contained in the simply connected complex Lie group  $G_{\mathbb{C}}$  with Lie algebra  $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ .

Let  $\tau : G \rightarrow G$  be a nontrivial involution commuting with the fixed Cartan involution  $\theta$ . We assume that  $\tau$  is not a Cartan involution, so  $H$  is not compact. As usual we denote the differential of  $\tau$  by the same letter. Let  $\mathfrak{h} := \{X \in \mathfrak{g} \mid \tau(X) = X\}$  and  $\mathfrak{q} := \{X \in \mathfrak{g} \mid \tau(X) = -X\}$ . Then  $\mathfrak{h}$  is the Lie algebra of  $H$ . We have

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q} = \mathfrak{h}_k \oplus \mathfrak{h}_p \oplus \mathfrak{q}_k \oplus \mathfrak{q}_p$$

where the subscript  $k$ , respectively  $p$ , denotes intersection with  $\mathfrak{k}$ , respectively  $\mathfrak{p}$ . An element  $X \in \mathfrak{g}$  is called *hyperbolic* if  $\text{ad}(X)$  is semisimple with real eigenvalues.

**Definition 5.1.** Assume that  $G/H$  is simple. Then  $G/H$  is called *noncompactly causal*, in short NCC, if there exists an open convex  $H$ -invariant cone  $\Omega \neq \emptyset$  in  $\mathfrak{q}$ , containing no affine lines, such that each  $X \in \Omega$  is hyperbolic.

We assume from now on that  $G/H$  is NCC and that  $\Omega$  is an  $H$ -invariant cone as in Definition 5.1. We shall assume that  $\Omega$  is maximal. We refer to [HÓ96] for information on causal symmetric spaces, and recall only the necessary facts. We can always choose a maximal abelian subspace  $\mathfrak{a} \subset \mathfrak{q}_p$  which is maximal abelian in  $\mathfrak{q}$  and  $\mathfrak{p}$ . Let  $\Omega_A := \Omega \cap \mathfrak{a}$ . Then  $\Omega = \text{Ad}(H)(\Omega_A)$ , and there exists a unique element  $X_0 \in \mathfrak{q}_p^{H \cap K} \cap \Omega_A$  such that  $\text{ad}(X_0)$  has eigenvalues 0, 1 and  $-1$ . Furthermore  $\Gamma(\Omega) := \exp(\Omega)H = H \exp(\Omega)$  is an open semigroup, diffeomorphic to  $\Omega \times H$ . The set  $\Sigma$  of roots decomposes into two disjoint sets:  $\Sigma_0 := \{\alpha \in \Sigma \mid \alpha(X_0) = 0\}$  and  $\Sigma_n := \{\alpha \in \Sigma \mid \alpha(X_0) \neq 0\}$ . Furthermore  $\Sigma_0$  is the set of roots of  $\mathfrak{a}$  in  $\mathfrak{g}^{\theta\tau} = \mathfrak{h}_k \oplus \mathfrak{q}_p$ . Let  $W_0$  be the corresponding Weyl group, i.e.,  $W_0$  is generated by the reflections  $r_\alpha$ ,  $\alpha \in \Sigma_0$  and  $-$  because of our assumption  $G \subset G_{\mathbb{C}}$  – we have  $W_0 = N_{K \cap H}(A)/Z_{K \cap H}(A)$ . Let  $\Sigma_n^+ := \{\alpha \in \Sigma_n \mid \alpha(X_0) = 1\}$ . Then, since  $X_0$  is  $K \cap H$ -invariant, it follows that  $\Sigma_n^+$  is  $W_0$ -invariant. We choose our positive set of roots such that  $\Sigma^+ = \Sigma_0^+ \cup \Sigma_n^+$ , where  $\Sigma_0^+$  is a positive system in  $\Sigma_0$ . It is important to remark that  $\Sigma$  is reduced, that is  $2\alpha \notin \Sigma$  for all  $\alpha \in \Sigma$ . We denote by  $\Pi_0 \subset \Pi$  the set of simple roots in  $\Sigma_0^+$ . Then  $\Pi \setminus \Pi_0$  contains one element. Let  $\mathfrak{n}_0 := \bigoplus_{\alpha \in \Sigma_0^+} \mathfrak{g}_\alpha$  and  $\mathfrak{n}_+ := \bigoplus_{\alpha \in \Sigma_n^+} \mathfrak{g}_\alpha$ . Similarly we set  $N_0 := \exp(\mathfrak{n}_0)$  and  $N_+ := \exp(\mathfrak{n}_+)$ . Let  $A_0 := \exp(\Omega_A) = \Gamma(\Omega) \cap A$ . Then we have the following theorem.

**Theorem 5.2.** *Let the notation be as above. Then the following holds.*

- (1)  $A_0$  is  $W_0$ -invariant;
- (2)  $A_0 = \left(\bigcup_{w \in W_0} w(A^+)\right)^o$ ;
- (3)  $\Gamma(\Omega) \subset HAN$ ;
- (4)  $\Gamma(\Omega) = \text{Ad}(H)(A_0)$ ;

(5)  $M \subset H \cap K$  and

$$H \times A \times N \ni (h, a, n) \mapsto han \in G$$

is a diffeomorphism onto an open subset of  $G$ ;

(6) The group  $N_+$  is abelian,  $G^{\theta}$  normalizes  $N_+$ , and  $N_0 \subset G^{\theta\tau}$ ;

(7) The group  $N$  can be represented as semidirect product  $N = N_0N_+ = N_+N_0$ .

Notice that (3) gives a generalized Iwasawa decomposition for the  $H$ -invariant domain  $\Gamma(\Omega)$  and (4) is a generalized Cartan decomposition for  $\Gamma(\Omega)$ . In particular we note that  $H$ -biinvariant functions on  $\Gamma(\Omega)$  are uniquely determined by their  $W_0$ -invariant restriction to  $A_0$ . By (5) we can define an analytic map  $a_H : HAN \rightarrow A$  by  $x \in Ha_H(x)N$ . For  $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$  let

$$\varphi_{\Pi_0}(\lambda, s) = \int_H a_H(sh)^{\lambda-\rho} dh, \quad s \in \Gamma(\Omega), \quad (31)$$

whenever  $H \ni h \mapsto a_H(sh)^{\lambda-\rho} \in \mathbb{C}$  is integrable. Denote by  $\mathbb{D}_{G/H}(A_0)$  the space of differential operators on  $A_0$  gotten by taking  $H$ -radial component of  $G$ -invariant differential operators on  $G/H$ . Let  $\bar{N} = \theta(N) = \exp(\oplus_{\alpha \in \Sigma^+} \mathfrak{g}_{-\alpha})$ . Then  $\bar{N} = \bar{N}_+\bar{N}_0$ , with the obvious notation. The main results in [FHÓ94] can now be stated as follows.

**Theorem 5.3.** *Let the notation be as above. Then the following holds.*

(1) There exist an open convex set  $\emptyset \neq \mathcal{E} \subset \mathfrak{a}_{\mathbb{C}}^*$  such that  $\varphi_{\Pi_0}(\lambda, a)$  is well defined for all  $\lambda \in \mathcal{E}$  and  $a \in A_0$ .

(2) Assume that  $\lambda \in \mathcal{E}$ , then  $\varphi_{\Pi_0}(\lambda, \cdot)|_{A_0}$  is a joint eigenfunction for  $\mathbb{D}_{G/H}(A_0)$ .

(3) Let  $s, t \in \Gamma(\Omega)$ . Then  $H \ni h \mapsto \varphi_{\Pi_0}(\lambda, sht) \in \mathbb{C}$  is integrable, and

$$\int_H \varphi_{\Pi_0}(\lambda, sht) dh = \varphi_{\Pi_0}(\lambda, s)\varphi_{\Pi_0}(\lambda, t).$$

(4) There exists an open convex set  $\emptyset \neq \mathcal{E}_0 \subset \mathcal{E}$  such that if  $\lambda \in \mathcal{E}_0$ , then the integral

$$c_{\Pi_0}(\lambda) := \int_{\bar{N} \cap HAN} a_H(\bar{n})^{-\lambda-\rho} d\bar{n}$$

exists and

$$\lim_{A^+ \ni a \rightarrow \infty} a^{\rho-\lambda} \varphi_{\Pi_0}(\lambda, a) = c_{\Pi_0}(\lambda).$$

Because of the decomposition  $N = N_0N_+$  it follows easily that the  $c$ -function can be written as a product,  $c_{\Pi_0}(\lambda) = c_{\Pi_0}^+(\lambda)c_{\Pi_0}^-(\lambda)$ , where

$$c_{\Pi_0}^+(\lambda) = \int_{N_0} a_H(\bar{n})^{-\lambda-\rho_0} d\bar{n}, \quad \text{with} \quad \rho_0 := \frac{1}{2} \sum_{\alpha \in \Sigma_0^+} m_{\alpha}\alpha,$$

is the Harish-Chandra  $c$ -function for the Riemannian symmetric space  $G^{\theta\tau}/H \cap K$  – even if  $G^{\theta\tau}$  is not semisimple, Harish-Chandra's theory can still be worked out on  $G^{\theta\tau}/H \cap K$ . See the remark at the beginning of Section 3 –. Moreover,  $c_{\Pi_0}^-(\lambda)$  is related to the geometry of  $G/H$  by

$$c_{\Pi_0}^-(\lambda) = \int_{\bar{N}_+ \cap HAN} a_H(\bar{n})^{-\lambda-\rho} d\bar{n}.$$

Furthermore  $c_{\Pi_0}^-(w\lambda) = c_{\Pi_0}^-(\lambda)$  for all  $w \in W_0$ . We remark, that we are not using the standard notation from [FHÓ94],[KÓ02],[KÓ03],[Ó197]. Instead we use here the same notation as for the general case in the next section, but we do not include the multiplicity function  $m$  at this point. The connection to the notation in above references is:  $c_{\Pi_0}^+(\lambda) = c_0(\lambda)$  and  $c_{\Pi_0}^-(\lambda) = c_{\Omega}(\lambda)$ .

Notice that the radial part of the Casimir operator is elliptic on  $A^+$  and hence  $\varphi_{\Pi_0}(\lambda, \cdot)|_{A^+}$  is real analytic. By Theorem 3.4 there exists, for generic spectral parameters, complex numbers  $c_{w,\lambda}$ ,

$w \in W$ , such that  $\varphi_{\Pi_0}(\lambda, a) = \sum_{w \in W} c_{w, \lambda} \Phi_{w\lambda}(m; a)$  for all  $a \in A^+$ . It was proved in [Ó197] that the coefficients  $c_{w, \lambda}$  are given by the  $c$ -function, in fact

$$\varphi_{\Pi_0}(\lambda, a) = \sum_{w \in W_0} c_{\Pi_0}(w\lambda) \Phi_{w\lambda}(m; a) = c_{\Pi_0}^-(\lambda) \sum_{w \in W_0} c_{\Pi_0}^+(w\lambda) \Phi_{w\lambda}(m; a), \quad a \in A^+. \quad (32)$$

From the integral formula for the  $c$ -function it follows easily that  $c_{\Pi_0}(\lambda)$  extends to a meromorphic function on  $\mathfrak{a}_{\mathbb{C}}^*$ . Thus (32) implies, in particular, that  $\varphi_{\Pi_0}(\lambda, a)$ , as a function of the spectral parameter  $\lambda$ , extends to a meromorphic function on  $\mathfrak{a}_{\mathbb{C}}^*$ . Using the methods of Heckmann and Opdam it was also proved in [Ó197] that  $\varphi_{\Pi_0}(\lambda, a)$  is smooth on all of  $A_0$ .

**Example 5.4** (The rank one case, [FHÓ94]). The noncompactly causal symmetric spaces of rank one are all locally isomorphic to  $\mathrm{SO}_0(1, n)/\mathrm{SO}_0(1, n-1)$ . In this case  $\Sigma = \Sigma_n = \{\alpha, -\alpha\}$ . In particular  $W_0 = \{1\}$ . We identify  $\mathfrak{a}_{\mathbb{C}}^*$  with  $\mathbb{C}$  by setting  $\alpha \equiv 1$ . Then in particular  $\rho = (n-1)/2$ . Set  $a_t = \exp(tX_0)$ . Then  $\mathbb{R}^+ \ni t \mapsto a_t \in A_0$  is a diffeomorphism. Notice, that  $\mathbb{D}_{G/H}(A_0)$  is generated by the radial part of the Casimir element, and the corresponding differential equations on  $A_0$  are the same as for the Riemannian case. A simple calculation (cf. [FHÓ94]) shows that

$$c_{\Pi_0}^-(\lambda) = \kappa \frac{\Gamma(\rho)\Gamma(-\lambda - \rho + 1)}{\Gamma(-\lambda + 1)} = \kappa B(\rho, \lambda - \rho + 1)$$

where  $\kappa$  is a nonzero constant. Finally we get:

$$\varphi_{\Pi_0}(\lambda, a_t) = c_{\Pi_0}^-(\lambda) (2 \cosh t)^{\lambda - \rho} {}_2F_1 \left( \frac{-\lambda - \rho}{2}, \frac{-\lambda + \rho + 1}{2}; 1 - \lambda; \frac{1}{\cosh^2 t} \right).$$

**Example 5.5** (The complex case, [FHÓ94]). The space  $G_{\mathbb{C}}/G$  is NCC if and only if  $G/K$  is a bounded symmetric domain. In this case

$$\varphi_{\Pi_0}(\lambda, a) = \kappa_1 c_{\Pi_0}^-(\lambda) \frac{\sum_{w \in W_0} (\det w) a^{w\lambda}}{\prod_{\alpha \in \Sigma_0^+} \lambda_{\alpha} \prod_{\alpha \in \Sigma^+} \sinh(\alpha(\log a))}$$

and

$$c_{\Pi_0}^-(\lambda) = \kappa_2 \prod_{\alpha \in \Sigma_n^+} \lambda_{\alpha}^{-1}$$

where  $\kappa_1$  and  $\kappa_2$  are nonzero constants.

The only unknown part in (32) is so far the  $c_{\Pi_0}^-$ -function. It was worked out for the rank one case in [FHÓ94], for the Cayley type spaces by Faraut in [Far95], and for  $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(p, q)$  by Graczyk [Gra97]. The general case was then solved in [KÓ02], Theorem III.5, see also [KÓ03].

**Theorem 5.6** (Product formula for the  $c$ -function). *For  $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$  and  $\alpha \in \Sigma$  let  $\lambda_{\alpha}$  be as in (18). Then the following product formula holds for the function  $c_{\Pi_0}^-$ :*

$$c_{\Pi_0}^-(\lambda) = \kappa \prod_{\alpha \in \Sigma_n^+} B(m_{\alpha}/2, -\lambda_{\alpha} - m_{\alpha}/2 + 1) = \kappa \prod_{\alpha \in \Sigma_n^+} \frac{\Gamma(m_{\alpha}/2) \Gamma(-\lambda_{\alpha} - m_{\alpha}/2 + 1)}{\Gamma(-\lambda_{\alpha} + 1)},$$

where  $\kappa$  is a nonzero constant.

Notice that, if  $m_{\alpha}$  is even for all  $\alpha$ , then  $1/c_{\Pi_0}^-(\lambda)$  is a polynomial:

$$\frac{1}{c_{\Pi_0}^-(\lambda)} = \gamma \prod_{\alpha \in \Sigma_n^+} \prod_{j=0}^{(m_{\alpha}/2)-1} (\lambda_{\alpha} + j)$$

where  $\gamma := (-1)^{\sum_{\alpha} m_{\alpha}/2} [\kappa \prod_{\alpha \in \Sigma_n^+} \Gamma(m_{\alpha}/2)]^{-1}$ .

Define now the *spherical Fourier-Laplace transform*  $(\mathcal{F}f)(\lambda)$  of a compactly supported function on  $A_0$  by

$$(\mathcal{F}f)(\lambda) = \int_{A_0} f(a)\varphi_{\pi_0}(\lambda, a)\delta(a) da. \quad (33)$$

Then we have the following inversion formula [Ó197].

**Theorem 5.7.** *Let  $c(\lambda)$  be the Harish-Chandra  $c$ -function for the Riemannian symmetric space  $G/K$ , and let  $\varphi_\lambda$  the spherical function on  $G/K$ . Then there exists a constant  $\eta$  such that for all  $f \in C_c^\infty(A_0)$  the spherical Fourier-Laplace transform  $\mathcal{F}f$  of  $f$  is inverted according to*

$$f(a) = \eta \int_{i\mathfrak{a}^*} (\mathcal{F}f)(\lambda)\varphi_{-\lambda}(a)\frac{d\lambda}{c_{\pi_0}(\lambda)c(-\lambda)}.$$

Paley-Wiener type theorems for the spherical Laplace-Fourier transform were first considered for rank one spaces in [AO01], and have then been proven for noncompactly causal symmetric spaces with even multiplicity [AÓ00] and for some Cayley-type spaces [AU02]. We do not state the exact theorems here. In the even multiplicity case they are contained in the more general results for the  $\Theta$ -spherical transform, which shall be discussed in the next section. We would just like to point out that the general Paley-Wiener type theorem has not been worked out so far.

## 6. THE $\Theta$ -SPHERICAL FUNCTIONS

The theory of  $\Theta$ -functions extends Heckman-Opdam's theory of hypergeometric functions associated with root systems so that it encloses special geometric instances which we have discussed: the Harish-Chandra's theory of spherical functions on Riemannian symmetric spaces of noncompact type, and the theory of spherical functions on NCC symmetric spaces. The  $\Theta$ -spherical functions, as the Heckman-Opdam's hypergeometric functions, are special functions associated with root systems. Because of Examples 3.5 and 5.4, they can be considered as geometrically motivated multivariable generalizations of the Jacobi functions.

Both spherical functions on Riemannian and NCC symmetric spaces admit expansion with respects to the Harish-Chandra series. This suggests, exactly as in the case of hypergeometric functions associated with root systems, to use linear combinations of Harish-Chandra series for defining their common generalizations. Comparing the expansion (32) to (16), one notices that the summation over the Weyl group  $W$  is replaced by a summation over its parabolic subgroup  $W_0$ . The coefficients for the Harish-Chandra series are also modified. A unified theory can therefore be constructed by considering suitable linear combinations of Harish-Chandra series over parabolic subgroups of  $W$ .

For simplicity of exposition we shall assume in the following that the root system  $\Sigma$  is reduced, i.e. that  $2\alpha \notin \Sigma$  for all  $\alpha \in \Sigma$ . This condition is for instance always satisfied by the root systems of NCC symmetric spaces. We refer to [ÓP01], [Pa02a] and [Pa02b] for the theory of  $\Theta$ -spherical functions on general root systems and for further details. We keep the notation introduced in Section 4.

Let  $\Theta$  be an arbitrary subset of positive simple roots in a fundamental system  $\Pi \subset \Sigma^+$ , and  $W_\Theta$  the parabolic subgroup of  $W$  generated by the reflections  $r_\alpha$ ,  $\alpha \in \Theta$ . Then  $W_\Theta$  is the Weyl group of the root system  $\langle \Theta \rangle := \mathbb{Z}\Theta \cap \Sigma$ . Set  $\langle \Theta \rangle^+ := \langle \Theta \rangle \cap \Sigma^+$ . Notice that  $W_\Theta(\Sigma^+ \setminus \langle \Theta \rangle^+) \subset \Sigma^+ \setminus \langle \Theta \rangle^+$ . We refer to [Hu90] for additional information on parabolic subgroups of Weyl groups.

We introduce the  $c$ -functions

$$c_{\Theta}^{+}(m; \lambda) := \prod_{\alpha \in \langle \Theta \rangle^{+}} \frac{\Gamma(\lambda_{\alpha})}{\Gamma(\lambda_{\alpha} + m_{\alpha}/2)},$$

$$c_{\Theta}^{-}(m; \lambda) := \prod_{\alpha \in \Sigma^{+} \setminus \langle \Theta \rangle^{+}} \frac{\Gamma(-\lambda_{\alpha} - m_{\alpha}/2 + 1)}{\Gamma(-\lambda_{\alpha} + 1)},$$

with the conventions that products over empty sets are equal to 1, i.e.

$$c_{\emptyset}^{+} = c_{\Pi}^{-} := 1.$$

Notice that (up to a constant depending on  $m$  but not on  $\lambda$ ) the function  $c_{\Theta}^{+}$  is Harish-Chandra's  $c$ -function for the root system  $\langle \Theta \rangle$ , whereas  $c_{\Theta}^{-}$  is modelled on the function  $c_{\Pi_0}^{-}$  of Theorem 5.6. Observe also that  $c_{\Theta}^{-}(m; \lambda)$  is a  $W_{\Theta}$ -invariant function of  $\lambda \in \mathfrak{a}_{\mathbb{C}}^{*}$ .

When all multiplicities  $m_{\alpha}$  are even, all these  $c$ -functions are reciprocals of polynomials. Indeed, in this case

$$c_{\Theta}^{+}(m; \lambda) := \prod_{\alpha \in \langle \Theta \rangle^{+}} \prod_{h=0}^{m_{\alpha}/2-1} \frac{1}{\lambda_{\alpha} + h},$$

$$c_{\Theta}^{-}(m; \lambda) := (-1)^{d(\Theta, m)} \prod_{\alpha \in \Sigma^{+} \setminus \langle \Theta \rangle^{+}} \prod_{h=0}^{m_{\alpha}/2-1} \frac{1}{\lambda_{\alpha} + h},$$

where

$$d(\Theta, m) := \frac{1}{2} \sum_{\alpha \in \Sigma^{+} \setminus \langle \Theta \rangle^{+}} m_{\alpha}. \quad (34)$$

**Definition 6.1.** Let  $\Theta \subset \Pi$  and let  $U^{+}$  be a tubular neighborhood of  $A^{+}$  in  $A_{\mathbb{C}}$  on which, for generic  $\lambda \in \mathfrak{a}_{\mathbb{C}}^{*}$ , each series defining  $\Phi_{w\lambda}(m; a)$  converges to a holomorphic function of  $a$  (cf. comments on p. 6). The function on  $U^{+}$  defined for generic  $\lambda \in \mathfrak{a}_{\mathbb{C}}^{*}$  by

$$\varphi_{\Theta}(m; \lambda, a) := c_{\Theta}^{-}(m; \lambda) \sum_{w \in W_{\Theta}} c_{\Theta}^{+}(m; w\lambda) \Phi_{w\lambda}(m; a), \quad a \in U^{+} \quad (35)$$

is called the  $\Theta$ -spherical function of spectral parameter  $\lambda$ .

As a linear combination of the Harish-Chandra series  $\Phi_{w\lambda}(m; a)$ , the  $\Theta$ -spherical function of spectral parameter  $\lambda$  is, by construction, a solution of the hypergeometric system (29).

We now illustrate how to recover our motivating examples from the context of  $\Theta$ -spherical functions.

- Example 6.2.** (1) If  $m = 0$ , then  $\Phi_{\lambda}(0; a) = a^{\lambda}$ . We thus obtain the Euclidean case (with different symmetries induced by the choice of  $\Theta$ ).
- (2) When  $\Theta = \Pi$ , then  $\varphi_{\Theta}(m; \lambda, a)$  coincides – up to the normalization factor  $c_{\Pi}^{+}(m; \rho(m))$  – with Heckman-Opdam's hypergeometric function of spectral parameter  $\lambda$ , hence with Harish-Chandra's spherical function of spectral parameter  $\lambda$  when the triple  $(\mathfrak{a}, \Sigma, m)$  is geometric.
- (3) Suppose  $(\mathfrak{a}, \Sigma, m)$  corresponds to the Riemannian dual of an NCC symmetric space  $G/H$  according to Section 5. If  $\Theta = \Pi_0$  is the set of positive compact simple roots, then  $\varphi_{\Pi_0}(m; \lambda, a)$  coincides – up to the normalization factor  $c_{\Pi_0}^{+}(m; \rho(m))$  – with the spherical function  $\varphi_{\Pi_0}(\lambda, a)$  on  $G/H$ . Notice also that  $W_{\Pi_0} = W_0$ .

As in the theory of Heckman and Opdam, the first crucial problems are to understand the domain on which the  $\Theta$ -spherical functions extend as functions of both  $\lambda$  and  $a$ , and to study the nature of this extension. Notice that in the NCC case of Section 5 it was proved that the spherical functions

are meromorphic in  $\lambda \in \mathfrak{a}_c^*$ , but there was no description of the  $\lambda$ -singular set. The  $\lambda$ -singular set of the  $\Theta$ -spherical functions – and hence also of the spherical functions on NCC symmetric spaces – turns out to be described by the numerator  $n_\Theta^-$  of the function  $c_\Theta^-$ :

$$n_\Theta^-(m; \lambda) := \prod_{\alpha \in \Sigma^+ \setminus \langle \Theta \rangle^+} \Gamma\left(-\lambda_\alpha - \frac{m_\alpha}{2} + 1\right).$$

In the  $a$ -variable the situation is very similar to the one depicted for the special case of spherical functions on NCC symmetric spaces. The domain of extension will be

$$A_\Theta := \exp(\mathfrak{a}_\Theta) \quad \text{with} \quad \mathfrak{a}_\Theta := \left( \bigcup_{w \in W_\Theta} w(\mathfrak{a}^+) \right)^o. \quad (36)$$

The proof of the following theorem can be found in [ÓP01], Theorem 8.3, or [Pa02a], Theorem 3.5.

**Theorem 6.3.** *Let the notation be as above. Then there exists a  $W_\Theta$ -invariant tubular neighborhood  $U_\Theta$  of  $A_\Theta$  in  $A_c$  such that the function*

$$(\lambda, a) \mapsto \frac{\varphi_\Theta(m; \lambda, a)}{n_\Theta^-(m; \lambda)}$$

*extends as a  $W_\Theta$ -invariant holomorphic function of  $(\lambda, a) \in \mathfrak{a}_c^* \times U_\Theta$ .*

Observe that (if  $\Sigma$  is reduced as we are assuming here) the  $\lambda$ -singularities are at most of first order and located along a locally finite (in general infinite) family of complex affine hyperplanes in  $\mathfrak{a}_c^*$ . We shall see shortly that the situation greatly simplifies when all multiplicities are even.

**Definition 6.4.** Let  $\Theta \subset \Pi$ . The  $\Theta$ -spherical transform of  $f \in C_c^\infty(A_\Theta)^{W_\Theta}$  associated with the data  $(\mathfrak{a}, \Sigma, m)$  is the  $W_\Theta$ -invariant function  $\mathcal{F}_\Theta f(m)$  on  $\mathfrak{a}_c^*$  defined for  $\lambda \in \mathfrak{a}_c^*$  by

$$\mathcal{F}_\Theta f(m; \lambda) := \frac{1}{|W_\Theta|} \int_{A_\Theta} f(a) \varphi_\Theta(m; \lambda, a) \delta(m; a) da = \int_{A^+} f(a) \varphi_\Theta(m; \lambda, a) \delta(m; a) da, \quad (37)$$

where  $da$  is a suitable normalization of the Haar measure on  $A$ .

For every  $\lambda \in \mathfrak{a}_c^*$  let  $E_\Theta(m; \lambda)$  be the  $W_\Theta$ -invariant function on  $A_\Theta$  defined by requiring that the equality

$$E_\Theta(m; \lambda, a) = \frac{c_\Theta^-(m; \lambda) c_\Theta^+(m; \lambda)}{c_\Pi^+(m; \lambda)} \varphi_\Pi(m; \lambda, a), \quad a \in A^+. \quad (38)$$

Then  $E_\Theta(m; \lambda, a)$  is a  $W_\Theta$ -invariant meromorphic function of  $\lambda \in \mathfrak{a}_c^*$ . Notice that, if  $m$  is an even multiplicity function, then  $E_\Theta(m; \lambda, a) = \pm \varphi_\Pi(m; \lambda, a)$  for all  $\Theta$ . The inversion of the  $\Theta$ -spherical transform on  $C_c^\infty(A_\Theta)^{W_\Theta}$  is provided by the following theorem. See [Pa02b], Theorem 4.5.

**Theorem 6.5.** *Let  $f \in C_c^\infty(A_\Theta)^{W_\Theta}$ . Then for all  $a \in A_\Theta$  we have*

$$f(a) = \kappa \frac{|W|}{|W_\Theta|} \int_{i\mathfrak{a}^*} \mathcal{F}_\Theta f(m; \lambda) E_\Theta(m; -\lambda, a) |c_\Theta(m; \lambda)|^{-2} d\lambda, \quad (39)$$

*where  $d\lambda$  is a suitable normalization of the Lebesgue measure on  $i\mathfrak{a}^*$  and  $\kappa$  is a positive constant depending on the normalizations of the measures.*

A Paley-Wiener theorem for the  $\Theta$ -spherical transform is a characterization of the image under  $\mathcal{F}_\Theta(m)$  of the set  $C_c^\infty(A_\Theta)^{W_\Theta}$ . A particularly interesting higher-rank situation in which this theorem has been worked out is the even multiplicity case.

The *even multiplicity case* corresponds to triples  $(\mathfrak{a}, \Sigma, m)$  for which  $\Sigma$  is a reduced root system and  $m_\alpha \in 2\mathbb{N}_0$  for all  $\alpha \in \Sigma$ . In the geometric case of Riemannian symmetric spaces of noncompact type, this situation corresponds to spaces  $G/K$  in which the Lie algebra  $\mathfrak{g}$  of  $G$  possesses a unique

conjugacy class of Cartan subalgebras. Moreover, as already observed in Examples 3.6 and 5.5, multiplicity functions with constant value 2 correspond to  $\mathfrak{g}$ 's admitting a complex structure.

The class of symmetric spaces for which the theory of  $\Theta$ -spherical functions is relevant consists of the  $K_\varepsilon$  symmetric spaces of Oshima and Sekiguchi [OS80]. The most important examples among them are our basic examples, that is the Riemannian symmetric spaces of noncompact type and the NCC symmetric spaces. Let  $G/H$  be a  $K_\varepsilon$  symmetric space with even multiplicity functions with  $G$  a connected, noncompact, simple Lie group. Then, according to classification ([OS80]; see also Appendix A below), either  $\mathfrak{g}$  admits complex structure or the root system of  $G/K$  is of type  $A_n$ . In the latter case,  $G/H$  is either Riemannian or NCC. Furthermore, all irreducible  $K_\varepsilon$  spaces are determined by a signature, corresponding to the choice of a set  $\Theta \subset \Pi$  of positive simple roots so that either  $\Theta = \Pi$  (Riemannian case) or  $|\Pi \setminus \Theta| = 1$ . We shall therefore state the Paley-Wiener theorem for the  $\Theta$ -spherical transform in even multiplicities with the additional assumptions on  $(\mathfrak{a}, \Sigma, m)$  that  $\Sigma$  is a root system of type  $A_n$  and that  $\Theta$  consists of at least  $n - 1$  positive simple roots. As just observed, this is the most significant case from the point of view of the harmonic analysis on symmetric spaces with even multiplicities. By only dealing with the above situation, we can explain the proving methods without entering into the technical difficulties which appear in the general even multiplicity case. For the latter, we refer the reader to [ÓP03].

One of the simplifications occurring in the case of  $A_n$  is that there is only one Weyl group orbit in  $\Sigma$ . This forces the multiplicity function  $m$  to be constant. In the following we shall denote by the same symbol  $m$  the constant even multiplicity function and its value.

Before stating the Paley-Wiener theorem, let us add some important properties of the  $\Theta$ -spherical functions in the even multiplicity case. Compared with the general case, the even multiplicity case (not necessarily  $A_n$ ) is greatly simplified by the fact that the  $\lambda$ -singularities are located only along a *finite* family of complex affine hyperplanes. Moreover – and this will turn out to be one of the key ingredients in proving the Paley-Wiener theorem – there are explicit formulas relating, by means of differential operators, the  $\Theta$ -spherical functions to exponential functions. The main tool for this are *Opdam's shift operators*.

Let  $\mathbb{C}[A_c]$  be the algebra of regular functions on  $A_c$  introduced in Section 4, and let  $\Delta$  be the Weyl denominator as in (21). The algebra  $\mathbb{C}_\Delta[A_c] = \bigcup_{k \in \mathbb{Z}} \Delta^k \mathbb{C}[A_c]$  is the localization of  $\mathbb{C}[A_c]$  along  $\Delta$ . Shift operators are  $W$ -invariant differential operators on  $A$  with coefficients in  $\mathbb{C}_\Delta[A_c]$ . They have a characteristic asymptotic expansion on  $A^+$  and commute with the operator  $L(m)$ . For every even multiplicity function  $m$  there are shift operators “lowering” the Harish-Chandra series  $\Phi_\lambda(m; a)$  to the exponential function  $a^\lambda$  (which is the Harish-Chandra series for  $m = 0$ ), or, vice versa, “raising”  $a^\lambda$  to  $\Phi_\lambda(m; a)$ .

**Lemma 6.6** (Opdam). *Let  $m \in 2\mathbb{N}_0$  be even. Then there exists unique shift operators  $D_-(m)$  and  $D_+(m)$  so that for all  $(\lambda, a) \in (\mathfrak{a}_c^* \setminus P) \times A^+$*

$$D_-(m)\Phi_\lambda(m; a) = \frac{1}{c_{\mathbb{H}}^+(m; \lambda)} a^\lambda, \quad (40)$$

$$D_+(m)a^\lambda = \frac{1}{c_{\mathbb{H}}^+(m; -\lambda)} \Phi_\lambda(m; a). \quad (41)$$

The shift operators  $D_\pm(m)$  of Lemma 6.6 can be constructed by composing fundamental shift operators of shifts  $\pm 2$ . Let  $G_+(m)$  and  $G_-(m)$  respectively denote the fundamental shift operator mapping  $\Phi_\lambda(m; a)$  to  $\Phi_\lambda(m+2; a)$  and the fundamental shift operator mapping  $\Phi_\lambda(m; a)$  to  $\Phi_\lambda(m-2; a)$ . Then  $G_+(m)$  is related to the formal adjoint of  $G_-(m)$  by the relation

$$G_+(m) := \Delta^{-(2+m)} \circ G_-^*(m+2) \circ \Delta^m. \quad (42)$$

We refer to Proposition 4.4 in [Heck97] for general explicit formulas. In the geometric case of Riemannian symmetric spaces,  $D_+(m)$  is a constant multiple of the differential operator inverting

the Abel transform. Further results and explicit formulas have been obtained in this context by Beerends ([Bee88], [Bee87]), Hba [Hba87], and Vretrare [Vre84].

**Example 6.7** (The root system  $A_1$ ). In the rank-one  $A_1$  case there is only one positive root  $\alpha$ . With the identifications introduced in Example 3.5, the shift relations of Lemma 6.6 correspond to the classical differentiation formula for the hypergeometric function

$$\frac{d}{dz} {}_2F_1(a, b; c; z) = \frac{ab}{c} {}_2F_1(a+1, b+1; c+1; z)$$

(cf. e.g. [Er53], 2.8(20)). Seen as differential operators on  $\mathfrak{a}_{\mathbb{C}} \equiv \mathbb{C}$  the elementary shift operators of shift 2 are therefore

$$\begin{aligned} G_-(m) &= \Delta(z) \frac{d}{dz} + (m-1)(e^z + e^{-z}), \\ G_+(m) &= -\frac{1}{\Delta(z)} \frac{d}{dz} \end{aligned}$$

with  $\Delta(z) := e^z - e^{-z}$ . Since  $G_+(m)$  is independent of  $m$ , it follows in particular that

$$D_+(m) = G_+(2)^{m/2}.$$

**Example 6.8** (The root system  $A_2$ ). The fundamental shift operators  $G_+(m)$  for the root system  $A_2$  were determined by Beerends [Bee88]. They are

$$G_+(m) = \Delta^{-1} \left[ \prod_{\alpha \in \Sigma^+} \partial_{\alpha} + \frac{m-2}{2} \sum_{\alpha \in \Sigma^+} \eta(\alpha) \circ \partial_{\alpha} \circ \coth \alpha \circ \partial_{\alpha} \right]$$

where

$$\eta(\alpha) = \prod_{\beta \in \Sigma^+ \setminus \{\alpha\}} \langle \alpha, \beta \rangle.$$

The shift operators  $G_-(m)$  can then be deduced from (42). Notice that the formula simplifies in the complex case, since

$$G_+(0) = \Delta^{-1} \prod_{\alpha \in \Sigma^+} \partial_{\alpha}.$$

Composition yields for instance

$$\begin{aligned} D_+(4) = G_+(2) \circ G_+(0) &= \Delta^{-2} \left[ \partial_{\alpha} \partial_{\beta} \partial_{\gamma} - \sum' |\alpha|^2 \coth(\alpha) \partial_{\beta} \partial_{\gamma} + \sum' |\alpha|^2 |\beta|^2 \coth(\alpha) \coth(\beta) \partial_{\gamma} \right. \\ &\quad \left. - |\alpha|^2 |\beta|^2 |\gamma|^2 \coth(\alpha) \coth(\beta) \coth(\gamma) - \frac{|\alpha|^2 |\beta|^2 |\gamma|^2}{2\Delta} \right] \circ \partial_{\alpha} \partial_{\beta} \partial_{\gamma}, \end{aligned}$$

where  $\sum'$  denotes the sum over the cyclic permutations of the three positive roots  $\alpha, \beta, \gamma = \alpha + \beta$ . The operator  $D_+(4)$  was first computed by Hba [Hba87] as the inverse of the Abel transform on the Riemannian symmetric space  $\mathrm{SL}(3, \mathbb{H})/\mathrm{Sp}(3)$ .

Lemma 6.6 together with Definition 6.1 immediately yields explicit formulas for the  $\Theta$ -spherical functions, at least on  $A^+$ . The difficulty is to extend these formulas to the domain  $A_{\Theta}$ , since the shift operators have singular coefficients on the set  $\{\Delta = 0\}$ . The definition of shift operators ensures that these singularities can be cancelled by multiplication by a suitable power  $\Delta^k$ , but it gives no estimates on  $k$ . Direct estimates from Heckman's explicit formulas for the fundamental shift operators are hard. First of all, these formulas are rather involved in high rank. Moreover, in the composition of shift operators of step 2 many simplifications occur. In fact the exponent  $k$  is lower than one could guess from the form of the single step-2 parts. This occurs for instance for the operator  $D_+(4)$  in the above Example 6.8.

An argument for estimating the value of  $k$  has been presented in [ÓP03], Theorem 4.10. It does not rely on explicit formulas for the shift operators, but on general estimates for the Harish-Chandra series and their derivatives. It states that  $k$  is smaller or equal to  $m$ . This in particular implies that  $D_m := \Delta^m D_+(m)$  extends as  $W$ -invariant differential operator on  $A$  with coefficients in  $\mathbb{C}[A_{\mathbb{C}}]$ .

The regularity properties of the  $\Theta$ -spherical functions in even multiplicities are collected in the following theorem. We state it for the case in which  $m \in 2\mathbb{N}_0$  is a constant multiplicity function, but it can be extended with minor modifications to arbitrary even multiplicities. We refer the reader to [ÓP02] and [ÓP04] for more information.

**Theorem 6.9.** *Suppose  $m \in 2\mathbb{N}_0$ . Let  $\Theta \subset \Pi$  and define*

$$e_{\Theta}^{-}(m; \lambda) := \prod_{\alpha \in \Sigma^+ \setminus \langle \Theta \rangle^+} \prod_{k=-m/2+1}^{m/2-1} (\lambda_{\alpha} - k) \quad (43)$$

for  $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ . We convene that empty products are equal to one.

- (1) *There is a  $W_{\Theta}$ -invariant tubular neighborhood  $U_{\Theta}$  in  $A_{\mathbb{C}}$  of  $A_{\Theta}$  such that the function*

$$e_{\Theta}^{-}(m; \lambda) \varphi_{\Theta}(m; \lambda, a)$$

*extends as a  $W_{\Theta}$ -invariant holomorphic function of  $(\lambda, a) \in \mathfrak{a}_{\mathbb{C}}^* \times U_{\Theta}$ .*

- (2) *There is a  $W$ -invariant differential operator with coefficients in  $\mathbb{C}[A_{\mathbb{C}}]$ , namely  $D_m = \Delta^m D_+(m)$ , so that for all  $(\lambda, a) \in \mathfrak{a}_{\mathbb{C}}^* \times U_{\Theta}$*

$$\Delta^m(a) \varphi_{\Theta}(m; \lambda, a) = (-1)^{d(\Theta, m)} \left[ \prod_{\alpha \in \Sigma^+} \prod_{k=0}^{m/2-1} (k^2 - \lambda_{\alpha}^2) \right]^{-1} D_m \left( \sum_{w \in W_{\Theta}} a^{w\lambda} \right) \quad (44)$$

*with  $d(\Theta, m)$  as in (34).*

- (3) *If  $\Theta = \Pi$ , then  $\Delta^m(a) \varphi_{\Pi}(m; \lambda, a)$  extends as a holomorphic function on  $\mathfrak{a}_{\mathbb{C}}^* \times U_{\Pi}$  by means of the formula*

$$\Delta^m(a) \varphi_{\Pi}(m; \lambda, a) = \left[ \prod_{\alpha \in \Sigma^+} \prod_{k=0}^{m_{\alpha}/2-1} (k^2 - \lambda_{\alpha}^2) \right]^{-1} D_m \left( \sum_{w \in W} e^{w\lambda(\log a)} \right). \quad (45)$$

*If, moreover,  $\lambda \in P$ , then  $\Delta^m(a) \varphi_{\Pi}(m; \lambda, a)$  extends by (45) as  $W$ -invariant entire function on  $A_{\mathbb{C}}$ .*

Observe that in the complex case

$$D_+(m) = \Delta^{-1} \prod_{\alpha \in \Sigma^+} \partial_{\alpha}. \quad (46)$$

In the geometric case, Formula (45) reduces therefore to the classical formula by Harish-Chandra of Example 3.6, whereas (44) with  $\Theta = \Pi_0$  reduces to the results of [FHÓ94] explained in Example 5.5.

The description of the image of the compactly supported functions under the  $\Theta$ -spherical transform is given by means of the following definition (which is given for the general even multiplicity case).

**Definition 6.10** (Paley-Wiener space). Let  $m$  be an even multiplicity function on the root system  $\Sigma$ , and let  $\Theta \subset \Pi$  be a fixed set of positive simple roots. Let  $C$  be a compact, convex and  $W_{\Theta}$ -invariant subset of  $\mathfrak{a}_{\Theta}$ . The *Paley-Wiener space*  $\text{PW}_{\Theta}(m; C)$  is the space of all  $W_{\Theta}$ -invariant meromorphic functions  $g : \mathfrak{a}_{\mathbb{C}}^* \rightarrow \mathbb{C}$  satisfying the following properties:

1.  $e_{\Theta}^{-}(m; \lambda)g(\lambda)$  is a rapidly decreasing entire function of exponential type  $C$ , that is for every  $N \in \mathbb{N}$  there is a constant  $C_N \geq 0$  such that

$$|e_{\Theta}^{-}(m; \lambda)g(\lambda)| \leq C_N(1 + |\lambda|)^{-N} e^{qC(\operatorname{Re} \lambda)}$$

for all  $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ .

2. The function

$$P_{\Theta}^{\operatorname{av}} g(\lambda) := \sum_{w \in W_{\Theta} \backslash W} g(w\lambda) \quad (47)$$

extends to an entire function on  $\mathfrak{a}_{\mathbb{C}}^*$ .

Condition 2 is automatically satisfied in the Euclidean case  $m = 0$ , in the complex case  $m = 2$ , and when  $\Theta = \Pi$ . Indeed,  $e_{\Theta}^{-} \equiv 1$  in the Euclidean case and when  $\Theta = \Pi$ . For the complex case, observe that Condition 1 in Definition 6.10 implies that  $P_{\Theta}^{\operatorname{av}} g$  is  $W$ -invariant and has at most first order singularities along each hyperplane  $\{\lambda \in \mathfrak{a}_{\mathbb{C}}^* \mid \lambda_{\alpha} = 0\}$  with  $\alpha \in \Sigma^+$ . This is only possible when the singularities are in fact removable.

**Theorem 6.11** (Paley-Wiener theorem). *Let  $\Sigma$  be a root system of type  $A_n$  with multiplicity function  $m$ , and let  $\Theta \subset \Pi$  be a set of positive simple root so that  $|\Theta| \geq n - 1$ . Suppose  $C$  is a compact, convex and  $W_{\Theta}$ -invariant subset of  $\mathfrak{a}_{\Theta}$ . Then the  $\Theta$ -spherical transform  $\mathcal{F}_{\Theta}(m)$  maps  $C_c^{\infty}(C)^{W_{\Theta}}$  bijectively onto  $\operatorname{PW}_{\Theta}(m; C)$ .*

Notice first that the case  $\Theta = \Pi$  is contained in the Paley-Wiener theorem for the Opdam transform (Theorem 4.2). An elementary proof of the even multiplicity case with  $\Theta = \Pi$  can also be found in [ÓP03], Corollary 10.2. In the following we shall therefore suppose that  $\Pi \setminus \Theta$  consists of a single element, say  $\Pi \setminus \Theta = \{\beta\}$ .

The fact that the  $\Theta$ -spherical transform maps  $C_c^{\infty}(C)^{W_{\Theta}}$  into the Paley-Wiener space  $\operatorname{PW}_{\Theta}(m; C)$  depends mainly on the explicit formula for the spherical functions. Suppose  $f \in C_c^{\infty}(C)^{W_{\Theta}}$ . Observe that  $e_{\Theta}^{-}(m; \lambda)\mathcal{F}_{\Theta}f(m; \lambda)$  is entire by Theorem 6.9. For  $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$  set

$$e_{\Theta}^{+}(m; \lambda) := (-1)^{m|\langle \Theta \rangle^+|/2} \prod_{\alpha \in \Sigma^+ \setminus \langle \Theta \rangle^+} \prod_{k=-m/2+1}^{m/2-1} (\lambda_{\alpha} - k) \quad (48)$$

(with the convention that empty products are equal to 1). The factor  $\Delta^m$  used in the definition of  $D_m$  appears as density of the measure inside the  $\Theta$ -spherical transform. Hence, Theorem 6.9 and the  $W$ -invariance of  $D_m$  imply

$$\pi(\lambda)e_{\Theta}^{+}(m; \lambda)e_{\Theta}^{-}(m; \lambda)\mathcal{F}_{\Theta}f(m; \lambda) = [\mathcal{F}_A(D_m^* f)](\lambda). \quad (49)$$

The regularity of  $D_m$  ensures that  $D_m^* f$  is a  $W_{\Theta}$ -invariant smooth function with compact support in  $\exp C$ . The classical Paley-Wiener theorem for the Euclidean Fourier transform  $\mathcal{F}_A$  (see Section 2) implies therefore that the function in (49) belongs to  $\operatorname{PW}(C)$ . Now, a classical result of Malgrange states for an entire function  $F$  and a polynomial  $p$  that  $pF \in \operatorname{PW}(C)$  if and only if  $F \in \operatorname{PW}(C)$ . See e.g. [Hel94], Lemma 5.13. This proves the necessity of Condition 1 for  $\mathcal{F}_{\Theta}f$ .

Condition 2 follows for the Paley-Wiener theorem for the Opdam transform. Indeed, every function  $f \in C_c^{\infty}(A_{\Theta})^{W_{\Theta}}$  can be uniquely extended to a  $W$ -invariant function  $f_{\Pi} \in C_c^{\infty}(A)^W$ . The definition of  $\Theta$ -spherical functions yields the relation

$$\varphi_{\Pi}(m; \lambda, a) = (-1)^{d(\Theta, m)} \sum_{W_{\Theta} \backslash W} \varphi_{\Theta}(m; w\lambda, a),$$

where  $d(\Theta, m)$  is as in (34). Consequently

$$(\mathcal{F}_{\Pi} f_{\Pi})(m; \lambda) = (-1)^{d(\Theta, m)} \sum_{W_{\Theta} \backslash W} (\mathcal{F}_{\Theta} f)(m; w\lambda) = (-1)^{d(\Theta, m)} (P_{\Theta}^{\operatorname{av}} \mathcal{F}_{\Theta} f)(m; \lambda).$$

One should remark that in the above arguments the assumptions on  $\Sigma$  and  $\Theta$  play no role. Indeed, the  $\Theta$ -spherical transform maps  $C_c^\infty(C)^{W_\Theta}$  into  $\text{PW}_\Theta(m; C)$  for every root system  $\Sigma$ , every even multiplicity function on  $\Sigma$ , and every choice of  $\Theta \subset \Pi$ . The assumptions made in Theorem 6.11 enter only in the proof of the surjectivity of the transform, which we now outline.

The inversion formula and the explicit formulas for the  $\Theta$ -spherical functions suggest the following definition of wave packets. The *wave packet* of  $g \in \text{PW}_\Theta(m; C)$  is the function  $\mathcal{I}g = \mathcal{I}g(m) : A \rightarrow \mathbb{C}$  defined by

$$(\mathcal{I}g)(a) := \int_{i\mathfrak{a}^*} g(\lambda) \varphi_\Pi(m; -\lambda, a) |c_\Pi^+(m; \lambda)|^{-2} d\lambda \quad (50)$$

The  $\Theta$ -*wave-packet* of  $g$  is the function on  $A_\Theta$  obtained by restriction of  $\mathcal{I}g$  to  $A_\Theta$ , that is

$$\mathcal{I}_\Theta g = \mathcal{I}g \circ \iota_\Theta, \quad (51)$$

where  $\iota_\Theta : A_\Theta \hookrightarrow A$  is the inclusion map.

Notice that  $\mathcal{I}g = \frac{|W_\Theta|}{|W|} \mathcal{I}P_\Theta^{\text{av}} g$ . Hence, for some constant  $\kappa$ ,

$$\Delta^m(a) \mathcal{I}g(a) = \kappa D_m \mathcal{F}_A^{-1}(P_\Theta^{\text{av}} g)(a).$$

Condition 2 in Definition 6.10 and the classical Paley-Wiener theorem therefore ensure that the wave packet  $\mathcal{I}g$  has support contained in  $\exp(\text{conv}(W(C)))$ , where  $\text{conv}(W(C))$  denotes the convex hull of the Weyl group orbit of  $C$ .

The next task is to show that  $\mathcal{I}_\Theta g$  is smooth and compactly supported in  $A_\Theta$ . On the Lie algebra level, we need to separate points in the interior of  $\mathfrak{a}_\Theta$  from points on its boundary. This can be done by means of suitable elements in

$$\mathfrak{a}_\Theta^*(m) = \{\lambda \in \mathfrak{a}_\Theta^* \mid \lambda_\alpha \geq m/2 \text{ for all } \alpha \in \Sigma^+ \setminus \langle \Theta \rangle^+\}.$$

Here  $\mathfrak{a}_\Theta^* := \{\lambda \in \mathfrak{a}^* \mid \lambda(H) \geq 0 \text{ for all } H \in \mathfrak{a}_\Theta\} = \sum_{\alpha \in \Sigma^+ \setminus \langle \Theta \rangle^+} \mathbb{R}_0^+ \alpha$  is the dual cone of  $\mathfrak{a}_\Theta$ . The set  $\mathfrak{a}_\Theta^*(m)$  is introduced because it is a “large” closed subset of  $\mathfrak{a}_\Theta^*$  which is “away” from the possible singularities of every  $g \in \text{PW}_\Theta(m; C)$ . Indeed, each  $g \in \text{PW}_\Theta(m; C)$  is holomorphic in a neighborhood of the convex set  $i\mathfrak{a}^* - \mathfrak{a}_\Theta^*(m)$ . Furthermore, for every  $N \in \mathbb{N}$ , there is a constant  $C_N > 0$  such that for all  $\lambda \in i\mathfrak{a}^*$  and  $\mu \in -\mathfrak{a}_\Theta^*(m)$

$$|g(\lambda + \mu)| \leq C_N (1 + |\lambda|)^{-N} e^{q_C(\mu)}. \quad (52)$$

This allows to shift the contour of integration and get for all  $\mu \in -\mathfrak{a}_\Theta^*(m)$  and  $a \in A$ ,

$$\Delta^m(a) \mathcal{I}g(a) = \sum_{w \in W} D_m \int_{i\mathfrak{a}^*} g(\lambda + \mu) a^{-w(\lambda + \mu)} d\lambda. \quad (53)$$

The assumption that  $\Sigma$  is of type  $A_n$  ensures that  $\langle \beta, \alpha \rangle \geq 0$  for all  $\alpha \in \Sigma^+ \setminus \langle \Theta \rangle^+$ . This is used to prove that, for  $a \in A_\Theta$ , only the summands of (53) corresponding to  $w \in W_\Theta$  are nonzero. Hence for all  $\mu \in -\mathfrak{a}_\Theta^*(m)$  and  $a \in A_\Theta$  we have

$$\Delta^m(a) \mathcal{I}_\Theta g(a) = |W_\Theta| D_m \int_{i\mathfrak{a}^*} g(\lambda + \mu) a^{-(\lambda + \mu)} d\lambda. \quad (54)$$

Formula (54) yields the shift of contour of integration to show that the support of  $\mathcal{I}_\Theta g$  is a compact subset of  $A_\Theta$ .

The final step, which proves that the support of  $\mathcal{I}_\Theta g$  is indeed contained in  $\exp C$ , requires a certain application of Holmgren’s theorem. Holmgren’s theorem has been employed in the proof of Paley-Wiener type theorems also in [vdBS97], but, to be able to apply it to our situation, several adjustments are required.

The basic difficulty in working with  $\mathcal{I}_\Theta g$  is due to the possible  $\lambda$ -singularities of  $g$ . Of course one would like to replace  $g$  with  $e_\Theta^-(m; \lambda)g(\lambda)$ , which is entire. For this, the trick is to use suitable differential operators. The polynomial

$$q(m; \lambda) := \prod_{\alpha \in \Sigma} \prod_{k=-m/2+1}^{m/2-1} (\lambda_\alpha - k)$$

is divided by  $e_\Theta^-(m; \lambda)$  and belongs to  $S(\mathfrak{a}_\mathbb{C})^W$ . Let  $D(m; q) \in \mathbb{D}(\mathfrak{a}, \Sigma, m)$  denote the associated differential operator. Since the  $\Theta$ -spherical functions solve (29), we have on  $A^+$

$$D(m; q)\varphi_\Theta(m; \lambda, a) = q(m; \lambda)\varphi_\Theta(m; \lambda, a). \quad (55)$$

The differential operator  $D(m; q)$  might be singular on the set  $\{\Delta = 0\}$ , but we can always choose  $k \geq m$  so that  $D_q := \Delta^k D(m; q)$  is a  $W$ -invariant differential operator on  $A$  with real analytic coefficients. Moreover, (55) gives, for some constant  $\kappa$

$$D_q(\mathcal{I}_\Theta g)(a) = \kappa \Delta^{k-m}(a) D_m \mathcal{F}_A^{-1}(qg)(a)$$

with  $qg \in PW(C)$ . The classical Paley-Wiener theorem now ensures that  $\text{supp } D_q(\mathcal{I}_\Theta g) \subset \exp C$ . We have already proven that  $\mathcal{I}_\Theta g$  is compactly supported. If the leading symbol of  $D_q$  never vanished, then Holmgren's uniqueness theorem would imply that

$$\text{conv}(\text{supp } D_q(\mathcal{I}_\Theta g)) = \text{conv}(\text{supp } \mathcal{I}_\Theta g). \quad (56)$$

The problem is that, by construction, our differential operator  $D_q$  has zeros along hyperplanes determined by  $\alpha = 0$  with  $\alpha \in \Sigma$ . The necessary extension of Holmgren's uniqueness theorem was accomplished in [ÓP04]. Hence (56) in fact holds. For details we refer the reader to [ÓP03] and [ÓP04].

#### APPENDIX A. $K_\varepsilon$ -SYMMETRIC SPACES WITH EVEN MULTIPLICITIES

In this appendix we report the infinitesimal classification of  $K_\varepsilon$ -symmetric spaces with even multiplicities by listing the  $K_\varepsilon$ -symmetric pairs  $(\mathfrak{g}, \mathfrak{h})$  with even multiplicities for which  $\mathfrak{g}$  is simple and noncompact. The list has been extracted from the classification due to Oshima and Sekiguchi [OS80]. It is presented in three tables respectively collecting (for the even multiplicity case) the Riemannian symmetric pairs (Table 1), the non-compactly causal (NCC) symmetric pairs (Table 2) and the other  $K_\varepsilon$  symmetric pairs (Table 3). A non-Riemannian  $K_\varepsilon$ -symmetric pair is said to be of type  $K_\varepsilon I$  if its signature  $\varepsilon$  comes from a gradation of first kind according to [Ka96]. Otherwise it is said to be of type  $K_\varepsilon II$ . The symmetric pairs of type  $K_\varepsilon I$  coincide with the NCC symmetric pairs. Table 3 therefore collects all symmetric pairs with even multiplicities of type  $K_\varepsilon II$ .

The restricted root system  $\Sigma$  of a  $K_\varepsilon$ -symmetric pair with even multiplicities has at most two root lengths. The classification below shows that all multiplicities  $m_\alpha$  of  $\Sigma$  are equal, and moreover that they are all equal to 2 for symmetric pairs of type  $K_\varepsilon II$ . The restricted root system and multiplicities of a  $K_\varepsilon$ -symmetric pair  $(\mathfrak{g}, \mathfrak{h})$  coincide with those of the corresponding Riemannian dual symmetric pair  $(\mathfrak{g}, \mathfrak{k})$ . They are explicitly reported in Tables 2 and 3 for the reader's convenience.

If  $\Sigma$  is of type  $X_n$  (with  $X_n \in \{A_n, B_n, C_n, \dots\}$ ), then the index  $n$  denotes the real rank of  $\mathfrak{g}$ . The range for  $n$  is chosen to avoid overlappings due to isomorphisms of symmetric spaces. These isomorphisms arise from isomorphisms of the lower dimensional complex Lie algebras. We refer to [Hel78], Ch. X, §6, for more information. After each table below we report the relevant symmetric pair isomorphisms.

$\mathfrak{g}$	$\mathfrak{h} = \mathfrak{k}$	$\Sigma$	$m_\alpha$	
$\mathfrak{sl}(n, \mathbb{C})$	$\mathfrak{su}(n)$	$A_{n-1}$	2	$n \geq 2$
$\mathfrak{so}(2n+1, \mathbb{C})$	$\mathfrak{so}(2n+1)$	$B_n$	2	$n \geq 2$
$\mathfrak{sp}(n, \mathbb{C})$	$\mathfrak{sp}(n)$	$C_n$	2	$n \geq 3$
$\mathfrak{so}(2n, \mathbb{C})$	$\mathfrak{so}(2n)$	$D_n$	2	$n \geq 4$
$(\mathfrak{e}_6)_\mathbb{C}$	$\mathfrak{e}_6$	$E_6$	2	
$(\mathfrak{e}_7)_\mathbb{C}$	$\mathfrak{e}_7$	$E_7$	2	
$(\mathfrak{e}_8)_\mathbb{C}$	$\mathfrak{e}_8$	$E_8$	2	
$(\mathfrak{f}_4)_\mathbb{C}$	$\mathfrak{f}_4$	$F_4$	2	
$(\mathfrak{g}_2)_\mathbb{C}$	$\mathfrak{g}_2$	$G_2$	2	
$\mathfrak{su}^*(2n)$	$\mathfrak{sp}(n)$	$A_{n-1}$	4	$n \geq 2$
$\mathfrak{e}_{6(-26)}$	$\mathfrak{f}_{4(-20)}$	$A_2$	8	
$\mathfrak{so}(2n+1, 1)$	$\mathfrak{so}(2n+1)$	$A_1$	$2n$	$n \geq 3$

TABLE 1. Riemannian symmetric pairs with even multiplicities.

Special isomorphisms of Riemannian symmetric spaces with even multiplicities are

$$\begin{aligned}
\mathfrak{so}(3, \mathbb{C}) = \mathfrak{sp}(1, \mathbb{C}) \approx \mathfrak{sl}(2, \mathbb{C}), & \quad \mathfrak{so}(3) = \mathfrak{sp}(1) \approx \mathfrak{su}(2); \\
\mathfrak{sp}(2, \mathbb{C}) \approx \mathfrak{so}(5, \mathbb{C}), & \quad \mathfrak{sp}(2) \approx \mathfrak{so}(5); \\
\mathfrak{so}(6, \mathbb{C}) \approx \mathfrak{sl}(4, \mathbb{C}), & \quad \mathfrak{so}(6) \approx \mathfrak{su}(4); \\
\mathfrak{so}(3, 1) \approx \mathfrak{sl}(2, \mathbb{C}), & \quad \mathfrak{so}(3) \approx \mathfrak{su}(2); \\
\mathfrak{so}(5, 1) \approx \mathfrak{su}^*(4), & \quad \mathfrak{so}(5) \approx \mathfrak{sp}(2).
\end{aligned}$$

The Lie algebra  $\mathfrak{so}(2, \mathbb{C})$  is not semisimple. Observe also that  $\mathfrak{so}(4, \mathbb{C}) \cong \mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C})$  is not simple. The structure of its homogeneous spaces can be therefore deduced from the structure of the homogeneous spaces of  $\mathfrak{sl}(2, \mathbb{C})$ .

In the next table we list all the non-compactly causal, or  $K_\varepsilon I$ , symmetric pairs with even multiplicities. The third column reports the subalgebra of  $\mathfrak{g}$  fixed by  $\theta\theta_\varepsilon$ , where  $\theta_\varepsilon$  is the involution associated with the  $K_\varepsilon$ -pair  $(\mathfrak{g}, \mathfrak{h})$ .

Special isomorphisms of NCC symmetric pairs with even multiplicities are

$$\begin{aligned}
\mathfrak{so}(3, \mathbb{C}) = \mathfrak{sp}(1, \mathbb{C}) \approx \mathfrak{sl}(2, \mathbb{C}), & \quad \mathfrak{so}(1, 2) \approx \mathfrak{sp}(1, \mathbb{R}) \approx \mathfrak{su}(1, 1); \\
\mathfrak{sp}(2, \mathbb{C}) \approx \mathfrak{so}(5, \mathbb{C}), & \quad \mathfrak{sp}(2, \mathbb{R}) \approx \mathfrak{so}(3, 2); \\
\mathfrak{so}(6, \mathbb{C}) \approx \mathfrak{sl}(4, \mathbb{C}), & \quad \mathfrak{so}(4, 2) \approx \mathfrak{su}(2, 2); \\
\mathfrak{so}(6, \mathbb{C}) \approx \mathfrak{sl}(4, \mathbb{C}), & \quad \mathfrak{so}^*(6) \approx \mathfrak{su}(3, 1); \\
\mathfrak{so}(3, 1) \approx \mathfrak{sl}(2, \mathbb{C}), & \quad \mathfrak{so}(2, 1) \approx \mathfrak{su}(1, 1); \\
\mathfrak{so}(5, 1) \approx \mathfrak{su}^*(4), & \quad \mathfrak{so}(4, 1) \approx \mathfrak{sp}(1, 1); \\
\mathfrak{so}(8, \mathbb{C}) = \mathfrak{so}(8, \mathbb{C}), & \quad \mathfrak{so}^*(8) \approx \mathfrak{so}(2, 6);
\end{aligned}$$

$\mathfrak{g}$	$\mathfrak{h}$	$\mathfrak{g}^{\theta\theta_\varepsilon}$	$\Sigma$	$m_\alpha$	
$\mathfrak{sl}(n, \mathbb{C})$	$\mathfrak{su}(n-j, j)$	$\mathfrak{sl}(n-j, \mathbb{C}) \times \mathfrak{sl}(j, \mathbb{C}) \times \mathbb{C}$	$A_{n-1}$	2	$n \geq 2, 1 \leq j \leq [n/2]$
$\mathfrak{so}(2n+1, \mathbb{C})$	$\mathfrak{so}(2n-1, 2)$	$\mathfrak{so}(2n-1, \mathbb{C}) \times \mathbb{C}$	$B_n$	2	$n \geq 2$
$\mathfrak{sp}(n, \mathbb{C})$	$\mathfrak{sp}(n, \mathbb{R})$	$\mathfrak{gl}(n, \mathbb{C})$	$C_n$	2	$n \geq 3$
$\mathfrak{so}(2n, \mathbb{C})$	$\mathfrak{so}(2n-2, 2)$	$\mathfrak{so}(2n-2, \mathbb{C}) \times \mathbb{C}$	$D_n$	2	$n \geq 4$
$\mathfrak{so}(2n, \mathbb{C})$	$\mathfrak{so}^*(2n)$	$\mathfrak{gl}(n, \mathbb{C})$	$D_n$	2	$n \geq 5$
$(\mathfrak{e}_6)_{\mathbb{C}}$	$\mathfrak{e}_{6(-14)}$	$\mathfrak{so}(10, \mathbb{C}) \times \mathbb{C}$	$E_6$	2	
$(\mathfrak{e}_7)_{\mathbb{C}}$	$\mathfrak{e}_{7(-25)}$	$(\mathfrak{e}_6)_{\mathbb{C}} \times \mathbb{C}$	$E_7$	2	
$\mathfrak{su}^*(2n)$	$\mathfrak{sp}(n-j, j)$	$\mathfrak{su}^*(2(n-j)) \times \mathfrak{su}^*(2j) \times \mathbb{R}$	$A_{n-1}$	4	$n \geq 2, 1 \leq j \leq [n/2]$
$\mathfrak{e}_{6(-26)}$	$\mathfrak{f}_{4(-20)}$	$\mathfrak{so}(9, 1) \times \mathbb{R}$	$A_2$	8	
$\mathfrak{so}(2n+1, 1)$	$\mathfrak{so}(2n, 1)$	$\mathfrak{so}(2n+1) \times \mathbb{R}$	$A_1$	$2n$	$n \geq 3$

TABLE 2. Non-compactly causal symmetric pairs with even multiplicities.

The last table contains all the other  $K_\varepsilon$ -symmetric pairs, i.e. those of type  $K_\varepsilon II$ , with even multiplicities.

$\mathfrak{g}$	$\mathfrak{h}$	$\mathfrak{g}^{\theta\theta_\varepsilon}$	$\Sigma$	$m_\alpha$	
$\mathfrak{so}(2n+1, \mathbb{C})$	$\mathfrak{so}(2(n-j)+1, 2j)$	$\mathfrak{so}(2(n-j)+1, \mathbb{C}) \times \mathfrak{so}(2j, \mathbb{C})$	$B_n$	2	$n \geq 2, 2 \leq j \leq n$
$\mathfrak{sp}(n, \mathbb{C})$	$\mathfrak{sp}(n-j, j)$	$\mathfrak{sp}(n-j, \mathbb{C}) \times \mathfrak{sp}(j, \mathbb{C})$	$C_n$	2	$n \geq 3, 1 \leq j \leq [n/2]$
$\mathfrak{so}(2n, \mathbb{C})$	$\mathfrak{so}(2(n-j), 2j)$	$\mathfrak{so}(2(n-j), \mathbb{C}) \times \mathfrak{so}(2j, \mathbb{C})$	$D_n$	2	$n \geq 4, 2 \leq j \leq [n/2]$
$(\mathfrak{e}_6)_{\mathbb{C}}$	$\mathfrak{e}_{6(2)}$	$\mathfrak{sl}(6, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C})$	$E_6$	2	
$(\mathfrak{e}_7)_{\mathbb{C}}$	$\mathfrak{e}_{7(7)}$	$\mathfrak{sl}(8, \mathbb{C})$	$E_7$	2	
$(\mathfrak{e}_7)_{\mathbb{C}}$	$\mathfrak{e}_{7(-5)}$	$\mathfrak{so}(12, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C})$	$E_7$	2	
$(\mathfrak{e}_8)_{\mathbb{C}}$	$\mathfrak{e}_{8(8)}$	$\mathfrak{so}(16, \mathbb{C})$	$E_8$	2	
$(\mathfrak{e}_8)_{\mathbb{C}}$	$\mathfrak{e}_{8(-24)}$	$\mathfrak{e}_{7\mathbb{C}} \times \mathfrak{sl}(2, \mathbb{C})$	$E_8$	2	
$(\mathfrak{f}_4)_{\mathbb{C}}$	$\mathfrak{f}_{4(4)}$	$\mathfrak{sp}(3, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C})$	$F_4$	2	
$(\mathfrak{f}_4)_{\mathbb{C}}$	$\mathfrak{f}_{4(-20)}$	$\mathfrak{so}(9, \mathbb{C})$	$F_4$	2	
$(\mathfrak{g}_2)_{\mathbb{C}}$	$\mathfrak{g}_{2(2)}$	$\mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C})$	$G_2$	2	

TABLE 3. Other  $K_\varepsilon$ -symmetric pairs with even multiplicities.

A special isomorphism of  $K_\varepsilon II$  symmetric pairs with even multiplicities is

$$\mathfrak{sp}(2, \mathbb{C}) \approx \mathfrak{so}(5, \mathbb{C}), \quad \mathfrak{sp}(1, 1) \approx \mathfrak{so}(1, 4).$$

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DEPARTMENT OF MATHEMATICS, LOUISIANA STATE UNIVERSITY, BATON ROUGE, LA 70803, U.S.A.  
*E-mail address:* olafsson@math.lsu.edu

FAKULTÄT FÜR MATHEMATIK, RUHR-UNIVERSITÄT BOCHUM, UNIVERSITÄTSTRASSE 150, GEBÄUDE NA 4/65,  
 44780 GERMANY.  
*E-mail address:* pasquale@cplx.ruhr-uni-bochum.de