LAGUERRE FUNCTIONS ON SYMMETRIC CONES AND RECURSION RELATIONS IN THE REAL CASE

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Abstract. In this article we derive differential recursion relations for the Laguerre functions on the cone Ω of positive definite real matrices. The highest weight representations of the group $\mathrm{Sp}(n,\mathbb{R})$ play a fundamental role. Each such representation acts on a Hilbert space of holomorphic functions on the tube domain $\Omega + i\mathrm{Sym}(n,\mathbb{R})$. We then use the Laplace transform to carry the Lie algebra action over to $L^2(\Omega, d\mu_\nu)$. The differential recursion relations result by restricting to a distinguished three dimensional subalgebra, which is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$.

Introduction

The theory of special functions has its origins in the late eighteenth and early nineteenth centuries when it was seen that the algebraic, exponential, and trigonometric functions (and their inverses) were not adequate to express results to differential equations that arose in the context of some important physical problems. New functions arose to which we have associated names like Bessel, Hermite, Jacobi, Laguerre, and Legendre. Then there are the Gamma, Beta, Hypergeometric and many other families of special functions. By the latter half of the 19^{th} century these same functions arose in different contexts and their name 'special' began to take on greater meaning. Their functional properties were explored and included functional relations, differential and difference recursion relations, orthogonality relations, integral relations, and others.

The preface of Vilenkin's book [17] notes that the connection between special functions and group representations was first discovered by È. Cartan in the early part of the 20^{th} century. By the time Vilenkin's book appeared in the 1960's that interplay had been well established. The texts by Miller, Vilenkin, and Vilenkin and Klimyk [10, 17, 18], for example, well document the general philosophy. In short, group representation theory made it possible to express the classical special functions as matrix entries of a representation and to unify many of the disparate relationships mentioned above. The representation can then be used to derive differential equations and differential recursion relations for those functions.

In 1964 Simon Gindikin published his paper: 'Analysis on Homogeneous Domains', cf. [6]. This important paper developed special functions as part of analysis on homogeneous convex cones and built upon the earlier work of C. L. Siegel [16] on the cone of positive definite matrices. The Siegel integral of the first and second kind generalize to become the Beta and Gamma functions for the cone, respectively. Generalized hypergeometric functions are extended to homogeneous cones. Many important differential properties also extend.

Around the same time M. Koecher [7, 8] began to develop his analysis on symmetric cones and the complex tube domains associated with them. Jordan algebras proved to be a decisive tool for framing and obtaining many important fundamental results. The outstanding text by Faraut and Koranyi [5] documents this interaction (see also its extensive bibliography). Nevertheless, the representation theory

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of Hermitian groups, which are naturally associated with tube domains, is not used in any outstanding way.

In a series of papers [1, 2, 3, 4] the second and third authors (with Genkai Zhang in the first two referenced articles) use the representation theory of Hermitian groups in a decisive way to obtain differential and difference recursion relations on series of special functions. In the context of bounded symmetric domains the relevant special functions are generalized Meixner polynomials and in the context of tube domains over a symmetric cone the relevant special functions are Laguerre functions. These special functions exist in distinguished L^2-spaces, which are unitarily isomorphic to Hilbert spaces of holomorphic functions on either a bounded symmetric domain or a tube domain. The well-known representation theory that exists there then transfers to the corresponding L^2-space to produce differential and difference relations that exist among the special functions. One cannot downplay the essential role that Jordan algebras play in establishing and expressing many of the fundamental results obtained about orthogonal families of special functions defined on symmetric cones. Nevertheless, the theory of highest weight representations add fundamental new results not otherwise easily obtained.

In this present paper we will continue the themes outlined in the above mentioned papers for the Laguerre functions defined of the cone of positive definite real symmetric matrices. The underlying group is $\mathrm{Sp}(n,\mathbb{R})$ and its representation theory establishes new differential recursion relations that Laguerre functions satisfy. The case $n=1$ reduces to the classical Laguerre functions defined on $\mathbb{R}^+$. Briefly, the classical Laguerre polynomials are defined by the formula

$$L_m^\nu(x) = \sum_{k=0}^{m} \frac{\Gamma(m+\nu)}{\Gamma(k+\nu)} \binom{m}{k} (-x)^k$$

and the Laguerre functions are defined by

$$\ell_m^\nu(x) = e^{-x} L_m^\nu(2x).$$

The classical differential recursion relations are then expressed by the following three formulas:

(1) $xD^2 + \nu D - x)\ell_m^\nu(x) = -(2m+\nu)\ell_m^\nu(x)$
(2) $xD^2 + (2x+\nu)D + (t+\nu))\ell_m^\nu(x) = -2m(\nu+m-1)\ell_{m-1}^\nu(x)$
(3) $xD^2 - (2x-\nu)D + (t-\nu))\ell_m^\nu(x) = -2\ell_{m+1}^\nu(x)$.

It is these three formulas that we generalize via the representation theory of $Sp(n,\mathbb{R})$ to Laguerre functions defined on the cone of positive definite real symmetric matrices. (Some definitions of L_m^ν include a factor of $1/n!$. This is the case in [1] but not [2]. The inclusion of this factor changes the differential recursion relations slightly.)

This article is organized as follows: In the first section we introduce some standard Jordan algebra notation. In particular, we introduce the Laguerre functions and polynomials. Even though this material and most of the material in Sections 2 and 3 hold in general for simple Euclidean Jordan algebras, we specialize to the case $J = \mathrm{Sym}(n,\mathbb{R})$, the Jordan algebra of symmetric real $n\times n$-matrices. In Section 2 we introduce the tube domain $T(\Omega) = \Omega + i\mathrm{Sym}(n,\mathbb{R})$, where Ω is the open self dual cone of positive definite matrices. We also discuss the structure of the group $\mathrm{Sp}(n,\mathbb{R})$ and its Lie algebra $\mathfrak{sp}(n,\mathbb{R})$. Some important subalgebras of $\mathfrak{sp}(n,\mathbb{C})$ are introduced. This structure is later used to construct the differential operators that give rise to the differential equations satisfied by the Laguerre functions.

Section 3 is devoted to the discussion of the highest weight representations $(\pi_\nu, \mathcal{H}_\nu(T(\Omega)))$. We also introduce the Laplace transform as a special case of the *restriction principle* introduced in [13]. In Section 4 we describe how the Lie algebra, $\mathfrak{sp}(n,\mathbb{C})$ acts on $\mathcal{H}_\nu(T(\Omega))$. In particular, Proposition 4.2 gives an explicit formula for the action of the derived representation for each of the three subalgebras $\mathfrak{k}_\mathbb{C}$, $\mathfrak{p}^+$, and $\mathfrak{p}^-$, whose direct sum is $\mathfrak{sp}(n,\mathbb{C})$. It should be noted, however, that not all of this information is needed to establish the differential recursion relations for the Laguerre functions. In fact, only the action of three elements are needed. The action of $\mathfrak{sp}(n,\mathbb{C})$ on $L^2(\Omega, d\mu_\nu)$ is described in Section 5. The result is the following theorem:

Theorem 5.2. *For $f \in L^2_\nu(\Omega)$ a smooth vector we have:*

(1) $\lambda_\nu(X)f(x) = \mathrm{tr}[(bx + (ax - xa - \nu b)\nabla - x\nabla b\nabla]f(x),\ X = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \in \mathfrak{k}_{\mathbb{C}}$

(2) $\lambda_\nu(X)f(x) = \mathrm{tr}[(\nu a + ax + (ax + xa + \nu a)\nabla + x\nabla a\nabla]f(x),\ X = \begin{pmatrix} a & a \\ -a & -a \end{pmatrix} \in \mathfrak{p}^+$

(3) $\lambda_\nu(X)f(x) = \mathrm{tr}[(\nu a - ax + (ax + xa - \nu a)\nabla - x\nabla a\nabla]f(x),\ X = \begin{pmatrix} a & -a \\ a & -a \end{pmatrix} \in \mathfrak{p}^-$

Here we use ∇ to denote the gradient, $\mathfrak{k}_{\mathbb{C}}$ is the complexification of the Lie subalgebra $\mathfrak{u}(n) \subset \mathfrak{sp}(n,\mathbb{R})$, and $\mathfrak{p}^\pm$ are certain Abelian subalgebras of $\mathfrak{sp}(n,\mathbb{C})$ on which $\mathfrak{k}_{\mathbb{C}}$ acts. We explain the main ideas for the special case of $\mathrm{Sp}(1,\mathbb{R}) \simeq \mathrm{SL}(2,\mathbb{R})$ and present the lengthy proof of the theorem in the Appendix.

Specializing the above results to the elements

$$\mathrm{X} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \in \mathfrak{p}^+ , \quad \mathrm{Y} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \in \mathfrak{p}^- , \quad \text{and} \quad \mathrm{Z} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{k}_{\mathbb{C}} ,$$

where 1 stands for the $n \times n$ identity matrix, using properties of highest weight representations, and employing Lemma 5.5 of [2] we get our main result:

Theorem 6.3. *The Laguerre functions are related by the following differential recursion relations:*

(1) $\mathrm{tr}(-x\nabla\nabla - \nu\nabla + x)\ell^\nu_{\mathbf{m}}(x) = (n\nu + 2|\mathbf{m}|)\ell^\nu_{\mathbf{m}}(x)$.

(2) $\mathrm{tr}(x\nabla\nabla + (\nu I + 2x)\nabla + (\nu I + x))\ell^\nu_{\mathbf{m}}(x) = -2\sum_{j=1}^r \begin{pmatrix} \mathbf{m} \\ \mathbf{m} - \mathbf{e_j} \end{pmatrix} (m_j - 1 + \nu - (j-1))\ell^\nu_{\mathbf{m}-\mathbf{e_j}}(x)$

(3) $\mathrm{tr}(-x\nabla\nabla + (-\nu I + 2x)\nabla + (\nu I - x))\ell^\nu_{\mathbf{m}}(x) = 2\sum_{j=1}^r c_{\mathbf{m}}(j)\ell^\nu_{\mathbf{m}+\mathbf{e_j}}(x)$.

1. The Jordan Algebra of Real Symmetric Matrices

In this section we introduce the Jordan algebra J of real symmetric matrices. We then discuss the space of L-invariant polynomial functions on J and the Γ-function associated to the cone of symmetric positive matrices. Finally we introduce the generalized Laguerre functions and polynomials.

1.1. **The Jordan Algebra $J = \mathrm{Sym}(n,\mathbb{R})$.** We denote by J the vector space of all real symmetric $n \times n$ matrices. The multiplication $x \circ y = \frac{1}{2}(xy + yx)$ and the inner product $(x\,|\,y) = \mathrm{tr}(xy)$ turn J into a real Euclidean simple Jordan algebra. The determinant and trace functions for J are the usual determinant and trace of an $n \times n$ matrix and will be denoted det and tr, respectively. Observe that $\dim J := d = \frac{n(n+1)}{2}$. Let Ω denote the interior of the cone of squares: $\{x^2 \mid x \in J\}$. Then Ω is the set of all positive definite matrices in J. Let

$$H(\Omega) = \{g \in \mathrm{GL}(J) \mid g\Omega = \Omega\}$$

and let H be the connected component of the identity of $H(\Omega)$. Then H can be identified with $\mathrm{GL}(n,\mathbb{R})_+$ (where $+$ indicates positive determinant) acting on Ω by the formula

$$g \cdot x = gxg^t, \quad g \in H,\ x \in \Omega.$$

This action is transitive and, since Ω is self-dual, it follows that Ω is a symmetric cone. Let L be the stability subgroup of the identity $e \in \Omega$. Then $L = \mathrm{SO}(n,\mathbb{R})$ and

$$\Omega \simeq H/L\,. \tag{1.1}$$

Let $E_{i,i}$ be the diagonal $n \times n$ matrix with 1 in the (i,i)-position and zeros elsewhere. Then $(E_{1,1}, \ldots, E_{n,n})$ is a Jordan frame for J. Let $J^{(k)}$ be the $+1$-eigenspace of the idempotent $E_{1,1}+\cdots+E_{k,k}$ acting on J by multiplication. Each $J^{(k)}$ is a Jordan subalgebra and we have

$$J^{(1)} \subset J^{(2)} \subset \cdots \subset J^{(n)} = J.$$

If $\det_k$ is the determinant function for J_k and P_k is orthogonal projection of J onto $J^{(k)}$ then the function $\Delta_k(x) = \det_k P_k(x)$ is the usual k^{th} principal minor for an $n \times n$ symmetric matrix; it is homogeneous of degree k. In particular $\Delta(x) := \Delta_n(x) = \det(x)$. Note also that

$$\Delta(h \cdot x) = \det(h)^2 \Delta(x)\,, \qquad \forall h \in H\,. \tag{1.2}$$

For $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{C}^n$ we write $\mathbf{m} \geq 0$ if each m_i is a nonnegative integer and $m_1 \geq m_2 \geq \cdots \geq m_n \geq 0$. We let $\Lambda = \{\mathbf{m} \mid \mathbf{m} \geq 0\}$. For each $\mathbf{m} \in \Lambda$ define

$$\Delta_{\mathbf{m}} = \Delta_1^{m_1-m_2} \Delta_2^{m_2-m_3} \cdots \Delta_{n-1}^{m_{n-1}-m_n} \Delta_n^{m_n}.$$

These are the *generalized power functions*. It is not hard to see that the degree of $\Delta_{\mathbf{m}}$ is $|\mathbf{m}| := m_1 + \cdots + m_n$. Observe that each generalized power function extends to a holomorphic polynomial on $J_{\mathbb{C}} = \mathrm{Sym}(n, \mathbb{C})$ in a unique way.

1.2. **The L-invariant Polynomials.** For each $\mathbf{m} \in \Lambda$ we define an L-invariant polynomial, $\psi_{\mathbf{m}}$, on $J_{\mathbb{C}}$ by

$$\psi_{\mathbf{m}}(z) = \int_L \Delta_{\mathbf{m}}(lz)\, dl, \quad z \in J_{\mathbb{C}}$$

where dl is normalized Haar measure on L. A well known theorem of Schmid (c.f. [15]) gives

Lemma 1.1. *If $\mathcal{P}(J_{\mathbb{C}})$ is the space of all polynomial functions on $J_{\mathbb{C}}$ and $\mathcal{P}(J_{\mathbb{C}})^L$ denotes the space of L-invariant polynomials then $\{\psi_{\mathbf{m}} \mid \mathbf{m} \geq 0\}$ is a basis of $\mathcal{P}(J_{\mathbb{C}})^L$. Furthermore, if $\mathcal{P}_k(J_{\mathbb{C}})^L$ denotes the space of L-invariant polynomials of degree less than or equal k then $\{\psi_{\mathbf{m}} \mid |\mathbf{m}| \leq k\}$ is a basis of $\mathcal{P}_k(J_{\mathbb{C}})^L$.*

This lemma implies among other things that $\psi_{\mathbf{m}}(e+x)$ is a linear combination of $\psi_{\mathbf{n}}$, $|\mathbf{n}| \leq |\mathbf{m}|$. The *generalized binomial coefficients*, $\binom{\mathbf{m}}{\mathbf{n}}$, are defined by the equation:

$$\psi_{\mathbf{m}}(e+x) = \sum_{|\mathbf{n}| \leq |\mathbf{m}|} \binom{\mathbf{m}}{\mathbf{n}} \psi_{\mathbf{n}}(x).$$

1.3. **The Generalized Gamma Function.** For $\mathbf{m} \in \mathbb{C}^n$ we define $\Delta_{\mathbf{m}}(x)$, $x \in \Omega$, by the same formula given above for $\mathbf{m} \in \Lambda$. The *generalized Gamma function* is defined by

$$\Gamma_{\Omega}(\mathbf{m}) = \int_{\Omega} e^{-\operatorname{tr} x} \Delta_{\mathbf{m}}(x) \Delta(x)^{\frac{-(n+1)}{2}}\, dx.$$

Conditions for convergence of this integral are given in the proposition below. If λ is a real number we will associate the multi-index $(\lambda, \cdots, \lambda)$ and denote it by λ as well. The context of use should not cause confusion. Thus we define

$$(\lambda)_{\mathbf{m}} = \frac{\Gamma_{\Omega}(\lambda + \mathbf{m})}{\Gamma_{\Omega}(\lambda)}.$$

For later reference we note the following facts about the generalized Gamma function:

Proposition 1.2. *Let the notation be as above. Then the following holds:*

(1) *If* $\mathbf{m} = (m_1, m_2, \ldots, m_n) \in \mathbb{C}^n$ *then the integral defining the generalized Gamma function converges if* $\mathrm{Re}(m_j) > (j-1)\frac{1}{2}$*, for* $j = 1, \ldots, n$*, and in this case*

$$\Gamma_\Omega(\mathbf{m}) = (2\pi)^{\frac{n(n-1)}{4}} \prod_{i=1}^{n} \Gamma(m_i - (i-1)\frac{1}{2}),$$

where Γ *is the usual Gamma function. In particular it follows, that the* Γ*-function has a meromorphic continuation to all of* $\mathbb{C}^n$*.*

(2) *If* $\mathbf{e}_k$ *is an* n*-vector with* 1 *in the* k^{th} *position and* 0*'s elsewhere then the following holds for all* $\mathbf{m} \in \mathbb{C}^n$*:*

(a) $\dfrac{\Gamma_\Omega(\mathbf{m})}{\Gamma_\Omega(\mathbf{m} - \mathbf{e}_k)} = m_k - 1 - (k-1)\dfrac{1}{2}$;

(b) $\dfrac{\Gamma_\Omega(\mathbf{m} + \mathbf{e}_k)}{\Gamma_\Omega(\mathbf{m})} = m_k - (k-1)\dfrac{1}{2}$.

Proof. Part (2) follows immediately from (1) and part (1) is Theorem 7.1.1 of [5]. □

1.4. **The Generalized Laguerre Functions and Polynomials.** Let $\nu > 0$ and $\mathbf{m} \in \Lambda$. The *generalized Laguerre polynomial* is defined (cf. [5] p. 242) by the formula

$$L^\nu_{\mathbf{m}}(x) = (\nu)_{\mathbf{m}} \sum_{|\mathbf{n}| \le |\mathbf{m}|} \binom{\mathbf{m}}{\mathbf{n}} \frac{1}{(\nu)_{\mathbf{n}}} \psi_{\mathbf{n}}(-x), \quad x \in J$$

and the *generalized Laguerre function* is defined by

$$\ell^\nu_{\mathbf{m}}(x) = e^{-\operatorname{tr} x} L^\nu_{\mathbf{m}}(2x).$$

Remark 1.3. In the case $n = 1$, i.e. in the case $G \simeq Sp(1, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})$, the generalized Laguerre polynomials and functions defined above are precisely the classical Laguerre polynomials and functions defined on $\mathbb{R}^+$. We refer to [1] for the discussion of that case.

The determinant $\mathrm{Det}(h)$ of $h \in H$ acting on J is

$$\mathrm{Det}(h) = \det(h)^{n+1}.$$

It follows from (1.2) that the measure

$$d\mu_0(x) = \Delta(x)^{-\frac{n+1}{2}}\, dx$$

is H-invariant. Here dx is the Lebesgue measure on J. More generally, we set $d\mu_\nu(x) = \Delta(x)^{\nu - \frac{n+1}{2}} dx$ and define

$$L^2_\nu(\Omega) = L^2(\Omega, d\mu_\nu).$$

We observe that by (1.2) it follows that H acts unitarily on $L^2_\nu(\Omega)$ be the formula

$$\lambda_\nu(h) f(x) = \det(h)^\nu f(h^t \cdot x).$$

Theorem 1.4 ([2, 5])**.** *The set* $\{\ell^\nu_{\mathbf{m}} \mid \mathbf{m} \ge 0\}$ *forms a complete orthogonal system in* $L^2_\nu(\Omega)^L$*, the Hilbert space of* L*-invariant functions in* $L^2_\nu(\Omega)$*.*

In [1] it was shown, that the classical differential recursion relations and differential equations for the Laguerre functions on $\mathbb{R}^+$ follows from the representation theory of $\mathrm{Sp}(1, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})$. In [3] this was generalized to the space of complex Hermitian matrices. It is a goal of this article to extend this result to the generalized Laguerre functions defined of the cone of symmetric matrices. This indicates nicely what the more general results should be. Here we use heavily the structure of $\mathrm{Sp}(n, \mathbb{R})$ and its Lie algebra, but the proof of the general results should be more in the line of Jordan algebras.

2. The Tube Domain $T(\Omega)$, the Group $\mathrm{Sp}(n,\mathbb{R})$ and its Lie Algebra

In this section we introduce the tube domain $T(\Omega) = \Omega + i\mathrm{Sym}(n,\mathbb{R})$ and discuss the action of the group $\mathrm{Sp}(n,\mathbb{R})$ on this domain. We then discuss some important Lie subalgebras of $\mathfrak{sp}(n,\mathbb{C})$, the complexification of the Lie algebra of $\mathfrak{g} = \mathfrak{sp}(n,\mathbb{R})$. These subalgebras will show up again in Section 4 where we compute their action on Hilbert spaces of holomorphic functions on $T(\Omega)$ introduced in the next section. We then use that information to construct the Laguerre differential operators.

2.1. **The Group** $\mathrm{Sp}(n,\mathbb{R})$**.** Let $T(\Omega) = \Omega + iJ$ be the tube over Ω in $J_{\mathbb{C}}$, which we identify with the space of complex $n \times n$ symmetric matrices. Let $G_\circ$ be the group of biholomorphic diffeomorphisms on $T(\Omega)$. Then $G_\circ$ is a Lie group with Lie algebra isomorphic to $\mathfrak{sp}(n,\mathbb{R})$ and acts homogeneously on $T(\Omega)$. The group $\mathrm{Sp}(n,\mathbb{R})$ is isomorphic to a finite covering group of $G_\circ$ in the following precise way. The usual definition of $\mathrm{Sp}(n,\mathbb{R})$ is

$$\mathrm{Sp}(n,\mathbb{R}) = \left\{ g \in \mathrm{SL}(2n,\mathbb{R}) \mid gjg^t = j \right\},$$

where $j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SL}(2n,\mathbb{R})$. Each element $g \in \mathrm{Sp}(n,\mathbb{R})$ can be written in block form as

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, B, C, and D are real matrices. Defined in this way $\mathrm{Sp}(n,\mathbb{R})$ acts by linear fractional transformation on the *upper half plane* $J + i\Omega$. Let G be the group defined by

$$G = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid \begin{pmatrix} A & -iB \\ iC & D \end{pmatrix} \in \mathrm{Sp}(n,\mathbb{R}) \right\}.$$

This means then that the $(1,2)$ and $(2,1)$ entries of an element of G are purely imaginary matrices. Clearly G is isomorphic to $\mathrm{Sp}(n,\mathbb{R})$. It acts on the *right half plane* $T(\Omega) = \Omega + iJ$ by linear fractional transformations: if $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$ and $z \in T(\Omega)$ then

$$g \cdot z = (Az + B)(Cz + D)^{-1}.$$

It is also a finite covering group of $G_\circ$. For $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$ we have the following relations among A, B, C, and D:

$$\begin{array}{ll} A^tC - C^tA = 0 & AB^t - BA^t = 0 \\ A^tD - C^tB = I & AD^t - BC^t = I \\ B^tD - D^tB = 0 & CD^t - DC^t = 0 \\ B^tC - D^tA = -I & AD^t - BC^t = -I \end{array}$$

2.2. **Some Subgroups of G.** Let e be the $n \times n$-identity matrix. Then $e \in \Omega \subset T(\Omega)$. Let K be the stability subgroup of e in G. Then

$$K = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \in G \mid A \pm B \in U(n) \right\} \simeq U(n).$$

Here the isomorphism is given by

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix} \mapsto A + B.$$

The subgroup K is a maximal compact subgroup of G and G/K is naturally isomorphic to $T(\Omega)$ by the map $gK \to g \cdot e$. The connected component of the identity of the subgroup of G that leaves Ω invariant is isomorphic to H via the map

$$h \to \begin{pmatrix} h & 0 \\ 0 & (h^t)^{-1} \end{pmatrix}.$$

This map realizes L as a subgroup of G as well. In fact, we have

$$L = H \cap K,$$

via the above isomorphism.

2.3. **Lie Algebras.** If $P = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ then $G = P^{-1}\mathrm{Sp}(n, \mathbb{R})P$. From this it follows that the Lie algebra, $\mathfrak{g}$, of G is given by $\mathfrak{g} = P^{-1}\mathfrak{sp}(n, \mathbb{R})P$, and hence

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & ib \\ -ic & -a^t \end{pmatrix} \in \mathfrak{sl}(2n, \mathbb{C}) \mid a, b, c \text{ real}, b = b^t, c = c^t \right\}. \tag{2.1}$$

We define a *Cartan involution* on $\mathfrak{g}$ by

$$\theta(X) = -X^* .$$

It induces a decomposition of $\mathfrak{g}$ and $\mathfrak{g}_{\mathbb{C}}$ into ± 1-eigenspaces, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_{\mathbb{C}}$. The $+1$-eigenspace $\mathfrak{k}$ is the Lie algebra of K. These spaces are given by

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid \theta(X) = X\} = \left\{ \begin{pmatrix} a & ib \\ ib & a \end{pmatrix} \in \mathfrak{sl}(2n, \mathbb{C}) \mid a, b \text{ real}, a = -a^t, b = b^t, \mathrm{tr}(a) = 0 \right\}$$

and

$$\mathfrak{p} = \{X \in \mathfrak{g} \mid \theta(X) = -X\} = \left\{ \begin{pmatrix} a & ib \\ -ib & -a \end{pmatrix} \in \mathfrak{sl}(2n, \mathbb{C}) \mid a, b \text{ real}, a = a^t, b = b^t \right\}.$$

Their complexifications are given by

$$\mathfrak{k}_{\mathbb{C}} = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \in \mathfrak{sl}(2n, \mathbb{C}) \mid a, b \text{ complex}, a = -a^t, b = b^t, \mathrm{tr}(a) = 0 \right\}$$

and

$$\mathfrak{p}_{\mathbb{C}} = \left\{ \begin{pmatrix} a & b \\ -b & -a \end{pmatrix} \in \mathfrak{sl}(2n, \mathbb{C}) \mid a, b \text{ complex}, a = a^t, b = b^t \right\}.$$

It is clear that $K_{\mathbb{C}}$ acts on $\mathfrak{p}_{\mathbb{C}}$. This representation decomposes into two parts. For that let $\mathrm{z} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $\mathrm{z} \in \mathcal{Z}(\mathfrak{k}_{\mathbb{C}})$ and $\mathrm{ad}(\mathrm{z}) : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ has eigenvalues $0, 2, -2$. The 0-eigenspace is $\mathfrak{k}_{\mathbb{C}}$, the $+2$-eigenspace is denoted by $\mathfrak{p}^-$ and is given by

$$\mathfrak{p}^- = \left\{ \begin{pmatrix} a & -a \\ a & -a \end{pmatrix} \in \mathfrak{g}_{\mathbb{C}} \mid a = a^t \right\} \subset \mathfrak{p}_{\mathbb{C}},$$

and the -2-eigenspace is denoted by $\mathfrak{p}^+$ and is given by

$$\mathfrak{p}^+ = \left\{ \begin{pmatrix} a & a \\ -a & -a \end{pmatrix} \in \mathfrak{g}_{\mathbb{C}} \mid a = a^t \right\} \subset \mathfrak{p}_{\mathbb{C}}.$$

Each of the spaces $\mathfrak{p}^{\pm}$ are invariant under $K_{\mathbb{C}}$ and irreducible as $K_{\mathbb{C}}$ representation. Note that this is not necessarily the standard notation. In our notation the eigenvectors (in $\mathfrak{p}^+$) with -2-eigenvalue correspond to *annihilation* operators while the eigenvectors (in $\mathfrak{p}^-$) with $+2$-eigenvalue correspond to *creation* operators. These operators will be described is Section 4 below.

3. The Highest Weight Representations $(\pi_\nu, \mathcal{H}_\nu(T(\Omega)))$

In this section we introduce the highest weight representations $(\pi_\nu, \mathcal{H}_\nu(T(\Omega)))$ and state the main results needed. We also introduce the Laplace transform as a special case of the *restriction principle* introduced in [13].

3.1. **Unitary Representations of G in $\mathcal{O}(T(\Omega))$.** In this subsection we define a series of unitary representations of G on a Hilbert space of holomorphic functions on $T(\Omega)$. These representations are well known. Let $\tilde{G}$ be the universal covering group of G. Then $\tilde{G}$ acts on $T(\Omega)$ by $(g,z)\mapsto \kappa(g)\cdot z$ where $\kappa:\tilde{G}\to G$ is the canonical projection. For $\nu>n$ let $\mathcal{H}_\nu(T(\Omega))$ be the space of holomorphic functions $F:T(\Omega)\to\mathbb{C}$ such that

$$\|F\|_\nu^2 := \alpha_\nu \int_{T(\Omega)} |F(x+iy)|^2 \Delta(x)^{\nu-(n+1)}\,dxdy < \infty \tag{3.1}$$

where

$$\alpha_\nu = \frac{2^{n\nu}}{(4\pi)^d \Gamma_\Omega(\nu - \frac{n+1}{2})}\,.$$

Then $\mathcal{H}_\nu(T(\Omega))$ is a non-trivial Hilbert space with inner product

$$(F\,|\,G) = \alpha_\nu \int_{T(\Omega)} F(x+iy)\overline{G(x+iy)}\,\Delta(x)^{\nu-(n+1)}\,dxdy\,. \tag{3.2}$$

For $\nu\le n$ this space reduces to $\{0\}$. If $\nu=n+1$ this is the *Bergman space*. The space $\mathcal{H}_\nu(T(\Omega))$ is a *reproducing kernel Hilbert space*. This means that point evaluation

$$E_z : \mathcal{H}_\nu((T(\Omega))\to\mathbb{C}$$

given by $E_zF=F(z)$ is continuous, for every $z\in T(\Omega)$. This implies the existence of a kernel function $K_z\in\mathcal{H}_\nu(T(\Omega))$, such that $F(z)=(F\,|\,K_z)$ for all $F\in\mathcal{H}_\nu(T(\Omega))$ and $z\in T(\Omega)$. Set $K(z,w)=K_w(z)$. Then $K(z,w)$ is holomorphic in the first variable and antiholomorphic in the second variable. The function $K(z,w)$ is called *the reproducing kernel* for $\mathcal{H}_\nu(T(\Omega))$. We note that the Hilbert space is completely determined by the function $K(z,w)$. In particular, we have:

(1) The space of finite linear combinations $\mathcal{H}_\nu(T(\Omega))^o := \left\{\sum c_jK_{w_j}\mid c_j\in\mathbb{C},\, w_j\in T(\Omega)\right\}$ is dense in $\mathcal{H}_\nu(T(\Omega))$;
(2) The inner product in $\mathcal{H}_\nu(T(\Omega))^o$ is given by

$$\left(\sum_j c_jK_{w_j}\mid \sum_k d_kK_{z_k}\right) = \sum_{j,k} c_j\overline{d_k}K(z_k,w_j)\,.$$

We refer to [2, 9] for more details.

3.2. **The Unitary Representations $(\pi_\nu,\mathcal{H}_\nu(T(\Omega)))$.** For $g\in\tilde{G}$ and $z\in T(\Omega)$, let $J(g,z)$ be the *complex* Jacobian determinant of the action of $g\in\tilde{G}$ on $T(\Omega)$ at the point z. We will use the same notation for elements $g\in G$. A straightforward calculation gives

$$J(g,z)=\det(Cz+D)^{-n-1},\quad g=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in G \text{ and } z\in T(\Omega).$$

We also have the *cocycle relation*:

$$J(ab,z)=J(a,b\cdot z)J(b,z)$$

for all $a,b\in\tilde{G}$ and $z\in T(\Omega)$. It is well known that for $\nu>n$ the formula

$$\pi_\nu(g)f(z) = J(g^{-1},z)^{\frac{\nu}{n+1}}f(g^{-1}\cdot z) = \det(A-z\,C)^{-\nu}f(g^{-1}\cdot z) \tag{3.3}$$

defines a unitary irreducible representation of $\tilde{G}$. In [5], [14], and [19] it was shown that this unitary representation $(\pi_\nu,\mathcal{H}_\nu(T(\Omega)))$ has an analytic continuation to the half-interval $\nu>(n-1)\frac{1}{2}$. Here the representation π_ν is given by the same formula (3.3) but the formula for the *norm* in (3.1) is no longer valid. There are also finitely many equidistant values of ν that give rise to unitary representations, but they will not be of concern to us here.

In the following theorem we summarize what we have discussed and collect additional information from [2] and [5] (cf. p. 260, in particular, Theorem XIII.1.1 and Proposition XIII.1.2).

Theorem 3.1. *Let the notation be as above. Assume that for $\nu > n$ then the following hold:*

(1) *The space $\mathcal{H}_\nu(T(\Omega))$ is a reproducing Hilbert space.*
(2) *The reproducing kernel of $\mathcal{H}_\nu(T(\Omega))$ is given by*

$$K_\nu(z,w) = \Gamma_\Omega(\nu)\Delta\left(z+\bar{w}\right)^{-\nu}$$

(3) *If $\nu > \frac{1}{2}(n-1)$ then there exists a Hilbert space $\mathcal{H}_\nu(T(\Omega))$ of holomorphic functions on $T(\Omega)$ such that $K_\nu(z,w)$ defined in (2) is the reproducing kernel of that Hilbert space. The representation π_ν defines a unitary representation of $\tilde{G}$ on $\mathcal{H}_\nu$.*
(4) *If $\nu > \frac{1}{2}(n-1)$ then the functions*

$$q^\nu_{\mathbf{m}}(z) := \Delta(z+e)^{-\nu}\psi_{\mathbf{m}}\left(\frac{z-e}{z+e}\right), \qquad \mathbf{m}\in\mathbf{\Lambda},$$

form an orthogonal basis of $\mathcal{H}_\nu(T(\Omega))^L$, the space of L-invariant functions in $\mathcal{H}_\nu(T(\Omega))$.

3.3. **The Restriction Principle and the Laplace Transform.** The restriction principle [12, 11] is a general recipe to construct unitary maps between a reproducing kernel Hilbert space of holomorphic functions and L^2-spaces on a totally real submanifold. Suppose M is a complex manifold and $\mathbb{H}(M)$ is a reproducing kernel Hilbert space of holomorphic functions on M with kernel K. Suppose X is a totally real submanifold of M and a measure space for some measure μ.

Assume we have a holomorphic function D on M, such that D is positive on X, and such that the map

$$R : \mathbb{H}(M) \to L^2(X,\mu)\,,$$

given by $Rf(x) = D(x)f(x)$, is densely defined. As each f is holomorphic, its restriction to X is injective. It follows that R is an injective map. We call R a *restriction map*. Assume R is closed and has dense range. If $K(z,w) = K_w(z)$ is the reproducing kernel for $\mathbb{H}(M)$, and $f \in L^2(X, d\mu)$, then

$$\begin{aligned} R^*f(z) &= (R^*f\,|\,K_z)\\ &= (f\,|\,RK_z)\\ &= \int_X f(x)D(x)K(z,x)\,d\mu(x)\,. \end{aligned}$$

In particular, if we set $\Psi(x,y) = D(y)D(x)K(y,x)$, then RR^* is given by

$$RR^*f(y) = \int_X f(x)\Psi(x,y)\,d\mu(x)$$

and thus is an integral operator. Consider the polar decomposition of the operator R^*. We can write

$$R^* = U\sqrt{RR^*},$$

where U is a unitary operator

$$U : L^2(X,\mu) \to \mathbb{H}(M).$$

The unitary map U is sometimes called the *generalized Segal-Bargmann transform*. In many applications of the restriction principle, M and X will be homogeneous spaces with a group H acting on both. When the restriction map R is H-intertwining so will the unitary operator U. This is exactly what happens in the situation at hand. Here we can take $D = 1$ and define $R : \mathcal{H}_\nu(T(\Omega)) \to L^2_\nu(\Omega)$ by

$$Rf(x) = f(x)\,.$$

Then we obtain the following:

Theorem 3.2. *The map R is injective, densely defined and has dense range. The unitary part, U, of the polar decomposition of R^*: $R^* = U\sqrt{RR^*}$, is the Laplace transform given by*

$$Uf(z) = \mathcal{L}_\nu f(z) = \int_\Omega e^{-(z|x)} f(x)\, d\mu_\nu(x)\,.$$

Furthermore,

$$\mathcal{L}_\nu(\lambda_\nu(h)f) = \pi_\nu(h)\mathcal{L}_\nu(f)$$

for all $h \in H$. In particular, $\mathcal{L}_\nu$ induces an isomorphism $\mathcal{L}_\nu : L^2_\nu(\Omega)^L \to \mathcal{H}_\nu(T(\Omega))^L$. Moreover

$$\mathcal{L}_\nu(\ell^\nu_{\mathbf{m}}) = \Gamma_\Omega(\mathbf{m}+\nu)q^\nu_{\mathbf{m}}\,.$$

Proof. The first proof of this theorem was done for $\mathrm{SL}(2,\mathbb{R})$ in [1]. The general case is on pages 187-190 of [2]. □

Remark 3.3. Rossi and Vergne [14] obtained the unitarity of the Laplace transform using a result of Nussbaum.

The unitarity of the Laplace transform allows us to transfer the representation, π_ν, of G on $\mathcal{H}_\nu(T(\Omega))$ to an equivalent representation of G on $L^2_\nu(\Omega)$, which extends λ_ν by the above theorem, . We will denote the extension by λ_ν as well. It is possible to describe λ_ν on various subgroups of G whose product is dense in G. However, it is a difficult problem at best to describe a global realization of λ_ν on all of G. However, part of the point of this paper is to give a formula for the derived representation of λ_ν on the Lie algebra of G and its complexification. It is from the derived representation that new differential recursion relations arise that relate the generalized Laguerre functions.

4. The Action of $\mathfrak{g}_\mathbb{C}$

In this section we introduce some subalgebras of $\mathfrak{sp}(n,\mathbb{C})$, the complexification of the Lie algebra of G, and explain how they act in the Hilbert space $\mathcal{H}_\nu(T(\Omega))$.

4.1. **The Derived Representation on $\mathcal{H}_\nu(T(\Omega))$.** Denote by $\mathcal{H}_\nu(T(\Omega))^\infty$ the space of functions $F \in \mathcal{H}_\nu(T(\Omega))$ such that the map

$$\mathbb{R} \ni t \mapsto \pi_\nu(\exp tX)F \in \mathcal{H}_\nu(T(\Omega))$$

is smooth for all $X \in \mathfrak{g} = \mathfrak{sp}(n,\mathbb{R})$. If $f \in C^\infty_c(G)$, then $\pi_\nu(f)F = \int_G f(g)\pi_\nu(g)F\,dg$ is in $\mathcal{H}_\nu(T(\Omega))^\infty$ and it follows, that $\mathcal{H}_\nu(T(\Omega))^\infty$ is dense in $\mathcal{H}_\nu(T(\Omega))$. The Lie algebra representation, denoted also by π_ν, of $\mathfrak{g}$ on $\mathcal{H}_\nu(T(\Omega))^\infty$ is given, by differentiation as follows:

$$\begin{aligned}\pi_\nu(X)F &= \lim_{t\to 0}\frac{\pi_\nu(\exp tX)F - F}{t}\\ &= \frac{d}{dt}\pi_\nu(\exp(tX))F|_{t=0}.\end{aligned}$$

Note that the limit is taken in the Hilbert space norm in $\mathcal{H}_\nu(T(\Omega))$, but it is easy to see that if $F \in \mathcal{H}_\nu(T(\Omega))^\infty$, then in fact for $X \in \mathfrak{g}$:

$$\pi_\nu(X)F(z) = \frac{d}{dt}J(\exp(-tX),z)^{\frac{\nu}{n+1}}F(\exp(-tX)\cdot z)|_{t=0}\,, \tag{4.1}$$

for all $z \in T(\Omega)$. We extend this by complex linearity to $\mathfrak{g}_\mathbb{C}$.

Define D_w by

$$D_wF(z) = \frac{d}{dt}F(z+tw)|_{t=0} = F'(z)w$$

where F' denotes the derivative of F.

Lemma 4.1. *Suppose z, w are $n \times n$ matrices over $\mathbb{C}$ and z is invertible. Then*

$$D_w \det(z)^n = n \det(z)^n \operatorname{tr}(z^{-1}w).$$

Proof. This follows from the chain rule and the fact that

$$D_w \det(z) = \frac{d}{dt} \det(z + tw)|_{t=0} = \det z \frac{d}{dt} \det(1 + tz^{-1}w)|_{t=0} = \det(z) \operatorname{tr}(z^{-1}w) .$$

□

The following proposition expresses the relevant formulas on $\mathfrak{k}_{\mathbb{C}}$, $\mathfrak{p}^+$, and $\mathfrak{p}^-$. It's proof is a straightforward calculation using Lemma 4.1.

Proposition 4.2. *For each piece of the Lie algebra of $\mathfrak{g}_{\mathbb{C}}$ introduced in Subsection 2.3, we have:*

(1) $\pi_\nu(X)F(z) = \nu \operatorname{tr}(bz)F(z) + D_{za-az-b+zbz}F(z)$, $X = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \in \mathfrak{k}_{\mathbb{C}}$

(2) $\pi_\nu(X)F(z) = -\nu \operatorname{tr}(az + a)F(z) - D_{(za+az)+zaz+a}F(z)$, $X = \begin{pmatrix} a & a \\ -a & -a \end{pmatrix} \in \mathfrak{p}^+$.

(3) $\pi_\nu(X)F(z) = -\nu \operatorname{tr}(-az + a))F(z) + D_{-(za+az)+zaz+a}F(z)$, $X = \begin{pmatrix} a & -a \\ a & -a \end{pmatrix} \in \mathfrak{p}^-$

Remark 4.3. We note that these equations are closely related to the Jordan algebra structure of J. In particular, we have that $za + az = 2z \circ a$, where $\circ$ denotes the Jordan algebra product $a \circ b = \frac{1}{2}(ab + ba)$. Furthermore $zaz = P(z)a$ where P denotes the quadric representation (cf. [5], p 32). Finally we note, that $\operatorname{tr}(az) = \operatorname{tr}(z \circ a)$. These formulas therefore clearly indicate that the more general results are expressible in terms of Jordan algebraic constructs.

4.2. **Highest Weight Representations.** The fact that π_ν is a highest weight representation plays a decisive role in the recursion relations that we obtain. At this point we explain what this notion means.

We assume G is a Hermitian group, which means that G is simple and the maximal compact subgroup K has a one dimensional center. The Hermitian groups have been classified in terms of their Lie algebras. They are $\mathfrak{su}(p, q)$, $\mathfrak{sp}(n, \mathbb{R})$, $\mathfrak{so}^*(2n)$, $\mathfrak{so}(2, n)$, and two exceptional Lie algebras. The assumption that K has a one dimensional center implies that G/K is a bounded symmetric domain. In particular G/K is complex. It also implies that the complexification of the Lie algebra, $\mathfrak{g}_{\mathbb{C}}$, has a decomposition of the form $\mathfrak{g}_{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^-$, Specifically, $\mathfrak{p}^+$, $\mathfrak{k}_{\mathbb{C}}$, and $\mathfrak{p}^-$ are the -2, 0, 2-eigenspaces of ad(z), respectively, where z is in the center of $\mathfrak{k}_{\mathbb{C}}$.

Lemma 4.4. *We have the following inclusions:*

$$[\mathfrak{k}_{\mathbb{C}}, \mathfrak{p}^\pm] \subset \mathfrak{p}^\pm$$
$$[\mathfrak{p}^+, \mathfrak{p}^-] \subset \mathfrak{k}_{\mathbb{C}}$$

Proof. Suppose $X \in \mathfrak{p}^+$, $Y \in \mathfrak{p}^-$, and $Z \in \mathfrak{k}_{\mathbb{C}}$. Then

$$\mathrm{ad(z)}[X, Y] = [\mathrm{ad(z)}X, Y] + [X, ad(\mathrm{z})Y] = -2[X, Y] + 2[X, Y] = 0.$$

This implies $[X, Y] \in \mathfrak{k}_{\mathbb{C}}$. Similarly,

$$\mathrm{ad(z)}[Z, X] = [\mathrm{ad(z)}Z, X] + [Z, \mathrm{ad(z)}X] = -2[Z, X].$$

This implies $[Z, X] \in \mathfrak{p}^+$. A similar argument shows that $[Z, Y] \in \mathfrak{p}^-$. □

Suppose π that is an irreducible representation of G on a Hilbert Space $\mathbb{H}$. We say π is a *highest weight representation* if there is a nonzero vector $v \in \mathbb{H}$ such that

$$\pi(X)v = 0,$$

for all $X \in \mathfrak{p}^+$. Let $\mathbb{H}_\circ$ be the set of all such vectors. The following theorem is well known.

Theorem 4.5. *Suppose π is an irreducible unitary highest weight representation of G on $\mathbb{H}$ and $\mathbb{H}_\circ$ is defined as above. Then $(\pi|_K, \mathbb{H}_\circ)$ is irreducible. Furthermore, there is a scalar λ such that*

$$\pi(\mathrm{z})v = \lambda v,$$

for all $v \in \mathbb{H}_\circ$. If

$$\mathbb{H}_n = \{v \in \mathbb{H} \mid \pi(\mathrm{z})v = (\lambda + 2n)v\},$$

then

$$\mathbb{H} = \bigoplus_{n \geq 0} \mathbb{H}_n .$$

Additionally,

$$\begin{aligned} \pi(Z) &: \mathbb{H}_n \longrightarrow \mathbb{H}_n, \quad Z \in \mathfrak{k}_{\mathbb{C}}\,; \\ \pi(X) &: \mathbb{H}_n \longrightarrow \mathbb{H}_{n-1}, \quad X \in \mathfrak{p}^+\,; \\ \pi(Y) &: \mathbb{H}_n \longrightarrow \mathbb{H}_{n+1}, \quad Y \in \mathfrak{p}^-\,, \end{aligned}$$

where, in the case $n = 0$, $\mathbb{H}_{-1}$ is understood to be the $\{0\}$ space.

Proof. By Lemma 4.4, $\mathbb{H}_\circ$ is an invariant K-space. Suppose $\mathbb{V}_\circ$ is a nonzero invariant subspace of $\mathbb{H}_\circ$ and $\mathbb{W}_\circ$ is its orthogonal complement in $\mathbb{H}_\circ$. Define $\mathbb{V}_n$ inductively as follows:

$$\mathbb{V}_n = \operatorname{span}\left\{\pi(Y)v \mid Y \in \mathfrak{p}^-, v \in \mathbb{V}_{n-1}\right\}.$$

Let $\mathbb{V} = \oplus \mathbb{V}_n$. Define $\mathbb{W}_n$ in the same way as $\mathbb{V}_n$ and let $\mathbb{W} = \oplus \mathbb{W}_n$. Then, by Lemma 4.4, $\mathbb{V}$ and $\mathbb{W}$ are invariant $\mathfrak{g}_{\mathbb{C}}$ subspaces of $\mathbb{H}$. Since π is unitary $\mathbb{V}$ and $\mathbb{W}$ are orthogonal. However, since π is irreducible and $\mathbb{V}$ is nonzero, it follows that $\mathbb{V} = \mathbb{H}$ and hence $\mathbb{W} = 0$. This implies $\mathbb{W}_\circ = 0$ and thus $\pi|_K$ is irreducible. Since $\pi(\mathrm{z})$ commutes with $\pi(K)$ Schur's lemma implies that $\pi(\mathrm{z}) = \lambda$ on $\mathbb{H}_\circ$ for some scalar λ. Since $\mathbb{V}_\circ = \mathbb{H}_\circ$, induction, Lemma 4.4, and irreducibility of π implies that $\mathbb{V}_n = \mathbb{H}_n$. The remaining claims follow from Lemma 4.4. □

Remark 4.6. The operators $\pi(X)$, $X \in \mathfrak{p}^+$, are called *annihilation operators* because, for v in the algebraic direct sum $\bigoplus \mathbb{H}_n$, sufficiently many applications of $\pi(X)$ annihilates v. For $Y \in \mathfrak{p}^-$ the operators $\pi(Y)$ are called *creation operators*.

Remark 4.7. A straightforward calculation gives

$$\pi_\nu(X)q^\nu_{\mathbf{0}} = 0,$$

for all $X \in \mathfrak{p}^+$ and that $\mathcal{H}_\nu(T(\Omega))_\circ = \mathbb{C}q^\nu_{\mathbf{0}}$. Thus $(\pi_\nu, \mathcal{H}_\nu(T(\Omega)))$ is an irreducible unitary highest weight representation of G and by unitary equivalence so is $(\lambda_\nu, L^2_\nu(\Omega))$.

5. The Realization of λ_ν Acting on $L^2(\Omega, d\mu_\nu)$

In this section we determine explicitly the action of $\mathfrak{g}_{\mathbb{C}}$ on $L^2(\Omega, d\mu_\nu)$. More specifically, we define λ_ν via the Laplace transform by the following formula

$$\lambda_\nu(X) = \mathcal{L}_\nu^{-1}\pi_\nu(X)\mathcal{L}_\nu$$

and will determine explicit formulas for $\lambda_\nu(X)$, for $X \in \mathfrak{p}^+$, $X \in \mathfrak{k}_{\mathbb{C}}$, and $X \in \mathfrak{p}^-$.

5.1. **Preliminaries.** Let E_{ij} be the $n \times n$ matrix with a 1 in the (i,j) position and 0's elsewhere. Define $\tilde{E}_{i,j} = \frac{1}{2}(E_{i,j} + E_{j,i})$. Then the collection $\left\{\tilde{E}_{i,j} \mid 1 \leq i \leq j \leq n\right\}$ is a basis of J and $J_{\mathbb{C}}$, the real and complex symmetric matrices. Furthermore, $\left(\tilde{E}_{i,j} \,|\, \tilde{E}_{k,l}\right) = \frac{1}{2}(\delta_{jk}\delta_{il} + \delta_{jl}\delta_{ik})$, which implies $\left\{\tilde{E}_{i,j} \mid 1 \leq i \leq j \leq n\right\}$ is an orthogonal basis. Set $D_{i,j} = D_{\tilde{E}_{i,j}}$ and observe that $D_{i,j} = D_{j,i}$. The *gradient* of f, ∇f, is defined by

$$(\nabla f(x) \,|\, u) = D_u f(x).$$

Proposition 5.1. *Suppose $f, g \in L^2(\Omega, d\mu_\nu)$ are smooth and f vanishes on the boundary of the cone Ω. Let $1 \leq i, j \leq n$. Then*

(1)
$$\int_\Omega D_{i,j} f(s) g(s) ds = -\int_\Omega f(s) D_{i,j} g(s) ds.$$

(2)
$$\int_\Omega e^{-(z|s)} z_{i,j} f(s) ds = \int_\Omega e^{-(z|s)} D_{i,j} f(s) ds.$$

Proof. (1) is Stokes Theorem and (2) follows from (1) and the fact that $D_{i,j} e^{-(z|s)} = -e^{-(z|s)} z_{i,j}$, $z \in J_{\mathbb{C}}$. □

5.2. **The Representation λ_ν.** Recall that we determined the action of $\mathfrak{k}_{\mathbb{C}}$, $\mathfrak{p}^+$ and $\mathfrak{p}^-$ on $\mathcal{H}_\nu(T(\Omega))^\infty$ in Proposition 4.2. We denote the subspace of smooth vectors in $L^2_\nu(\Omega)$ by $L^2_\nu(\Omega)^\infty$. Thus $f \in L^2_\nu(\Omega)^\infty$ if and only if the map

$$\mathbb{R} \ni t \mapsto \lambda_\nu(\exp tX) f \in L^2_\nu(\Omega)$$

is smooth for all $X \in \mathfrak{g}$. Thus

$$L^2_\nu(\Omega)^\infty = \mathcal{L}_\nu^{-1}(\mathcal{H}_\nu(T(\Omega))^\infty)\,.$$

The action of $\mathfrak{g}$ on $L^2_\nu(\Omega)^\infty$ is, as usual, defined by

$$\lambda_\nu(X) f = \lim_{t \to 0} \frac{\lambda_\nu(\exp tX) f - f}{t},$$

for $X \in \mathfrak{g}$, and then by complex linearity the action extends to $\mathfrak{g}_{\mathbb{C}}$. The following theorem collects the corresponding equivalent action on the Hilbert space $L^2_\nu(\Omega)^\infty$. We remark again that these formulas can be stated in terms of the Jordan algebra structure of J indicating the extension of these results to other tube domains.

Theorem 5.2. *For $f \in L^2_\nu(\Omega)$ a smooth function we have:*

(1) $\lambda_\nu(X) f(x) = \mathrm{tr}[(bx + (ax - xa - \nu b)\nabla - x\nabla b\nabla] f(x)$, $X = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \in \mathfrak{k}_{\mathbb{C}}$

(2) $\lambda_\nu(X) f(x) = \mathrm{tr}[(\nu a + ax + (ax + xa + \nu a)\nabla + x\nabla a\nabla] f(x)$, $X = \begin{pmatrix} a & a \\ -a & -a \end{pmatrix} \in \mathfrak{p}^+$

(3) $\lambda_\nu(X) f(x) = \mathrm{tr}[(\nu a - ax + (ax + xa - \nu a)\nabla - x\nabla a\nabla] f(x)$, $X = \begin{pmatrix} a & -a \\ a & -a \end{pmatrix} \in \mathfrak{p}^-$

5.3. **The Case of $Sp(1,\mathbb{R})$.** The proof, which appears in the appendix, is very long and computational. However, to convey the main ideas of the proof we will discuss the simpler case of $\mathrm{Sp}(1,\mathbb{R})$, which is isomorphic to $\mathrm{SL}(2,\mathbb{R})$. A detailed account of this case (modelled on the upper half plane) is found in [1].

Let $G=\left\{\begin{pmatrix} a & ib\\ -ic & d\end{pmatrix} \mid \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{SL}(2,\mathbb{R})\right\}$. The group G acts on the right half plane $T(\mathbb{R}^+)$ by linear fractional transformations. The complexification, $\mathfrak{g}_{\mathbb{C}}$, of the Lie algebra of G is $\mathfrak{s}l(2,\mathbb{C})$, a three dimensional Lie algebra spanned by

$$\mathrm{Z}=\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix},\quad \mathrm{X}=\begin{pmatrix} 1 & 1\\ -1 & -1\end{pmatrix},\quad \text{and}\quad \mathrm{Y}=\begin{pmatrix} 1 & -1\\ 1 & -1\end{pmatrix}.$$

Proposition 4.2 for this case reads:

Theorem 5.3. *The action of $\mathfrak{sl}(2,\mathbb{R})$ on the right half-plane is given by:*

1. $\pi_\nu(\mathrm{Z})F(z)=\nu zF(z)+(z^2-1)F'(z)$
2. $\pi_\nu(\mathrm{X})F(z)=-\nu(z+1)F(z)-(z+1)^2F'(z)$
3. $\pi_\nu(\mathrm{Y})F(z)=\nu(z-1)F(z)+(z-1)^2F'(z)$

To find the corresponding action on $L^2_\nu(\mathbb{R}^+)$ we must compute the operators that corresponds to D_z, M_z, M_{z^2}, $M_z\circ D_z$ and $M_{z^2}\circ D_z$ in $L^2_\nu(\mathbb{R}^+)^\infty$. Here M stands for "multiplication operator". To do this, requires several uses of integration by parts, a special case of Stokes theorem. It was exactly this kind of computation that was done in [1] and we repeat it here:

For D_z we have:

$$\begin{aligned}\frac{d}{dz}\mathcal{L}_\nu(f)(z) &= \int_0^\infty \frac{de^{-zt}}{dz}f(t)t^{\nu-1}\,dt\\ &= \mathcal{L}_\nu(-tf(t))\end{aligned}$$

Thus $D_z\longleftrightarrow M_{-t}$.

For M_z we have

$$\begin{aligned}z\mathcal{L}_\nu(f)(z) &= \int_0^\infty -\frac{de^{-zt}}{dt}f(t)t^{\nu-1}\,dt\\ &= \int_0^\infty e^{-zt}\frac{d}{dt}(f(t)t^{\nu-1})\,dt\\ &= \int_0^\infty e^{-zt}(f'(t)+\frac{\nu-1}{t}f(t))\,t^{\nu-1}\,dt.\end{aligned}$$

Thus $M_z\longleftrightarrow D+M_{\frac{\nu-1}{t}}$.

We calculate M_{z^2} similarly and get

$$M_{z^2}\longleftrightarrow D^2+\frac{2(\nu-1)}{t}D+\frac{(\nu-1)(\nu-2)}{t^2}.$$

Thus:

Lemma 5.4. *Let the notation be as above. Then the following holds:*

1. $D_z\circ\mathcal{L}_\nu=-\mathcal{L}_\nu\circ(M_t)$;
2. $M_z\circ\mathcal{L}_\nu=\mathcal{L}_\nu\circ\left(D+\frac{\nu-1}{t}\right)$;
3. $M_z\circ D_z\circ\mathcal{L}_\nu=\mathcal{L}_\nu\circ(-tD-\nu)$;
4. $M_{z^2}\circ D_z\circ\mathcal{L}_\nu=\mathcal{L}_\nu\circ\left(-tD^2-2\nu D-\frac{\nu(\nu-1)}{t}\right)$.

Combining Theorem 5.3 and Lemma 5.4 gives

Lemma 5.5. *Let the notation be as above. Then the following holds:*

(1) $\lambda_\nu(\mathrm{Z}) = -tD^2 - \nu D + t$;
(2) $\lambda_\nu(\mathrm{X}) = tD^2 + (\nu + 2t)D + (\nu + t)$;
(3) $\lambda_\nu(\mathrm{Y}) = -tD^2 + (-\nu + 2t)D + (\nu - t)$.

Note that this is Theorem 5.2 for this special case.

One more ingredient is necessary for determining the classical recursion relations. This is a direct calculation and given in the following lemma. We note at this point that such a direct calculation is not done in the general case; deeper properties of the representation theory must be used. (c.f. Proposition 6.1.)

Lemma 5.6. *Let* $q_m^\nu(z) = (z+1)^{-\nu}\left(\frac{z-1}{z+1}\right)^m$. *Then*

(1) $\pi_\nu(\mathrm{Z})q_m^\nu = (\nu + 2m)q_m^\nu$
(2) $\pi_\nu(\mathrm{X})q_m^\nu = -2mq_{m-1}^\nu$
(3) $\pi_\nu(\mathrm{Y})q_m^\nu = 2(\nu + m)q_{m+1}^\nu$

The combination of Lemmas 5.5 and 5.6 gives the classical recursion relations stated in the introduction and proves Theorem 6.3 for the classical case.

6. Differential Recursion Relations for ℓ_m^ν

We now turn our attention to differential recursion relations that exist among the generalized Laguerre functions. These relations are obtained by way of the highest weight representation λ_ν and generalize the classical case mentioned in the introduction.

We begin with some preliminaries and a result found in [2]. First we notice that in general the Lie algebra $\mathfrak{g}_\mathbb{C}$ does not map $(L^2_\nu(\Omega)^\infty)^L$ into itself. For the Laguerre functions the full Lie algebra is too big; we will in fact only need the much smaller Lie algebra $\mathfrak{g}_\mathbb{C}^L$, which maps $(L^2_\nu(\Omega)^\infty)^L$ into itself. It is well known, that in case $\mathfrak{g}$ is simple, then $\mathfrak{g}_\mathbb{C}^L \simeq \mathfrak{sl}(2,\mathbb{C})$. We choose $\mathrm{Z}, \mathrm{X}, \mathrm{Y}$ so that the isomorphism, which we will denote by φ, is given by

$$\mathrm{Z} \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathrm{X} \mapsto \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \quad \text{and} \quad \mathrm{Y} \mapsto \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.$$

Furthermore, we can assume that $\varphi(X^t) = \varphi(X)^t$. This shows that several calculations can in fact be reduced directly to $\mathfrak{sl}(2,\mathbb{C})$. We will come back to that later.

Define $Z^0 := \frac{1}{2}(\mathrm{X} + \mathrm{Y})$. Then $\varphi(Z^0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and Z^0 is in the center of $\mathfrak{h}$. For $\mathbf{m} \in \Lambda$ let

$$c_\mathbf{m}(j) = \prod_{j \neq k} \frac{m_j - m_k - \frac{1}{2}(j+1-k)}{m_j - m_k - \frac{1}{2}(j-k)}.$$

Then by Lemma 5.5 in [2] we have:

Proposition 6.1. *The action of* Z *and* Z^0 *is given by:*

(1) $\pi_\nu(\mathrm{Z})q_\mathbf{m}^\nu = (n\nu + 2\,|\mathbf{m}|)q_\mathbf{m}^\nu$.
(2) $\pi_\nu(-2Z^0)q_\mathbf{m}^\nu = \sum_{j=1}^r \binom{\mathbf{m}}{\mathbf{m}-\gamma_j} q_{\mathbf{m}-\mathbf{e}_j}^\nu - \sum_{j=1}^r (\nu + m_j - \frac{1}{2}(j-1)))c_\mathbf{m}(j)q_{\mathbf{m}+\mathbf{e}_j}^\nu$.

Corollary 6.2. *Let the notation be as above. Then the following holds:*

(1) $\lambda_\nu(\mathrm{Z})\ell_\mathbf{m}^\nu = (n\nu + 2|\mathbf{m}|)\ell_\mathbf{m}^\nu$.
(2) $\lambda_\nu(-2Z^\circ)\ell_\mathbf{m}^\nu = \sum_{j=1}^r \binom{\mathbf{m}}{\mathbf{m}-\mathbf{e}_j}(m_j - 1 + \nu - (j-1))\ell_{\mathbf{m}-\mathbf{e}_j}^\nu - \sum_{j=1}^r c_m(j)\ell_{\mathbf{m}+\mathbf{e}_j}^\nu$.

Proof. This statement follows from Proposition 6.1 and the following facts: $\lambda_\nu(X) = \mathcal{L}_\nu^{-1}\pi_\nu(X)\mathcal{L}_\nu$, $\mathcal{L}_\nu(\ell^\nu_{\mathbf{m}}) = \Gamma_\Omega(\mathbf{m}+\nu)q^\nu_{\mathbf{m}}$, and Proposition 1.2. In each of these formulas if either index $\mathbf{m}+\mathbf{e}_j$ or $\mathbf{m}-\mathbf{e}_j$ is not in Λ then it should be understood that the corresponding function does not appear. □

Theorem 6.3. *The Laguerre functions are related by the following differential recursion relations:*

(1) $\operatorname{tr}(-x\nabla\nabla - \nu\nabla + x)\ell^\nu_{\mathbf{m}}(x) = (n\nu + 2|\mathbf{m}|)\ell^\nu_{\mathbf{m}}(x)$.

(2) $\operatorname{tr}(x\nabla\nabla + (\nu I + 2x)\nabla + (\nu I + x))\ell^\nu_{\mathbf{m}}(x) = -2\sum_{j=1}^r \binom{\mathbf{m}}{\mathbf{m}-\mathbf{e_j}} (m_j - 1 + \nu - (j-1))\ell^\nu_{\mathbf{m}-\mathbf{e_j}}(x)$

(3) $\operatorname{tr}(-x\nabla\nabla + (-\nu I + 2x)\nabla + (\nu I - x))\ell^\nu_{\mathbf{m}}(x) = 2\sum_{j=1}^r c_{\mathbf{m}}(j)\ell^\nu_{\mathbf{m}+\mathbf{e_j}}(x)$.

Proof. If $\mathrm{Z} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ then substituting $a = 0$ and $b = 1$ into Proposition 5.2, part 1, gives

$$\lambda_\nu(\mathrm{Z}) = \operatorname{tr}(-x\nabla\nabla - \nu\nabla + x).$$

Combining this with part 1 of Corollary 6.2 gives the first formula.

Recall that $\mathrm{X} \in \mathfrak{p}^+$ and $\mathrm{Y} \in \mathfrak{p}^-$. According to Theorem 4.5 we have that $\lambda_\nu(\mathrm{X})\ell^\nu_{\mathbf{m}}$ has to be a linear combination of $\ell^\nu_{\mathbf{m}'}$, with $m'_j \le m_j$ for all j. Similarly, $\lambda_\nu(\mathrm{Y})\ell^\nu_{\mathbf{m}}$ has to be linear combination of those $\ell^\nu_{\mathbf{m}'}$ with $m'_j \ge m_j$. The statement follows now from Corollary 6.2 and the fact that $2Z^0 = \mathrm{X} + \mathrm{Y}$. □

7. Some Open Problems

There are still several open question that require further work. We mention three of these. One is a relation to the classical Laguerre polynomials, the other two are natural generalization of classical relations.

7.1. **Relation to Classical Laguerre Functions.** Every positive symmetric matrix A can be written as $A = kDk^{-1}$, where $k \in \mathrm{SO}(n)$ and $D = d(t_1, \ldots, t_n)$ is a diagonal matrix with $t_j > 0$. Thus, if

$$\Omega_1 = \left\{ d(\mathbf{t}) \mid \mathbf{t} \in (\mathbb{R}^+)^n \right\} \simeq (\mathbb{R}^+)^n$$

then

$$\Omega = L \cdot \Omega_1 \,.$$

As the Laguerre functions are L-invariant, it follows that they are uniquely determined by their restriction to Ω_1. Let $T(\Omega_1) := \{d(\mathbf{x}) + id(\mathbf{y}) \mid \mathbf{x} \in (\mathbb{R}^+)^n, \mathbf{y} \in \mathbb{R}^n\}$. Then $T(\Omega_1) \simeq (\mathbb{R}^+ + i\mathbb{R})^n$, and the group $\mathrm{SL}(2,\mathbb{R})^n$ acts transitively on the right hand side. But it is well known, that $\mathrm{SL}(2,\mathbb{R})^n$ can be realized as a closed subgroup of $\mathrm{Sp}(n,\mathbb{R})$. It follows therefore, that the generalized Laguerre functions can be written as a finite linear combinations of products of classical Laguerre functions. It is a natural problem to derive an exact formula.

7.2. **Relations in the λ-parameter.** It is well known that the classical Laguerre polynomials satisfy the following relations:

$$\begin{aligned} xL_n^\lambda &= (n+\lambda+1)L_n^{\lambda-1} - (n+1)L_{n+1}^{\lambda-1} \\ xL_n^\lambda &= (n+\lambda)L_{n-1}^{\lambda-1} - (n-x)L_n^{\lambda-1} \\ xL_n^{\lambda-1} &= L_n^\lambda - L_{n-1}^\lambda \,. \end{aligned}$$

In [1] it was shown, that these relations follows directly from the representation theory of $\mathfrak{sl}(2,\mathbb{R})$. It is therefore natural to look for similar relations for the generalized Laguerre polynomials and functions.

7.3. **Relations in the x, y Parameters.** Several other classical relations should be extended to the general case. We name here only the following

$$L_m^{\alpha+\beta+1}(x+y) = \sum_{n=0}^{m} L_n^{\alpha}(x) L_{m-n}^{\beta}(y) \, .$$

This relation is closely related to the decomposition of the tensor product of two highest weight representations and we expect that a similar relation can be derived also for the general case. Notice, however, that for general Laguerre polynomials the right hand side is L-invariant in the x and y variable while that is not the case on the left hand side. Thus any generalization will involve a projection (averaging over L) onto the L-invariant functions.

8. Appendix: Proof of Theorem (5.2)

Proof. We will prove the theorem for the case $x \in \mathfrak{p}^+$. The other two cases are done similarly. For convenience we let $m = \nu - \frac{n+1}{2}$.

Let $x = \begin{pmatrix} a & -a \\ a & -a \end{pmatrix} \in \mathfrak{p}^+$. By Proposition 4.2

$$\pi_\nu(x)\mathcal{L}_\nu f(z) = -\nu \operatorname{tr}(az+a)\mathcal{L}_\nu f(z) - D_{az+a+zaz+za}\mathcal{L}_\nu f(z), \quad X \in \mathfrak{p}^+.$$

Let

$$\begin{aligned} A &= -D_{az+za}\mathcal{L}_\nu f(z) \\ B &= -D_a\mathcal{L}_\nu f(z) \\ C &= -D_{zaz}\mathcal{L}_\nu f(z) \end{aligned}$$

Calculation of A.

$$\begin{aligned}
A &= -D_{az+za}\mathcal{L}_\nu f(z) \\
&= -\int_\Omega D_{az+za} e^{-(z|x)} f(x)\det(x)^m \, dx \\
&= \int_\Omega e^{-(z|x)}(az+za|x) f(x)\det(x)^m \, dx \\
&= \sum_{i,j,k}\int_\Omega e^{-(z|x)}(a_{ik}z_{kj} + z_{ik}a_{kj})(x_{ji}f(x)\det(x)^m)\, dx \\
&= \sum_{i,j,k}\int_\Omega e^{-(z|x)}(a_{ik}D_{kj} + a_{kj}D_{ik})(x_{ji}f(x)\det(x)^m)\, dx \\
&= \sum_{i,j,k}\int_\Omega e^{-(z|x)}(a_{ik}\frac{1}{2}(\delta_{kj}\delta_{ji}+\delta_{ki}\delta_{jj}) + a_{kj}\frac{1}{2}(\delta_{kj}\delta_{ii}+\delta_{ij}\delta_{ki}))f(x)\det(x)^m \, dx \\
&\quad + \sum_{i,j,k}\int_\Omega e^{-(z|x)}(a_{ik}x_{ji}D_{kj}f(x) + a_{kj}x_{ji}D_{ik}f(x))\det(x)^m \, dx \\
&\quad + \sum_{i,j,k}\int_\Omega e^{-(z|x)} f(x) m \det(x)^m (a_{ik}x_{ji}\operatorname{tr}(x^{-1}\tilde{E}_{kj}) + a_{kj}x_{ji}\operatorname{tr}(x^{-1}\tilde{E}_{ik}))dx \\
&= (n+1)\operatorname{tr}(a)\int_\Omega e^{-(z|x)}f(x)\det(x)^m \, dx \\
&\quad + \int_\Omega e^{-(z|x)}((ax+xa)_{ik}D_{ik}f(x))\det(x)^m \, dx \\
&\quad + \int_\Omega e^{-(z|x)} f(x) m \det(x)^m \operatorname{tr}(x^{-1}(xa+ax))dx \\
&= (2m+n+1)\operatorname{tr}(a)\int_\Omega e^{-(z|x)}f(x)\det(x)^m \, dx + \int_\Omega e^{-(z|x)}(\operatorname{tr}((ax+xa)\nabla f)(x))\det(x)^m \, dx \\
&= 2\nu \operatorname{tr}(a)\int_\Omega e^{-(z|x)}f(x)\det(x)^m \, dx + \int_\Omega e^{-(z|x)}(\operatorname{tr}((ax+xa)\nabla f)(x))\det(x)^m \, dx \\
&= 2\nu \operatorname{tr}(a)\mathcal{L}_\nu(f)(x) + \mathcal{L}_\nu((tr(ax+xa)\nabla)f)(x)
\end{aligned}$$

Calculation of B:.

$$\begin{aligned} B &= -D_a\mathcal{L}_\nu f(z) \\ &= \int_\Omega e^{-(z|x)}(a|x)f(x)\det(x)^m dx \\ &= \int_\Omega e^{-(z|x)}\operatorname{tr}(ax)f(x)\det(x)^m dx \\ &= \mathcal{L}_\nu(\operatorname{tr}(ax)f)(x) \end{aligned}$$

Calculation of C:.

$$\begin{aligned} C &= -D_{zaz}\mathcal{L}_\nu f(z) \\ &= \int_\Omega e^{-(z|x)}(zaz|x)f(x)\det(x)^m dx \\ &= \sum_{i,j,k,l}\int_\Omega e^{-(z|x)}a_{kl}z_{ik}z_{lj}x_{ji}f(x)\det(x)^m dx \\ &= \sum_{i,j,k}\int_\Omega e^{-(z|x)}a_{kl}z_{ik}D_{lj}(x_{ji}f(x)\det(x)^m)dx \\ &= \sum_{i,j,k,l}\int_\Omega e^{-(z|x)}a_{kl}z_{ik}\frac{1}{2}(\delta_{lj}\delta_{ji}+\delta_{li}\delta_{jj})f(x)\det(x)^m dx \\ &\quad+\sum_{i,j,k,l}\int_\Omega e^{-(z|x)}a_{kl}z_{ik}x_{ji}(D_{lj}f)(x)\det(x)^m dx \\ &\quad+\sum_{i,j,k,l}\int_\Omega e^{-(z|x)}a_{kl}z_{ik}x_{ji}f(x)m\det(x)^m\operatorname{tr}(x^{-1}\tilde{E}_{lj})dx \\ &= \sum_{i,k,l}\int_\Omega e^{-(z|x)}a_{kl}z_{ik}(\frac{n+1}{2}\delta_{li})f(x)\det(x)^m dx \\ &\quad+\sum_{i,j,k,l}\int_\Omega e^{-(z|x)}a_{kl}D_{ik}(x_{ji}D_{lj}f(x)\det(x)^m)dx \\ &\quad+\sum_{i,j,k,l}\int_\Omega e^{-(z|x)}a_{kl}z_{ik}x_{ji}f(x)m\det(x)^m\operatorname{tr}(x^{-1}\tilde{E}_{lj})dx \\ &= \frac{n+1}{2}\operatorname{tr}(az)\int_\Omega e^{-(z|x)}f(x)\det(x)^m dx \\ &\quad+\sum_{i,j,k,l}\int_\Omega e^{-(z|x)}a_{kl}\frac{1}{2}(\delta_{ij}\delta_{ki}+\delta_{ii}\delta_{kj})(D_{lj}f)(x)\det(x)^m dx \\ &\quad+\sum_{i,j,k,l}\int_\Omega e^{-(z|x)}a_{kl}x_{ji}(D_{ik}D_{lj}f)(x)\det(x)^m dx \\ &\quad+\sum_{i,j,k,l}\int_\Omega e^{-(z|x)}a_{kl}x_{ji}D_{lj}f(x)m\det(x)^m\operatorname{tr}(x^{-1}\tilde{E}_{ik})dx \\ &\quad+m\sum_{j,k}\int_\Omega e^{-(z|x)}a_{lk}z_{kl}\delta_{il}f(x)\det(x)^m dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{n+1}{2}\operatorname{tr}(az)\int_\Omega e^{-(z|x)} f(x)\det(x)^m dx \\
&\quad + \sum_{i,j,k,l}\int_\Omega e^{-(z|x)} a_{kl}\frac{1}{2}(\delta_{ij}\delta_{ki}+\delta_{ii}\delta_{kj})(D_{lj}f)(x)\det(x)^m dx \\
&\quad + \sum_{i,j,k,l}\int_\Omega e^{-(z|x)} a_{kl}x_{ji}(D_{ik}D_{lj}f)(x)\det(x)^m dx \\
&\quad + \sum_{i,j,k,l}\int_\Omega e^{-(z|x)} a_{kl}x_{ji}D_{lj}f(x) m\det(x)^m \operatorname{tr}(x^{-1}\tilde{E}_{ik})dx \\
&\quad + m\sum_{j,k}\int_\Omega e^{-(z|x)} a_{lk}z_{kl}\delta_{il}f(x)\det(x)^m dx \\
&= \nu\operatorname{tr}(az)\int_\Omega e^{-(z|x)} f(x)\det(x)^m dx \\
&\quad + \nu\int_\Omega e^{-(z|x)}(\operatorname{tr}(a\nabla)f)(x)\det(x)^m dx \\
&\quad + \int_\Omega e^{-(z|x)}((\operatorname{tr}(x\nabla a\nabla)f)(x)\det(x)^m dx \\
&= \nu\operatorname{tr}(az)\mathcal{L}_\nu f(x) + \nu\mathcal{L}_\nu(\operatorname{tr}(a\nabla)f)(x) + \mathcal{L}_\nu(\operatorname{tr}(x\nabla a\nabla)f)(x)
\end{aligned}
$$

$$
\begin{aligned}
\pi_\nu(x)\mathcal{L}_\nu f(z) &= -\nu\operatorname{tr}(az+a)\mathcal{L}_\nu f(z) - D_{az+a+zaz+za}\mathcal{L}_\nu f(z), \quad X\in\mathfrak{p}^+ \\
&= -\nu\operatorname{tr}(az+a)\mathcal{L}_\nu f(z) + 2\nu\operatorname{tr}(a)\mathcal{L}_\nu(f)(x) \\
&\quad + \mathcal{L}_\nu((tr(ax+xa)\nabla)f)(x) + \mathcal{L}_\nu(\operatorname{tr}(ax)f)(x) \\
&\quad + \nu\operatorname{tr}(az)\mathcal{L}_\nu f(x) + \nu\mathcal{L}_\nu(\operatorname{tr}(a\nabla)f)(x) \\
&\quad + \mathcal{L}_\nu(\operatorname{tr}(x\nabla a\nabla)f)(x) \\
&= \nu\operatorname{tr}(a)\mathcal{L}_\nu(f)(x) + \mathcal{L}_\nu((tr(ax+xa)\nabla)f)(x) \\
&\quad + \mathcal{L}_\nu(\operatorname{tr}(ax)f)(x) + \nu\mathcal{L}_\nu(\operatorname{tr}(a\nabla)f)(x) \\
&\quad + \mathcal{L}_\nu(\operatorname{tr}(x\nabla a\nabla)f)(x) \\
&= \mathcal{L}_\nu(\operatorname{tr}(\nu a + ax + (ax+xa+\nu a)\nabla + x\nabla a\nabla)f(x))
\end{aligned}
$$

Taking the inverse Laplace transform of each side gives the desired result.

□

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