THE RADON-GAUSS TRANSFORM

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Abstract. Gaussian measure is constructed for any given hyperplane in an infinite-dimensional Hilbert space, and this is used to define a generalization of the Radon transform to the infinite-dimensional setting, using Gauss measure instead of Lebesgue. An inversion formula is obtained.

1. Introduction

The purpose of this paper is to extend the theory of Radon transforms to infinite dimensions. Since there is no useful version of Lebesgue measure in infinite dimensions, and Gauss measure is the most useful standard measurea in this setting, we use Gauss measure as the background measure for the transform. We obtain an inversion formula and a support theorem (of course, there may be many different support theorems as there are in the finite dimensional case).

The Radon transform (invented by Radon in [3]; see reproduction in [1]) of a function f on $\mathbf{R}^n$ is a function which associates to each hyperplane $\xi \subset \mathbf{R}^n$ the value $\int_\xi f\,dm$, where m is Lebesgue measure on ξ. Our transform takes place in the setting of a real, infinite-dimensional, separable Hilbert space E_0. As is known (and we describe briefly in section 2), there is a probability space $(\Omega, \mathcal{F}, \mu)$ and a linear map $k \mapsto \hat{k}$, associating to each k in a dense linear subspace $E_{00} \subset E_0$ a measurable function $\hat{k}$ on Ω, such that each $\hat{k}$, viewed as a random variable, is Gaussian with mean 0 and variance $\|k\|^2$, and the random variables $\hat{k}$ generate the σ–algebra $\mathcal{F}$; this leads to a linear map $E_0 \to L^2(\mu) : k \mapsto \hat{k}$, with each $\hat{k}$ Gaussian of mean 0 and variance $\|k\|^2$. This is generally taken as the standard Gaussian measure "on" the Hilbert space E_0, though Ω is not equal to E_0 in any natural sense.

Let us now summarize some of the results and constructions of this paper, referring the the notation set up above. A hyperplane in E_0 is a subset of the form $\xi = pu + u^\perp$, where u is a unit vector in E_0 and p a non-negative real number; when p, which is the distance $d(0, \xi)$ of ξ from the origin, is positive, the hyperplane ξ determines p as $d(0, \xi)$ and u as the unit normal vector from the origin onto ξ.

In section 2, we show that on the $(\Omega, \mathcal{F})$ there is a probability measure μ_ξ, and the each function $\hat{k}$ is a Gaussian random variable with respect to μ_ξ with mean $p\langle k, u\rangle$ and variance $\langle k, P_{u^\perp} k\rangle$. At a coarse level (and we prove this), one can view μ_ξ as the Gauss measure μ *conditioned* to satisfy $\hat{u} = p$; however, the general construction of such conditional measures provide existence for almost every p whereas we construct μ_ξ as a probability measure for each given value of (p, u). One other issue to observe here is that, as we prove in Theorem 3.1 in section 3, the measures μ_ξ and μ are

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mutually singular, and so, *a priori*, the functions $\hat{k}$, when viewed as elements in $L^2(\mu)$, do not have meaning as elements of $L^2(\mu_\xi)$.

In section 4 we formally introduce the Radon-Gauss transform. In brief, if f is a suitable measurable function on $(\Omega, \mathcal{F})$, then its Radon-Gauss transform Gf associated to each hyperplane ξ in the Hilbert space E_0, the value

$$Gf(\xi) = \int_\Omega f\, d\mu_\xi$$

and work out a few examples. Then in section 5 we establish an inversion formula for the transform. In the finite-dimensional case, it is known (see, for example, [1]) that there is an inversion formula using powers of the Laplacian and another formula using the Fourier transform. We have not been able to give an appropriate meaning to an infinity-power of the Laplacian in our context (though the possibility of a meaningful definition remains), so we proceeded using the *Segal-Bargmann transform* S, and in Theorem 5.1 we establish a relation which allows inversion of G by inverting S.

2. Gaussian Measure on Hyperplanes: Construction

In this section we construct and study Gaussian measure on hyperplanes in Hilbert spaces. Although numerous avenues exist for this construction (from Minlos' theorem to ideas connected with Malliavin calculus), we choose a direct construction using Kolmogorov's method, which then provides a single measure space $(\Omega, \mathcal{F})$ on which all hyperplane measures will be defined simultaneously and also provides a linear space of measurable functions which are simultaneously dense in all the hyperplane L^2-spaces.

To begin with, let us consider the finite dimensional case. Consider a hyperplane ξ in $\mathbf{R}^N$. Then we can pick a unit normal vector $u \in \mathbf{R}^N$ such that

$$\xi = pu + u^\perp, \tag{2.1}$$

where $p \in \mathbf{R}$ is the distance of ξ from the origin. The standard Gaussian measure μ_ξ on ξ is given by

$$d\mu_\xi(y) = (2\pi)^{-\frac{N-1}{2}} e^{-|y-pu|^2/2} dy, \tag{2.2}$$

where $y \in \mathbf{R}^N$, but dy denotes Lebesgue measure on ξ. This measure is centered at the point pu, which is the point on ξ closest to the origin. Let $\hat{\mu}_\xi$ be the characteristic function for μ_ξ, viewed as a probability measure on $\mathbf{R}^N$. Then, for any $k \in \mathbf{R}^N$, we have

$$\begin{aligned}
\hat{\mu}_\xi(k) &= \int_\xi e^{i\langle k,y\rangle} (2\pi)^{-\frac{N-1}{2}} e^{-|y-pu|^2/2} dy \\
&= \int_{u^\perp} e^{i\langle k,pu+z\rangle} (2\pi)^{-\frac{N-1}{2}} e^{-|z|^2/2} dz \\
&= e^{ip\langle k,u\rangle} \int_{u^\perp} e^{i\langle k,z\rangle} (2\pi)^{-\frac{N-1}{2}} e^{-|z|^2/2} dz \\
&= e^{ip\langle k,u\rangle} e^{-|P_{u^\perp} k|^2/2},
\end{aligned}$$

Thus,

$$\hat{\mu}_\xi(k) = e^{ip\langle k,u\rangle - |P_{u^\perp} k|^2/2}, \tag{2.3}$$

where $P_{u^\perp} k$ is the orthogonal projection of k onto $u^\perp$.

Now we move to the infinite dimensional situation. Suppose E_0 is a separable, real infinite-dimensional Hilbert space. Let $\{e_n\}_{n\in\mathbf{P}}$ be an orthonormal basis of E_0, where $\mathbf{P}$ is the set of positive integers:

$$\mathbf{P} = \{1, 2, 3, ...\}$$

Let $(\Omega, \mathcal{F}, \mu)$ be the probability space, where

$$\Omega = \mathbf{R}^{\mathbf{P}}, \tag{2.4}$$

$\mathcal{F}$ is the product σ–algebra, and μ is the product of the standard Gaussian measure $(2\pi)^{-1/2}e^{-x^2/2}dx$. The coordinate projections

$$X_j : \Omega \to \mathbf{R} : \omega \mapsto \omega_j$$

are independent standard Gaussian random variables. If x is an element in the Hilbert space E_0 we have then the random variable $\hat{x}$ on Ω given by

$$\hat{x} = \sum_{j\in\mathbf{P}} \langle x, e_j\rangle X_j, \tag{2.5}$$

which is an $L^2(\mu)$–convergent series since

$$\sum_{j\in\mathbf{P}} \langle x, e_j\rangle^2 = \|x\|^2 < \infty$$

The series for $\hat{x}$ converges everywhere on Ω if x is in the linear span (as opposed to the closed linear span) of the vectors $e_1, e_2, ...$

Motivated by the finite-dimensional case, we will prove:

Theorem 2.1. *Suppose E_0 is a real, separable, infinite-dimensional Hilbert space. Let ξ be any closed hyperplane in E, given as $\xi = pu + u^\perp$, where $u \in E_0$ is a unit vector and $p \in \mathbf{R}$. Then there is a probability measure μ_ξ on the product space $(\Omega, \mathcal{F})$, and, for each $k \in E_0$, a Gaussian random variable $\hat{k}$ on $(\Omega, \mathcal{F}, \mu_\xi)$, depending linearly on k, such that*

$$\int_\Omega e^{i\hat{k}}\, d\mu_\xi = e^{ip\langle k,u\rangle - \frac{1}{2}\|P_{u^\perp}k\|^2} \tag{2.6}$$

for all $k \in E_0$.

We will carry out a direct construction using Kolmogorov's method of producing a measure on the infinite product through a consistent family of measures on finite-dimensional "sub-products." It is important to note that all the probability measures μ_ξ are defined on the *same* measurable space $(\Omega, \mathcal{F})$. If Ω is allowed to depend on ξ then the construction is much simpler. It should also be noted that random variables $\hat{k}$, for $k \in E_0$, were constructed for $L^2(\mu)$ but those random variables do not specify well-defined elements in $L^2(\mu_\xi)$, since μ_ξ, as we shall see later, lives on a set of μ–measure 0.

Proof. As before, let $\{e_j\}_{j\in\mathbf{P}}$ be an orthonormal basis of E_0. For $N \in \mathbf{P}$ let $u_N \in \mathbf{R}^N$ be the vector given by the first N components of u:

$$u_N = (\langle u, e_1\rangle, ..., \langle u, e_N\rangle) \tag{2.7}$$

First we construct on $\mathbf{R}^N$ the measure μ_N whose characteristic function is given by

$$\hat{\mu}_N(k) \stackrel{\text{def}}{=} \int_{\mathbf{R}^N} e^{i\langle k,x\rangle}\, d\mu_N(x) = e^{ip\langle k,u_N\rangle - \frac{1}{2}\left(\|k\|^2 - |\langle k,u_N\rangle|^2\right)} \tag{2.8}$$

The quadratic part of the exponent on the right can be recast as follows:

$$\|k\|^2 - |\langle k, u_N\rangle|^2 = \langle k, B_N k\rangle, \tag{2.9}$$

where

$$B_N = \left(1 - \|u_N\|^2\right)I + \|u_N\|^2 P_{u_N^\perp} \tag{2.10}$$

If $\|u_N\| < 1$ and then $B_N > 0$. In this case, μ_N is the characteristic function of the Gaussian measure μ_N on $\mathbf{R}^N$ given by:

$$d\mu_N(x) = \left(2\pi\right)^{-N/2}(\det B_N)^{-1/2} e^{-\frac{1}{2}\langle x - pu_N, B_N^{-1}(x-pu_N)\rangle}\, dx, \tag{2.11}$$

for, using $x = pu_N + B_N^{1/2} z$,

$$\begin{aligned}\int_{\mathbf{R}^N} e^{i\langle k,x\rangle}\, d\mu_N(x) &= e^{i\langle k,pu_N\rangle}\int_{\mathbf{R}^N} e^{i\langle B_N^{1/2}k,z\rangle - \frac{1}{2}\langle z,z\rangle}\,\frac{dz}{(2\pi)^{N/2}}\\ &= e^{i\langle k,pu_N\rangle} e^{-\frac{1}{2}\langle k,B_N k\rangle},\end{aligned}$$

which matches (2.8).

If $\|u_N\| = 1$ then $B_N = P_{u_N^\perp}$ (which is not invertible), and in this case μ_N is the hyperplane Gaussian μ_ξ considered in (2.2) and (2.3). (If Ω were allowed to depend on ξ then we could simply start with an orthonormal basis for which $e_1 = u$, and we would be done at this stage.) At the other extreme, if $u_N = 0$ then μ_N is just standard Gauss measure on $\mathbf{R}^N$.

The expression (2.8) for $\hat{\mu}_N(k)$ shows a consistency property: if $k = (k_1, ..., k_N)$ and $k' = (k, 0)$, then

$$\hat{\mu}_{N+1}(k') = \hat{\mu}_N(k)$$

This implies that for any Borel set $A \subset \mathbf{R}^N$ we have

$$\mu_{N+1}(A \times \mathbf{R}) = \mu_N(A),$$

and so Kolomogorov's theorem on existence of probability measures implies that there is a probability measure μ_ξ on $\mathbf{R}^{\mathbf{P}}$, with the property that, for every $N \in \mathbf{P}$,

$$\mu_\xi\left(p_N^{-1}(A)\right) = \mu_N(A),$$

for all Borel $A \subset \mathbf{R}^N$, with $p_N : \mathbf{R}^{\mathbf{P}} \to \mathbf{R}^N$ being the projection on the first N components.

Let E_{00} be the subspace of E_0 given by the linear span of the vectors $e_1, e_2, ...$ Let $k \in E_{00}$, and $k_j = \langle k, e_j\rangle$. Suppose that $k_j = 0$ for $j > N$. Let $\hat{k}$ denote the random variable on Ω given by

$$\hat{k}(x) = \sum_{j\in\mathbf{P}} k_j x_j$$

Then:

$$\begin{aligned}\int_\Omega e^{i\hat{k}} d\mu_\xi &= \int_{\mathbf{R}^N} e^{i(k_1x_1+\cdots+k_Nx_N)}\, d\mu_N(x)\\ &= \hat{\mu}_N\left((k_1, ..., k_N)\right)\\ &= e^{ip\langle k,u_N\rangle - \frac{1}{2}\left(\|k\|^2 - |\langle k,u_N\rangle|^2\right)}\\ &= e^{ip\langle k,u\rangle - \frac{1}{2}\left(\|k\|^2 - |\langle k,u\rangle|^2\right)}\end{aligned}$$

Thus,

$$\int_\Omega e^{i\hat{k}} d\mu_\xi = e^{ip\langle k,u_N\rangle - \frac{1}{2}\langle k, P_{u^\perp} k\rangle} \tag{2.12}$$

for all $k \in E_{00}$. Note that for $k \in E_{00}$, the function $\hat{k}$ is defined everywhere on Ω, and the relation (2.12) implies

$$\int_\Omega e^{it\hat{k}} d\mu_\xi = e^{itp\langle k,u\rangle - \frac{1}{2}t^2\langle k, P_{u^\perp} k\rangle}, \tag{2.13}$$

which shows that, with respect to the probability measure μ_ξ, the random variable $\hat{k}$ has Gaussian distribution with mean $p\langle k,u\rangle$ and variance $\langle k, P_{u^\perp} k\rangle$. (This is because a random variable X is Gaussian with mean m and variance v if and only if its characteristic function is $e^{itm - t^2 v/2}$.) Consequently,

$$\|\hat{k}\|^2_{L^2(\mu_\xi)} = p^2\langle k,u\rangle^2 + \langle k, P_{u^\perp} k\rangle \leq (p^2+1)\|k\|^2 \tag{2.14}$$

This implies that the linear mapping

$$E_{00} \to L^2(\mu_\xi) : k \mapsto \hat{k}$$

extends to a continuous linear mapping

$$E_0 \to L^2(\mu_\xi) : k \mapsto \hat{k}$$

For $k \in E_0$, we can choose a sequence of points $k'(n) \in E_{00}$ which converges to E_0. Then, by (2.14), $\hat{k'(n)} \to \hat{k}$ in $L^2(\mu_\xi)$, and so, by examining characteristic functions, it follows that $\hat{k}$ is Gaussian. Moreover, equation (2.12) implies (2.6). □

Here is a somewhat more general construction:

Theorem 2.2. *Let E_0 be a real, separable, infinite-dimensional Hilbert space, $U : E_0 \to \mathbf{R}^n$ a linear map onto $\mathbf{R}^n$, and let $p \in \mathbf{R}^n$. Then there is a unique measure $\mu_{p,U}$ on $(\Omega, \mathcal{F})$ and a linear map $E_0 \to L^2(\Omega, \mathcal{F}, \mu_{U,p}) : k \mapsto \hat{k}$ such that, for all $k \in E_0$:*

$$\int_\Omega e^{i\hat{k}}\, d\mu_{U,p} = e^{i\langle k, U_R^{-1}(p)\rangle - \frac{1}{2}\|k_\perp\|^2}, \tag{2.15}$$

where $k_\perp$ is the orthogonal projection of k onto $\ker U$, and $U_R^{-1} : \mathbf{R}^n \to (\ker U)^\perp$ is the isomorphism which inverts $U|(\ker U)^\perp$.

Proof. The argument being quite similar to the previous proof, we only sketch the essential elements. For any integer $N \geq 1$ consider the Gaussian measure $\mu_{U,p,N}$ on $\mathbf{R}^N$ specified by

$$\int_{\mathbf{R}^N} e^{i\sum_{r=1}^N k_r x_r}\, d\mu_{U,p,N}(x) = e^{i\left\langle \sum_{r=1}^N k_r e_r, U_R^{-1}(p)\right\rangle - \frac{1}{2}\sum_{r,s=1}^N k_r k_s \langle (e_r)_\perp, (e_s)_\perp\rangle}, \tag{2.16}$$

for all $(k_1, ..., k_N) \in \mathbf{R}^N$. Consistency of the system $\{\mu_{U,p,N}\}_{N\geq 1}$ follows by observing that

$$\hat{\mu}_{U,p,N+1}(k_1, ..., k_N, 0) = \hat{\mu}_{U,p,N}(k_1, ..., k_N) \tag{2.17}$$

Kolmogorov's theorem then gives the existence of a measure $\mu_{U,p}$ on $(\Omega, \mathcal{F})$ for which

$$\int_\Omega e^{i\hat{k}} d\mu_{U,p} = \hat{\mu}_{U,p,N}(k_1, ..., k_N) \tag{2.18}$$

holds for every $k \in E_0$ of the form $k = k_1 e_1 + \cdots + k_N e_N$. Using (2.16) we then have the desired formula (2.15) for all k in the subspace E_{00} spanned by the vectors $e_1, e_2, ...$ The case of general $k \in E_0$ since both sides in (2.15) are continuous in $k \in E_0$. □

3. Gaussian Measure on Hyperplanes: Properties

We proceed with the same setting as in the preceding section: E_0 is a real, separable, Hilbert space, $\Omega = \mathbf{R}^{\mathbf{P}}$, where $\mathbf{P} = \{1, 2, 3, ...\}$, and $\mathcal{F}$ is the product sigma–algebra of subsets of Ω. Let $\{e_n\}_{n\in\mathbf{P}}$ be an orthonormal basis of E_0, and E_{00} the subset of E_0 which is the linear span of the vectors $e_1, e_2, ...$ For each $k \in E_{00}$ we then have a measurable function $\hat{k}$ on Ω given by

$$\hat{k}(x) = \sum_{j\in\mathbf{P}} \langle k, e_j\rangle x_j,$$

where $x = (x_j)_{j\in\mathbf{P}} \in \Omega$. A typical hyperplane in E_0 will be denoted ξ, and is given as

$$\xi = pu + u^{\perp},$$

for some unit vector $u \in E_0$ and $p \in \mathbf{R}$. In the preceding section we constructed a probability measure μ_ξ on $(\Omega, \mathcal{F})$ for which

$$\int_\Omega e^{i\hat{k}}\, d\mu_\xi = e^{ip\langle k,u\rangle - \frac{1}{2}\|P_{u^\perp}k\|^2} \tag{3.1}$$

for all $k \in E_{00}$, thereby making $\hat{k}$ a Gaussian random variable on $(\Omega, \mathcal{F}, \mu_\xi)$. We then showed that there is a bounded linear map

$$E_0 \to L^2(\mu_\xi) : k \mapsto \hat{k}$$

such that equation (3.1) continues to hold for all $k \in E_0$.

Intuitively, it seems that μ_ξ should "live" on the subspace ξ. However, just as the product Gaussian measure does not live on E_0, we cannot expect μ_ξ to literally assign mass 1 to ξ viewed somehow as a subset of Ω. The correct version of this is:

Proposition 3.1. *Suppose E_0 is a real, separable Hilbert space, and ξ a hyperplane given by $\xi = pu + u^{\perp}$, where $p \in \mathbf{R}$ and u is a unit vector in E_0. Let μ_ξ be the probability measure on $(\Omega, \mathcal{F})$ from Theorem 2.1, and, for each $k \in E_0$, let $\hat{k}$ be the random variable described in Theorem 2.1. Then*

$$\hat{u}(x) = p \text{ for } \mu_\xi\text{–almost-every } x \in \Omega$$

It follows, as a consequence, that μ_ξ assigns full measure 1 to a set whose μ–measure is 0, where μ is the product Gaussian measure on $(\Omega, \mathcal{F})$.

Proof. From (2.6) we have

$$\begin{aligned}\int_\Omega e^{it\hat{u}}\, d\mu_\xi &= e^{ip\langle u,tu\rangle - \frac{1}{2}\|P_{u^\perp}tu\|^2}\\ &= e^{ipt}\\ &= \int_{\mathbf{R}} e^{its}\, d\delta_p(s),\end{aligned}$$

where δ_p is the delta measure with $\delta_p(\{p\}) = 1$. So, since the characteristic function of a random variable uniquely specifies the distribution, it follows that the random variable $\hat{u}$ has the distribution δ_p, i.e. $\hat{u}$ has the constant value p almost everywhere.

Consider any unit vector $u \in E_{00}$. Then, relative to the probability measure μ, the function $\hat{u}$ is a Gaussian random variable for variable $\|u\|^2 > 0$. On the other hand, the *same* function $\hat{u}$, is almost every-where constant with respect to μ_ξ. Thus the set $\hat{u}^{-1}(p)$ has μ–measure 0 but μ_ξ–measure 1. $\square$

We record the corresponding fact for the measures $\mu_{U,p}$ which were constructed in Theorem 2.2. As in that theorem, we have our separable Hilbert space E_0 and a linear surjection

$$U : E_0 \to \mathbf{R}^n$$

Now let $b_1, ..., b_n$ be the standard basis of $\mathbf{R}^n$. The linear map U can be expressed as

$$U(v) = \sum_{r=1}^{n} \langle U_r, v\rangle b_r \qquad \text{for all } v \in E_0 \tag{3.2}$$

for a unique set of vectors $U_1, ..., U_n \in E_0$. Then we have:

Theorem 3.2. *With notation and hypotheses as just described, and $p \in \mathbf{R}^n$, let $\hat{U}$ be the $\mathbf{R}^n$–valued random variable on $(\Omega, \mathcal{F}, \mu_{U,p})$ given by*

$$\hat{U} = \sum_{r=1}^{n} \hat{U}_r b_r.$$

Then

$$\hat{U} = p \qquad \mu_{U,p}\text{–almost-everywhere}$$

Proof. Let $U^* : \mathbf{R}^n \to E_0$ be the adjoint map satisfying $\langle U^*y, v\rangle = \langle y, Uv\rangle$ for all $y \in \mathbf{R}^n$ and $v \in E_0$. Then $U^*b_r = U_r$ for each $r \in \{1, 2, ..., n\}$. As is readily checked, the image of U^* is orthogonal to $\ker U$, and so, in particular, each U_r is orthogonal to $\ker U$. Then, using the characteristic-function formula for $\mu_{U,p}$ given in (2.15) we have

$$\int_\Omega e^{i\hat{U}_r}\, d\mu_{U,p} = e^{i\langle U^*b_r, U_R^{-1}(p)\rangle - \frac{1}{2}\|(U_r)_\perp\|^2}, \tag{3.3}$$

where $(U_r)_\perp$, being the orthogonal projection of U_r onto $\ker U$, is in fact 0. The first term in the exponent on the right equals

$$i\langle b_r, UU_R^{-1}(p)\rangle = i\langle b_r, p\rangle.$$

So

$$\int_\Omega e^{i\hat{U}_r}\, d\mu_{U,p} = e^{i\langle b_r, p\rangle}, \tag{3.4}$$

which proves that $\hat{U}_r$ is $\mu_{U,p}$–almost-everywhere equal to the constant value $\langle b_r, p\rangle$. Thus, the $\mathbf{R}^n$–valued random variable $\hat{U}$ is $\mu_{U,p}$–almost-everywhere equal to the constant value $\sum_{r=1}^{n}\langle b_r, p\rangle b_r = p$. $\square$

Before we proceed to other properties of μ_ξ we prepare a general lemma:

Lemma 3.3. *Suppose $\{\mathcal{G}_n\}_{n\in\mathbf{P}}$ is an increasing family of sigma-algebras of subsets of a set Ω', and let $\mathcal{G} = \sigma(\cup_{n\in\mathbf{P}}\mathcal{G}_n)$, the sigma-algebra generated by all the collections $\mathcal{G}_n$. Then, for any finite measure ν on $\mathcal{G}$, the Hilbert space $L^2(\Omega', \mathcal{G}, \nu)$ has $\cup_{n\in\mathbf{P}}L^2(\Omega', \mathcal{G}_n, \nu)$ as a dense linear subspace.*

Proof. Suppose $f \in L^2(\Omega', \mathcal{G}, \nu)$ is orthogonal to $V \stackrel{\text{def}}{=} \cup_{n\in\mathbf{P}}L^2(\Omega', \mathcal{G}_n, \nu)$. Our objective is to show that f must be 0 in $L^2(\Omega', \mathcal{G}, \nu)$.

Let $\mathcal{L}$ be the collection of all sets $A \in \mathcal{G}$ for which f is orthogonal to 1_A, i.e. $\int_A f\, d\nu = 0$. Let $\mathcal{P}$ be the union of all the sigma–algebras $\mathcal{G}_n$; since the latter are an increasing family of sigma-algebras, it follows that $\mathcal{P}$ is closed under finite intersections. Moreover, $\mathcal{L}$ contains $\emptyset$ and Ω' (since these are both in $\mathcal{P}$), is closed under complements and countable disjoint unions. Therefore, by the Dynkin $\pi - \lambda$ theorem, it follows that $\mathcal{L} \supset \sigma(\mathcal{P})$; thus,

$$\mathcal{L} = \mathcal{G}.$$

This means $\int_A f\, d\nu = 0$ for all $A \in \mathcal{G}$. So $f = 0$ almost everywhere with respect to ν. □

We can now describe a collection of functions on Ω whose linear span is dense simultaneously in all $L^2(\nu)$, for every finite measure ν on $(\Omega, \mathcal{F})$:

Proposition 3.4. *The functions $e^{i\hat{k}}$, as k runs over E_{00}, span a dense subspace of $L^2(\nu)$, for any finite measure ν on $(\Omega, \mathcal{F})$.*

Proof. Let W be the closure of the linear subspace of $L^2(\mu_\xi)$ containing all the functions $e^{i\hat{k}}$, for $k \in E_{00}$. Let $f \in W^\perp$. Our objective is to show that $f = 0$.

Let $\mathcal{G}_N$ be the sigma–algebra of subsets of Ω generated by the functions $\hat{e}_1, ..., \hat{e}_N$, and let f_N be the orthogonal projection of f onto the closed subspace $L^2(\mathcal{G}_N, \nu)$. Then

$$f_N(x) = F_N(\hat{e}_1, ..., \hat{e}_N)$$

for some function $F \in L^2(\nu_N)$, with ν_N being the measure on $\mathbf{R}^N$ specified by

$$\nu_N(A) = \nu\big(p_N^{-1}(A)\big),$$

where A is any Borel subset of $\mathbf{R}^N$, and $p_N : \mathbf{R}^{\mathbf{P}} \to \mathbf{R}^N$ is the projection on the first N components.

Let $k \in E_{00}$, and suppose $k_n = 0$ for $n > N$. Then, using notation from (2.11), we have

$$\begin{aligned} 0 &= \int_\Omega e^{i\hat{k}} f\, d\nu \\ &= \int_\Omega e^{i\sum_{j=1}^N k_j \hat{e}_j} f_n\, d\nu \\ &= \int_{\mathbf{R}^N} e^{i\sum_{j=1}^N k_j x_j} F_N(x)\, d\nu_N(x) \end{aligned}$$

Since this holds for all $(k_1, ..., k_N) \in \mathbf{R}^N$, it follows that the complex measure specified by $F_N d\nu_N$ is 0, and this implies F_N is zero μ_N–almost-everywhere. Since this is true for every f_N, it follows by Lemma 3.3 that $f = 0$ in $L^2(\nu)$. □

The next fact we verify is that μ_ξ provides a disintegration of the Gaussian measure μ, as p runs over $\mathbf{R}$, with u any fixed unit vector.

Theorem 3.5. *Let f be a non-negative or complex-valued measurable function on $(\Omega, \mathcal{F})$, and u a unit vector in E_0. Then*

$$p \mapsto G_u f(p) \stackrel{\text{def}}{=} \int_\Omega f\, d\mu_{pu+u^\perp} \tag{3.5}$$

is a Borel measurable function on $\mathbf{R}$. Furthermore,

$$\int_\Omega f\, d\mu = \int_{\mathbf{R}} \left[\int_\Omega f\, d\mu_{pu+u^\perp}\right] e^{-p^2/2} \frac{dp}{\sqrt{2\pi}}, \tag{3.6}$$

whenever the left side exists.

Note that, in general, conditional expectations are well-defined only almost-everywhere, not pointwise, and so cannot be used to define the measure μ_ξ for a given ξ.

Proof. Let $\mathcal{F}_0$ be the set of all cylinder subsets of Ω, i.e. sets of the form

$$\{\omega \in \Omega : \big(\hat{e}_1(\omega), ..., \hat{e}_N(\omega)\big) \in A\},$$

with N ranging over $\mathbf{P} = \{1, 2, 3, ...\}$ and A over all Borel subsets of $\mathbf{R}^N$. Assume for the moment that the conclusions of our Theorem hold for all functions f of the form 1_C with $C \in \mathcal{F}_0$. Now the collection of all sets $B \in \mathcal{F}$ for which our Theorem holds for $f = 1_B$ forms a λ–system (closed under countable disjoint unions, and complements). By assumption, this λ–system contains the collection $\mathcal{F}_0$ which is a π–system (contains $\emptyset$ and is closed under complements). Then by the Dynkin $\pi - \lambda$ theorem, it follows that our Theorem holds for all f of the form 1_D with $D \in \mathcal{F}$. Hence, by linearity and monotone convergence, we have the full result.

The reasoning in the preceding paragraph shows that it will suffice to prove our Theorem for all functions f of the form $F(\hat{e}_1, ..., \hat{e}_N)$, with N ranging over $\mathbf{P}$ and F over all bounded functions on $\mathbf{R}^N$. We shall use notation from the proof of Theorem 2.1, where μ_ξ was constructed using the system of finite-dimensional measures μ_n on $\mathbf{R}^n$. In particular, for the particular f we have now,

$$\int_\Omega f \, d\mu_{pu+u^\perp} = \int_{\mathbf{R}^N} F \, d\mu_N, \tag{3.7}$$

where μ_N is the Gaussian measure on $\mathbf{R}^N$ specified by (2.8):

$$\hat{\mu}_N(k) = e^{ip\langle k,u_N\rangle - \frac{1}{2}\left(\|k\|^2 - |\langle k,u_N\rangle|^2\right)}, \tag{3.8}$$

and

$$u_N = \big(\langle u, e_1\rangle, ..., \langle u, e_N\rangle\big)$$

To prove measurability of the function Gf_u, we consider the two cases $\|u_N\| < 1$ and $\|u_N\| = 1$ separately. If $\|u_N\| = 1$ then, as seen in the proof of Theorem 2.1, the integral on the right in (3.7) is the integral of F over the hyperplane $\{x \in \mathbf{R}^N : \langle u_N, x\rangle = p\}$. It is, therefore, a Borel measurable function of p, by essentially the argument used in the proof of the measurability-part of the usual Fubini's theorem (the argument needs a slight, but straightforward, modification because the section under consideration is a general hyperplane, not necessarily orthogonal to a vector in the standard basis). Now consider the case $\|u_N\| < 1$. In this case, by (2.11),

$$\int_{\mathbf{R}^N} F \, d\mu_N = \int_{\mathbf{R}^N} F(x)\big(2\pi\big)^{-N/2}\big(\det B_N\big)^{-1/2} e^{-\frac{1}{2}\langle x-pu_N, B_N^{-1}(x-pu_N)\rangle} \, dx, \tag{3.9}$$

and this is certainly Borel measurable (indeed, continuous) in p.

Finally, we have to prove the disintegration formula (3.6) which, as we have seen, is equivalent to proving the special case where $f = F(\hat{e}_1, ..., \hat{e}_N)$, where F is any bounded Borel function on $\mathbf{R}^N$ and $N \in \mathbf{P}$. For this f the left side of (3.6) is the standard Gaussian integral

$$\int_{\mathbf{R}^N} F(x) \, (2\pi)^{N/2} e^{-\|x\|^2/2} \, dx, \tag{3.10}$$

and the right side is

$$\int_{\mathbf{R}}\left[\int_{\mathbf{R}^N} F(x)\,d\mu_N(x)\right](2\pi)^{-1/2}e^{-p^2/2}\,dp. \tag{3.11}$$

Thus we will be done if we can show that the Borel measure μ'_N on $\mathbf{R}^N$ specified through

$$\int_{\mathbf{R}^N} F\,d\mu'_N = \int_{\mathbf{R}}\left[\int_{\mathbf{R}^N} F(x)\,d\mu_N(x)\right](2\pi)^{-1/2}e^{-p^2/2}\,dp \tag{3.12}$$

for all bounded measurable F, is actually the standard Gaussian measure on $\mathbf{R}^N$. To establish this all we need to do is verify that $\int_{\mathbf{R}^N} e^{i\langle k,x\rangle}\,d\mu'_N(x)$ equals the standard Gaussian characteristic $e^{-\|k\|^2/2}$ for all $k \in \mathbf{R}^N$:

$$\begin{aligned}
\int_{\mathbf{R}^N} e^{i\langle k,x\rangle}\,d\mu'_N(x) &= \int_{\mathbf{R}}\left[\int_{\mathbf{R}^N} e^{i\langle k,x\rangle}\,d\mu_N(x)\right](2\pi)^{-1/2}e^{-p^2/2}\,dp \\
&= \int_{\mathbf{R}} e^{ip\langle k,u_N\rangle-\frac{1}{2}\left(\|k\|^2-\langle k,u_N\rangle^2\right)}\,e^{-p^2/2}\,\frac{dp}{\sqrt{2\pi}} \qquad \text{by (3.8)} \\
&= e^{-\frac{1}{2}\langle k,u_N\rangle^2-\frac{1}{2}\left(\|k\|^2-\langle k,u_N\rangle^2\right)} \\
&= e^{-\frac{1}{2}\|k\|^2}
\end{aligned}$$

This proves that μ'_N is in fact the standard Gaussian measure on $\mathbf{R}^N$, and hence proves the disintegration formula (3.6). ☐

As consequence we have

Corollary 3.6. *Let $f \in L^2(\Omega, \mathcal{F}, \mu)$, and u a fixed unit vector in E_0; let $G_u f(p) = Gf(pu + u^\perp)$. Then $G_u f$ is the image of f under the orthogonal projection of $L^2(\Omega, \mathcal{F}, \mu)$ onto the subspace $L^2(\Omega, \sigma(\hat{u}), \mu)$, where $\sigma(\hat{u})$ is the sigma–algebra generated by $\hat{u}$. Thus, $Gf(pu + u^\perp)$ is the conditional expectation $E_\mu[f|\hat{u} = p]$*

Proof. Let $f \in L^2(\mu)$ and $g \in L^2\big(\mathbf{R}; e^{-x^2/2}(2\pi)^{-1/2}dx\big)$. Now in the disintegration formula (3.6) use $g(\hat{u})f$ in place of f, to obtain the desired conclusions. ☐

4. The Radon-Gauss transform

Let μ be the standard product Gaussian measure on $(\Omega, \mathcal{F})$, where $\Omega = \mathbf{R}^{\mathbf{P}}$, with $\mathbf{P} = \{1, 2, 3, ...\}$, and $\mathcal{F}$ is the product sigma-algebra. Let E_0 be a real separable Hilbert space with orthonormal basis $(e_n)_{n\in\mathbf{P}}$, and E_{00} the linear subspace spanned by the vectors $e_1, e_2, ...$ As usual, for $k \in E_{00}$ we have the Gaussian random variable $\hat{k}$ on $(\Omega, \mathcal{F}, \mu)$ given by

$$\hat{k}(x) = \sum_{j\in\mathbf{P}} k_j x_j$$

and the linear map $k \mapsto \hat{k}$ extends to a linear isometry $E_0 \to L^2(\mu) : k \mapsto \hat{k}$, with $\hat{k}$ being again Gaussian with mean zero and variance $\|k\|^2$.

Definition 4.1. *Let f be a measurable function f on $(\Omega, \mathcal{F})$ such that $\int_\Omega f\,d\mu_\xi$ exists for each hyperplane ξ in E_0. Consider the function Gf which associates to each hyperplane ξ in E_0, the value*

$$(Gf)(\xi) = \int_\Omega f\,d\mu_\xi \tag{4.1}$$

We call Gf the <u>*Radon-Gauss transform*</u> *of f.*

Let us make a quick obervation:

Proposition 4.2. *Let $f \in L^2(\mu)$, and u a unit vector in the Hilbert space E_0. Then $f \in L^2(\mu_{pu+u^\perp})$ for almost every $p \in \mathbf{R}$ and*

$$\|f\|^2_{L^2(\mu)} = \int_{\mathbf{R}} \|f\|^2_{L^2(\mu_{pu+u^\perp})} \frac{e^{-p^2/2}}{\sqrt{2\pi}} dp \tag{4.2}$$

In particular, if f equals 0 μ–almost-everywhere, then $Gf(pu+u^\perp)$ equals 0 for almost every $p \in \mathbf{R}$.

Proof. This follows from the disintegration formula (3.6). □

Let us work out the Radon-Gauss transform of the "coherent state" ("re")normaized exponential function

$$c_k = e^{\hat{k} - \frac{1}{2}\|k\|^2}, \tag{4.3}$$

for any $k \in E_0$. Then for any unit vector u and real number p, we have:

$$\begin{aligned} Gf(pu+u^\perp) &= \int_\Omega e^{\hat{k}-\frac{1}{2}\|k\|^2}\, d\mu_\xi \\ &= e^{p\langle k,u\rangle - \frac{1}{2}\langle k,u\rangle^2}, \end{aligned}$$

which we obtain by splitting k as a part $k_\perp$ orthogonal to u and a part $\langle k,u\rangle u$ parallel to u.

5. An Inversion Formula

The behavior of the measure μ under translations $x \mapsto x + \tilde{k}$ gives rise to useful notions and questions. For a bounded measurable function f on Ω, and $k \in E_0$ define

$$Sf(k) = \int_\Omega f e^{\hat{k} - \|k\|^2/2}\, d\mu, \tag{5.1}$$

which is actually equal to $\int_\Omega f(x+\tilde{k})\, d\mu(x)$, where $\tilde{k} \in \mathbf{R}^{\mathbf{P}}$ is the coordinate vector

$$\tilde{k} = \big(\langle k, e_j\rangle\big)_{j\in\mathbf{P}}, \tag{5.2}$$

as is readily verified by taking f to be a cylinder function first.

The corresponding definition for functions on $\mathbf{R}$ is

$$S_{\mathbf{R}}g(t) = \int_{\mathbf{R}} g(y+t)e^{-y^2/2}\frac{dy}{\sqrt{2\pi}} = \int_{\mathbf{R}} g(y)e^{ty-y^2/2}\frac{dy}{\sqrt{2\pi}}$$

It can be shown that $S_{\mathbf{R}}g$ has a holomorphic extension to a function, also denoted $S_{\mathbf{R}}g$, on $\mathbf{C}$, and $S_{\mathbf{R}}$ then specifies a unitary isomorphism of $L^2(\mathbf{R}; e^{-p^2/2}/(2\pi)^{1/2})$ onto the Hilbert space of holomorphic functions on $\mathbf{C}$ which are square-integrable with respect to the Gaussian measure $e^{-|z|^2}|dz|/\pi$. This is the *Segal-Bargmann* transform, and exists also in the infinite-dimensional setting. The transform S is also useful in studying distributions in terms of their S–transforms. The S–transform, and its inversion, has been well-studied in the literature (see, for instance [2] for the S–transform on distributions on infinite-dimensional spaces).

Our next objective is

Theorem 5.1. *Let $f \in L^2(\Omega, \mathcal{F}, \mu)$, and, for each unit vector $u \in E_0$, let F_u be the function on $\mathbf{R}$ given by:*

$$F_u(p) = (Gf)(pu + u^\perp) \tag{5.3}$$

Then $F_u \in L^2\big(e^{-p^2/2}(2\pi)^{-1/2}dp\big)$ and

$$(S_{\mathbf{R}}F_u)(t) = (Sf)(tu) \tag{5.4}$$

Therefore, f may be recovered from Gf by inverting the S–transform in (5.4).

Proof. We have:

$$\begin{aligned}
(S_{\mathbf{R}}F_u)(t) &= (2\pi)^{-\frac{1}{2}} \int_{\mathbf{R}} F_u(p+t)e^{-\frac{p^2}{2}}\,dp \\
&= (2\pi)^{-\frac{1}{2}} \int_{\mathbf{R}} F_u(p)e^{tp-\frac{p^2}{2}-\frac{t^2}{2}}\,dp \\
&= \int_{\mathbf{R}} \left[\int_\Omega f\,d\mu_\xi\right] e^{tp-\frac{t^2}{2}}\,\frac{e^{-\frac{p^2}{2}}dp}{(2\pi)^{1/2}} \qquad \text{where } \xi = pu + u^\perp \\
&= \int_{\mathbf{R}} \left[\int_\Omega f e^{t\hat{u}-\frac{t^2}{2}}\,d\mu_\xi\right]\, e^{-p^2/2}\frac{dp}{\sqrt{2\pi}} \qquad \text{by Theorem 3.5} \\
&= \int_\Omega f e^{t\hat{u}-\frac{t^2}{2}}\,d\mu \qquad \text{by Proposition 3.1} \\
&= (Sf)(tu),
\end{aligned}$$

which completes the argument. ∎

We observe also that the preceding result allows the possibility of defining the conditional expectations $Gf(\xi)$ of a distribution f, by inverting the $S_{\mathbf{R}}$–transform.

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