

# Extension of the Fourier-Budan theorem to one-variable signomials

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## Abstract

Let  $f(x) = a_0x^{r_0} + a_1x^{r_1} + \cdots + a_kx^{r_k}$ , where each  $a_i \in \mathbb{R}$ , each  $r_k \in \mathbb{N} := \{0, 1, \dots\}$ , and  $r_0 < r_1 < \cdots < r_k$ . Suppose  $u < v$ . Let  $z(f, u, v)$  = the number of roots of  $f$  in  $(u, v]$ , counted with multiplicity. For any  $w \in \mathbb{R}$  and  $n \in \mathbb{N}$ , let  $s(f, w, n)$  = the number of sign-changes in the sequence  $f(w), f'(w), f''(w), \dots, f^{(n)}(w)$  (skipping over zeros). Then the Fourier-Budan Theorem says that  $z(f, u, v) \leq s(f, u, r_k) - s(f, v, r_k)$  and  $z(f, u, v) \equiv s(f, u, r_k) - s(f, v, r_k) \pmod{2}$ . In this paper we weaken the hypothesis of this theorem by allowing the exponents of  $f$  to be arbitrary real numbers; but we must then restrict  $u$  and  $v$  to be positive, to avoid non-real values of  $f(x)$ . Our conclusion is then that there exists an  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $z(f, u, v) \leq s(f, u, n) - s(f, v, n)$  and  $z(f, u, v) \equiv s(f, u, n) - s(f, v, n) \pmod{2}$ .

## 1 Introduction

Let  $f$  be a ‘signomial’ in one variable  $x$ ; i.e., let  $f$  be a function  $f : (0, \infty) \rightarrow \mathbb{R}$  of the form

$$f(x) = a_0x^{r_0} + a_1x^{r_1} + \cdots + a_kx^{r_k}, \quad (1.0.1)$$

where  $k \in \mathbb{N} := \{0, 1, 2, \dots\}$ ,  $a_i, r_i \in \mathbb{R}$ ,  $a_i \neq 0$ ,  $r_0 < r_1 < \cdots < r_k$ , and  $x$  is a real variable with  $x > 0$ .<sup>1</sup> Then  $f \neq 0$ , and  $f$  is real analytic on  $(0, \infty)$ .

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<sup>1</sup>In general, a *signomial* in  $n$  variables  $x_1, \dots, x_m$  is a finite sum of terms of the form  $ax_1^{e_1} \cdots x_n^{e_n}$ , where  $a, e_1, \dots, e_n \in \mathbb{R}$ . See [Delzell, 2004] for background on signomials.

Suppose  $0 < w \in \mathbb{R}$  and  $n \in \mathbb{N}$ , and write

$$s(f, w, n) = \begin{bmatrix} \# \text{ of sign-changes in the sequence} \\ f(w), f'(w), \dots, f^{(n)}(w) \\ \text{(skipping over zeros)} \end{bmatrix}. \quad (1.0.2)$$

Suppose also that  $u \in \mathbb{R}$  and  $0 < u < v$ , and write

$$z(f, u, v) = \begin{bmatrix} \# \text{ of zeros of } f \text{ in } (u, v], \\ \text{counted with multiplicity} \end{bmatrix}. \quad (1.0.3)$$

(For a definition of multiplicity, see §2 below.) Since  $f \not\equiv 0$ ,  $z(f, u, v) < \infty$ —say, by the fact that a nonzero (real) analytic function can have only finitely many zeros in the compact set  $[u, v]$ . (Alternatively, the finiteness of  $z(f, u, v)$  will be contained in the following theorem.)

**Theorem 1.1** (Generalization of the Fourier-Budan Theorem to one-variable signomials). *If  $0 < u < v$ , then there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,*

$$z(f, u, v) \leq s(f, u, n) - s(f, v, n), \quad \text{and} \quad (1.1.1)$$

$$z(f, u, v) \equiv s(f, u, n) - s(f, v, n) \pmod{2}. \quad (1.1.2)$$

The usual Fourier-Budan theorem makes the additional hypothesis that  $f$  in (1.0.1) is a polynomial, i.e., that each  $r_i \in \mathbb{N}$ ; and it makes the additional conclusion that the  $N$  in (1.1) may be taken to be  $r_k$ .

In Theorem 7.5 below, we shall see that in the case where  $r_i \notin \mathbb{N}$  for only one  $i$ , we may still take  $N$  in (1.1) to be  $\max\{0, \lceil r_k \rceil\}$  (where  $\lceil r_k \rceil$  denotes the “ceiling” of  $r_k$ , i.e., the least integer  $\geq r_k$ ). But if there exist two or more  $i$  for which  $r_i \notin \mathbb{N}$ , then the upper bound that Theorem 7.5 gives on the  $N$  in (1.1) is complicated.

Theorem 1.1 above does not generalize to arbitrary real analytic functions  $f$  on  $(0, \infty)$ :

**Example 1.2.** Let  $f(x) = \sin x$ , which is real analytic on  $(0, \infty)$  (or even on  $(-\infty, \infty)$ ). Since  $f$  and all its derivatives have period  $2\pi$ , for every  $n \geq 0$ ,

$$\begin{aligned} 0 &= s\left(f, \frac{\pi}{2}, n\right) - s\left(f, \frac{\pi}{2}, n\right) \\ &= s\left(f, \frac{\pi}{2}, n\right) - s\left(f, \frac{\pi}{2} + 2\pi, n\right) \\ &= s\left(f, \frac{\pi}{2}, n\right) - s\left(f, \frac{\pi}{2} + 4\pi, n\right) = \dots \\ &= s\left(f, \frac{\pi}{2}, n\right) - s\left(f, \frac{\pi}{2} + 2m\pi, n\right) = \dots, \end{aligned}$$

for every  $m \geq 0$ . But  $z(f, \frac{\pi}{2}, \frac{\pi}{2} + 2m\pi) = 2m$ . Thus (1.1.1) fails for each  $m \geq 1$ , regardless of the choice of  $n$  or  $N \in \mathbb{N}$ .

[... references to Descartes, Fourier, Budan, Laguerre, Xiaoshen Wang..., and Basu, Pollock, and Roy...]

We shall prove Theorem 1.1 in section 6, after presenting some preliminary results in sections 2–5.

## 2 The parity of $z(g, u, v)$ for real analytic $g$

Suppose  $g$  is a real analytic function on a neighborhood (in  $\mathbb{R}$ ) of  $[u, v]$ ; for brevity, we shall say, simply, that  $g$  is analytic on  $[u, v]$ . Suppose also that  $g \not\equiv 0$ . For  $w \in [u, v]$ , we define the *multiplicity* (or *order*) of  $g$  at  $w$  to be the least  $d \in \mathbb{N}$  such that  $g^{(d)}(w) \neq 0$  ( $d$  exists, since  $g \not\equiv 0$ ). This number equals the least  $d \in \mathbb{N}$  such that the function

$$h_d(x) := \frac{g(x)}{(x-w)^d} \quad (2.0.1)$$

(which is obviously analytic on  $[u, v] \setminus \{w\}$ ) extends analytically to all of  $[u, v]$ . For this  $d$ , we denote this extension also by  $h_d$ ; moreover,  $h_d(w) \neq 0$  (specifically,  $h_d(w) = g^{(d)}(w)/d!$ ).

With these explicit definitions, we now generalize (1.0.3) by defining  $z(g, u, v)$  to be the sum of the multiplicities of  $g$  at  $w$ , for all  $w \in (u, v]$ . (This sum is finite, since  $g$  has only finitely many zeros in  $[u, v]$ .)

For  $a \in \mathbb{R}$ , we write

$$\operatorname{sgn} a = \begin{cases} +1 & \text{if } a > 0, \\ 0 & \text{if } a = 0, \text{ and} \\ -1 & \text{if } a < 0. \end{cases}$$

From (2.0.1) we conclude that if the multiplicity of  $g$  at  $w$  is  $d$ , then there exist  $\epsilon_1, \epsilon_2 > 0$  such that

$$\begin{aligned} &g \text{ has constant sign on } (w - \epsilon_1, w) \\ &\quad \text{(namely, } (-1)^d \operatorname{sgn} h(w) = (-1)^d \operatorname{sgn} g^{(d)}(w)\text{), and} \\ &g \text{ has constant sign on } (w, w + \epsilon_2) \\ &\quad \text{(namely, } \operatorname{sgn} h(w) = \operatorname{sgn} g^{(d)}(w)\text{).} \end{aligned} \quad (2.0.2)$$

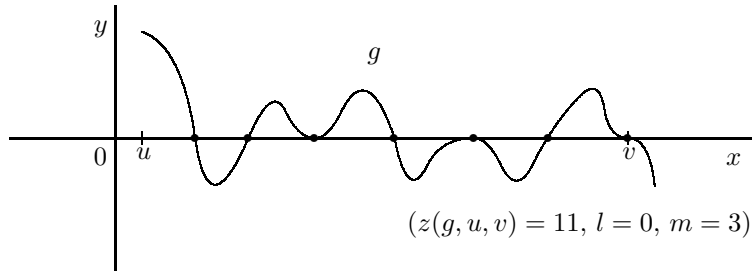
We write  $\operatorname{sgn} g(w^+) = \operatorname{sgn} h(w)$  and  $\operatorname{sgn} g(w^-) = (-1)^d \operatorname{sgn} h(w)$ .

**Lemma 2.1.** *Suppose  $u, v \in \mathbb{R}$ ,  $u < v$ , and  $g$  is real analytic on  $[u, v]$  (but  $g \not\equiv 0$ ). Let  $l, m \in \mathbb{N}$  be the multiplicities of  $g$  at  $u$  and  $v$ , respectively. Then*

$$g^{(l)}(u)g^{(m)}(v) > 0 \Leftrightarrow (-1)^m \operatorname{sgn} g(u^+) \operatorname{sgn} g(v^-) > 0 \Leftrightarrow z(g, u, v) \text{ is even.}$$

*Proof.* The first equivalence is immediate from (2.0.2).

One may convince oneself of the second equivalence by examining the sample graph below, which illustrates the essential possibilities. For a formal proof of the second equivalence, however, we use induction on  $z(g, u, v)$ , as follows.



If  $z(g, u, v) = 0$ , then  $(\Rightarrow)$  is trivial, since 0 is even. For  $(\Leftarrow)$ , note that  $m = 0$  in this case, and hence  $(-1)^m \operatorname{sgn} g(u^+) \operatorname{sgn} g(v^-) > 0$ , by the Intermediate-Value Theorem.

So suppose  $z(g, u, v) > 0$ , and suppose that the lemma holds with  $g$  replaced by any function  $h$  analytic on  $[u, v]$  for which  $z(h, u, v) < z(g, u, v)$ ; we shall prove the lemma for  $g$ .

Since  $z(g, u, v) > 0$ , there is some  $w \in (u, v]$  at which  $g$  has multiplicity  $\geq 1$ . Define  $h_1(x) = g(x)/(x-w)^1$ , which extends to an analytic function (still denoted by  $h_1$ ) on a neighborhood of  $[u, v]$ . Then  $z(h_1, u, v) = z(g, u, v) - 1$ .

*Case 1:*  $w \in (u, v)$ . Then the multiplicity of  $h_1$  at  $v$  is still  $m$ . Also,  $\operatorname{sgn} h_1(u^+) = -\operatorname{sgn} g(u^+)$  and  $\operatorname{sgn} h_1(v^-) = \operatorname{sgn} g(v^-)$ . Therefore

$$\begin{aligned} & (-1)^m \operatorname{sgn} g(u^+) \operatorname{sgn} g(v^-) > 0 \\ \Leftrightarrow & (-1)^m \operatorname{sgn} h_1(u^+) \operatorname{sgn} h_1(v^-) < 0 \\ \Leftrightarrow & z(h_1, u, v) \text{ is odd (by the inductive hypothesis)} \\ \Leftrightarrow & z(g, u, v) \text{ is even.} \end{aligned}$$

*Case 2:*  $w = v$ . Then  $m > 0$ , and the multiplicity of  $h_1$  at  $v$  is  $m - 1$ . Also,  $\operatorname{sgn} h_1(u^+) = -\operatorname{sgn} g(u^+)$  and  $\operatorname{sgn} h_1(v^-) = -\operatorname{sgn} g(v^-)$ . Therefore

$$\begin{aligned} & (-1)^m \operatorname{sgn} g(u^+) \operatorname{sgn} g(v^-) > 0 \\ \Leftrightarrow & (-1)^{m-1} \operatorname{sgn} h_1(u^+) \operatorname{sgn} h_1(v^-) < 0 \\ \Leftrightarrow & z(h_1, u, v) \text{ is odd (by the inductive hypothesis)} \\ \Leftrightarrow & z(g, u, v) \text{ is even.} \end{aligned} \quad \square$$

### 3 Variants of Rolle's Theorem for real analytic functions

Throughout this section, we adopt the following hypotheses and notation. We let  $u, v \in \mathbb{R}$ ,  $u < v$ , and  $g$  be real analytic on  $[u, v]$  (but  $g \not\equiv 0$ ). We let  $l$  and  $l'$  be the multiplicities of  $g$  and  $g'$ , respectively, at  $u$ , and we let  $m$  and  $m'$  be those at  $v$ . We write the distinct roots of  $g$  in  $(u, v]$  as  $z_1, \dots, z_e$  ( $e \geq 0$ ), in such a way that  $u < z_1 < z_2 < \dots < z_e \leq v$ .

**Lemma 3.1.** *Suppose  $e \geq 1$ . The number of roots (counting multiplicities) of  $g$  in  $[z_1, z_e]$  (or in  $(u, v]$ ), is at most 1 plus the number of roots of  $g'$  in  $[z_1, z_e]$ . (Consequently,  $z(g, u, v) \leq z(g', u, v) + 1$ , whether  $e \geq 1$  or not.)*

*Proof.* Since  $e \geq 1$ , the notation  $[z_1, z_e]$  is meaningful (and in case  $e = 1$ , it means  $\{z_1\}$ ). Write  $d_i$  for the multiplicity of  $g$  at  $z_i$ . Rolle's theorem<sup>2</sup> asserts that  $g'$  has at least one root in each open interval  $(z_i, z_{i+1})$ , for  $i = 1, 2, \dots, e-1$ . Then the number of roots of  $g'$  in  $[z_1, z_e]$  is at least

$$(e-1) + (d_1 - 1) + \dots + (d_e - 1) = d_1 + \dots + d_e - 1,$$

which is one less than the number of roots of  $g$  in  $[z_1, z_e]$ .  $\square$

**Lemma 3.2.** *Suppose  $e \geq 1$ . If  $g^{(l)}(u) \cdot (g')^{(l)}(u) > 0$ , then  $g'$  has at least one root in  $(u, z_1)$ .*

*Proof.* To see this, note first that

$$\begin{aligned} \operatorname{sgn} g'(u^+) &= \operatorname{sgn} g(u^+) \quad \text{by hypothesis (recall (2.0.2))} \\ &= \operatorname{sgn} g(z_1^-) \quad \text{by applying the Intermediate-Value Theorem to} \\ &\quad g \text{ on } [u + \epsilon, z_1 - \epsilon] \text{ for sufficiently small } \epsilon > 0 \\ &= -\operatorname{sgn} g'(z_1^-) \quad \text{since } g(z_1) = 0 \text{ (recall (2.0.2)).} \end{aligned}$$

Therefore, applying the Intermediate-Value Theorem to  $g'$  on  $[u + \epsilon, z_1 - \epsilon]$  for sufficiently small  $\epsilon > 0$ , we conclude that  $g'$  has at least one root in  $(u, z_1)$ .  $\square$

**Lemma 3.3.** *Suppose  $e \geq 1$ . If  $g^{(m)}(v) \cdot (g')^{(m)}(v) < 0$ , then  $g(v) \neq 0$  (i.e.,  $z_e \neq v$ ), and  $g'$  has at least one root in  $(z_e, v]$ .*

*Proof.* First,  $g(v) \neq 0$ , for otherwise  $m' = m - 1$  and  $g^{(m)}(v) = (g')^{(m)}(v)$ , contrary to hypothesis. Thus,  $m = 0$  and  $g^{(m)}(v) = g(v)$ . Second, if  $g'$  has no root in  $(z_e, v]$ , then, in particular,  $g'(v) \neq 0$ , whence  $m' = 0$  and  $(g')^{(m)}(v) = g'(v)$ . Thus

$$\begin{aligned} -\operatorname{sgn} g'(v) &= \operatorname{sgn} g(v) \quad \text{by hypothesis} \\ &= \operatorname{sgn} g(z_e^+) \quad \text{by applying the Intermediate-Value Theorem to} \\ &\quad g \text{ on } [z_e + \epsilon, v] \text{ for sufficiently small } \epsilon > 0 \\ &= \operatorname{sgn} g'(z_e^+) \quad \text{since } g(z_e) = 0 \text{ (recall (2.0.2)).} \end{aligned}$$

Therefore, applying the Intermediate-Value Theorem to  $g'$  on  $[z_e + \epsilon, v]$  for sufficiently small  $\epsilon > 0$ , we conclude that  $g'$  has at least one root in  $(z_e, v]$ , contradiction.  $\square$

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<sup>2</sup>[Wang, 2004] seemed to use to name ‘‘Rolle’s Theorem’’ to refer to the inequality in Lemma 3.1.

**Lemma 3.4.**

$$[g^{(l)}(u) \cdot (g')^{(l')}(u) > 0 \vee g^{(m)}(v) \cdot (g')^{(m')}(v) < 0] \Rightarrow z(g, u, v) \leq z(g', u, v); \quad (3.4.1)$$

$$[g^{(l)}(u) \cdot (g')^{(l')}(u) > 0 \wedge g^{(m)}(v) \cdot (g')^{(m')}(v) < 0] \Rightarrow z(g, u, v) \leq z(g', u, v) - 1. \quad (3.4.2)$$

*Proof.* In case  $e \geq 1$ , both (3.4.1) and (3.4.2) follow from (3.1), (3.2), and (3.3). In case  $e = 0$  (i.e.,  $z(g, u, v) = 0$ ), (3.4.1) is trivial. So it remains to prove (3.4.2) in case  $e = 0$ . In this case,  $g(v) \neq 0$  (i.e.,  $m = 0$ ), and  $\text{sgn } g(u^+) = \text{sgn } g(v)$ , by the Intermediate Value Theorem applied to  $g$  on  $[u + \epsilon, v]$  for sufficiently small  $\epsilon > 0$ . Since  $\text{sgn } g(u^+) = \text{sgn } g^{(l)}(u)$  (recall (2.0.2)), we have  $\text{sgn}(g')^{(l')}(u) = -\text{sgn}(g')^{(m')}(v)$ , by the hypothesis of (3.4.2). Equivalently,  $\text{sgn } g'(u^+) = -\text{sgn } g'(v^+)$ . Thus for every sufficiently small  $\epsilon > 0$ ,  $g'$  has a root in  $(u + \epsilon, v + \epsilon)$ , by the Intermediate-Value Theorem. Thus  $g'$  has a root in  $(u, v]$ , i.e.,  $z(g', u, v) \geq 1$ , which implies the conclusion of (3.4.2) in case  $e = 0$ .  $\square$

## 4 $s(g, w, n)$ versus $s(g', w, n - 1)$ for real analytic $g$

Suppose  $n \in \mathbb{N}$ ,  $w \in \mathbb{R}$ , and in a neighborhood of  $w$ , the function  $g$  is (real) analytic and  $\neq 0$ . Then we define the notation  $s(g, w, n)$  and  $s(g', w, n)$  as in (1.0.2), but with  $f$  replaced by  $g$  and  $g'$ , respectively.

**Lemma 4.1.** *If  $d$  and  $d'$  are the multiplicities at  $w$  of  $g$  and  $g'$ , respectively, then for all  $n > d'$ ,*

$$\text{sgn } g^{(d)}(w) = \text{sgn}(g')^{(d')}(w) \quad \Rightarrow \quad s(g, w, n) = s(g', w, n - 1), \quad \text{and} \quad (4.1.1)$$

$$\text{sgn } g^{(d)}(w) \neq \text{sgn}(g')^{(d')}(w) \quad \Rightarrow \quad s(g, w, n) = s(g', w, n - 1) + 1. \quad (4.1.2)$$

*Proof.* Recall that  $s(g, w, n)$  and  $s(g', w, n - 1)$  count the changes in the subsequence of *nonzero* signs in

$$\begin{aligned} &\text{sgn } g(w), \text{sgn } g'(w), \text{sgn } g''(w), \dots, \text{sgn } g^{(n)}(w) \quad \text{and in} \\ &\text{sgn } g'(w), \text{sgn } g''(w), \dots, \text{sgn } g^{(n)}(w), \end{aligned} \quad (4.1.3)$$

respectively.

*Case 1:*  $g(w) \neq 0$ . Then  $d = 0$ , and the second of the two above-mentioned subsequences arises from the first by dropping the first term,  $\text{sgn } g(w)$ . So if  $\text{sgn } g^{(d)}(w) \neq \text{sgn}(g')^{(d')}(w)$ , then  $s(g, w, n) = s(g', w, n - 1) + 1$  (using  $n > d'$ ), proving (4.1.2) in this case. (If  $1 \leq n \leq d'$ , then the two sequences (4.1.3) would be  $\text{sgn } g(w), 0, \dots, 0$  and  $0, \dots, 0$ , respectively, implying  $s(g, w, n) = s(g', w, n - 1) = 0$ .) If, on the other hand,  $\text{sgn } g^{(d)}(w) = \text{sgn}(g')^{(d')}(w)$ , then  $s(g, w, n) = s(g', w, n - 1)$ , even without using  $n > d'$ , proving (4.1.1) in this case.

*Case 2:*  $g(w) = 0$ . Then  $d > 0$  and  $d' = d - 1$ , whence  $g^{(d)}(w) = (g')^{(d-1)}(w) = (g')^{(d')}(w)$ , implying that the hypothesis of (4.1.1) (and not (4.1.2)) holds. The conclusion of (4.1.1) holds, too, since in this case the two above-mentioned subsequences are exactly the same (again, even without assuming  $n > d'$ ).  $\square$

## 5 Eventual alternation of $\text{sgn } f^{(n)}(w)$

For any  $r \in \mathbb{R}$ , we write  $\lceil r \rceil$  (the ‘‘ceiling’’ of  $r$ ) for the least integer  $\geq r$ , and  $\lfloor r \rfloor$  (the ‘‘floor’’ of  $r$ ) for the greatest integer  $\leq r$ .

**Lemma 5.1.** *For  $n \in \mathbb{N}$ ,  $r \in \mathbb{R} \setminus \mathbb{N}$ ,  $a \in \mathbb{R} \setminus \{0\}$ , and  $0 < w \in \mathbb{R}$ ,*

$$\text{sgn} \left( \frac{d^n}{dx^n} (ax^r) \Big|_{x=w} \right) = (-1)^{\max\{0, n - \max\{0, \lceil r \rceil\}\}} \text{sgn } a. \quad (5.1.1)$$

*Proof.* We use induction on  $n \geq 0$ . For  $n = 0$ ,  $\max\{0, n - \max\{0, \lceil r \rceil\}\} = 0$ , and so (5.1.1) is obvious (using  $w^r > 0$ ). For  $n > 0$ , we distinguish two cases.

*Case 1:*  $r < 0$ . Then  $\max\{0, \lceil r \rceil\} = 0$ , and so the righthand side of (5.1.1) begins alternating with  $n = 1, 2, \dots$  (as does the lefthand side).

*Case 2:*  $r > 0$ . Then  $\max\{0, \lceil r \rceil\} = \lceil r \rceil$ , and so the righthand side of (5.1.1) begins alternating with  $n = 1 + \lceil r \rceil, 2 + \lceil r \rceil, \dots$ , again in agreement with the lefthand side.  $\square$

**Proposition 5.2.** *Suppose  $f$  is as in (1.0.1), but  $f$  is not a polynomial—i.e.,  $r_i \notin \mathbb{N}$  for some  $i \in \{0, 1, \dots, k\}$ . Let  $i_0$  be the least such  $i$ . Suppose also that  $0 < v \in \mathbb{R}$ . Then there exists an  $M \in \mathbb{N}$  such that  $\forall n \geq M$  and  $\forall w \in (0, v)$ ,*

$$\text{sgn } f^{(n)}(w) = \text{sgn} \left( \frac{d^n}{dx^n} (a_{i_0} x^{r_{i_0}}) \Big|_{x=w} \right). \quad (5.2.1)$$

*Proof.* For  $i = 0, 1, \dots, k$  and  $x > 0$ , let  $g_i(x) = a_i x^{r_i}$ .

Determine  $t \in \mathbb{N}$  and  $i_0, i_1, \dots, i_t \in \mathbb{N}$  by the conditions that

$$\begin{aligned} 0 \leq i_0 < i_1 < \dots < i_t \leq k \text{ and} \\ \{r_{i_0}, r_{i_1}, \dots, r_{i_t}\} &= \{r_0, \dots, r_k\} \setminus \mathbb{N}. \end{aligned} \quad (5.2.2)$$

Thus  $r_{i_0} < r_{i_1} < \dots < r_{i_t}$  is the subsequence of  $r_0 < r_1 < \dots < r_k$  in (1.0.1) consisting of those elements  $r_i$  not in  $\mathbb{N}$ . Then

$$\forall n \geq \max\{0, 1 + \lceil r_k \rceil\}, \forall i \in \{0, 1, \dots, k\} \setminus \{i_0, i_1, \dots, i_t\}, g_i^{(n)}(x) \equiv 0. \quad (5.2.3)$$

Note that  $t \geq 0$ , by the hypothesis that some  $r_i \notin \mathbb{N}$ . We may even assume that  $t \geq 1$ , since if  $t = 0$ , then (5.2) is trivially true upon taking  $M = \max\{0, 1 + \lceil r_k \rceil\}$ , by (5.2.3).

We shall first prove that for  $s = 1, 2, \dots, t$ ,

$$\exists M_s \in \mathbb{N} \quad \forall n \geq M_s, \quad \forall w \in (0, v], \quad \left| \frac{g_{i_0}^{(n)}(w)}{g_{i_s}^{(n)}(w)} \right| > t. \quad (5.2.4)_s$$

Once (5.2.4)<sub>1</sub>, (5.2.4)<sub>2</sub>,  $\dots$ , and (5.2.4)<sub>t</sub> have been proved, it will follow that  $\forall n \geq \max\{M_1, M_2, \dots, M_t\}$ ,  $\forall s \in \{1, 2, \dots, t\}$ , and  $\forall w \in (0, v]$ ,

$$\begin{aligned} \frac{1}{t} |g_{i_0}^{(n)}(w)| &> |g_{i_s}^{(n)}(w)|; \text{ adding these for } s = 1, 2, \dots, t: \\ |g_{i_0}^{(n)}(w)| &> \sum_{s=1}^t |g_{i_s}^{(n)}(w)|. \end{aligned}$$

Then we shall be able to take

$$M = \max\{1 + \lceil r_k \rceil, M_1, \dots, M_t\} \quad (5.2.5)$$

in (5.2). For then  $\forall n \geq M$  and  $\forall w \in (0, v]$ ,

$$|g_{i_0}^{(n)}(w)| > \sum_{s=1}^t |g_{i_s}^{(n)}(w)| = \sum_{\substack{0 \leq i \leq k \\ i \neq i_0}} |g_i^{(n)}(w)| \geq \left| \sum_{\substack{0 \leq i \leq k \\ i \neq i_0}} g_i^{(n)}(w) \right|, \quad (5.2.3)$$

which implies (5.2.1).

So fix  $s \in \{1, 2, \dots, t\}$ . To find  $M_s$  as in (5.2.4)<sub>s</sub>, observe that for  $n \geq 1$ ,

$$\left| \frac{g_{i_0}^{(n)}(w)}{g_{i_s}^{(n)}(w)} \right| = \left| \frac{r_{i_0}(r_{i_0} - 1)(r_{i_0} - 2) \cdots (r_{i_0} - n + 1) a_{i_0} w^{r_{i_0}}}{r_{i_s}(r_{i_s} - 1)(r_{i_s} - 2) \cdots (r_{i_s} - n + 1) a_{i_s} w^{r_{i_s}}} \right|.$$

(The numerator and denominator are nonzero, since  $r_{i_0}, r_{i_s} \notin \mathbb{N}$ .) Let  $l_s = \max\{0, \lceil r_{i_s} \rceil\}$  ( $\leq \max\{0, 1 + \lceil r_k \rceil\}$ ), and let

$$\begin{aligned} L_s &= \begin{cases} 1 & \text{if } l_s = 0 \\ \left| \frac{r_{i_0}(r_{i_0} - 1)(r_{i_0} - 2) \cdots (r_{i_0} - l_s + 1)}{r_{i_s}(r_{i_s} - 1)(r_{i_s} - 2) \cdots (r_{i_s} - l_s + 1)} \right| & \text{if } l_s \geq 1 \end{cases} \\ &= \begin{cases} 1 & \text{if } l_s = 0 \\ \left| \frac{r_{i_0}(r_{i_0} - 1)(r_{i_0} - 2) \cdots (r_{i_0} - l_s + 1)}{r_{i_s}(r_{i_s} - 1)(r_{i_s} - 2) \cdots (r_{i_s} - l_s + 1)} \right| & \text{if } l_s \geq 1. \end{cases} \end{aligned} \quad (5.2.6)$$

Then for  $n > l_s$ ,

$$\left| \frac{g_{i_0}^{(n)}(w)}{g_{i_s}^{(n)}(w)} \right| = L_s \frac{(l_s - r_{i_0})(l_s + 1 - r_{i_0}) \cdots (n - 1 - r_{i_0}) |a_{i_0}| w^{r_{i_0}}}{(l_s - r_{i_s})(l_s + 1 - r_{i_s}) \cdots (n - 1 - r_{i_s}) |a_{i_s}| w^{r_{i_s}}},$$

---

<sup>3</sup>From the proof below, more will be evident:  $\lim_{n \rightarrow \infty} |g_{i_0}^{(n)}(w)/g_{i_s}^{(n)}(w)| = \infty$ , uniformly for all  $w \in (0, v]$  and for all  $s = 1, 2, \dots, t$ ; i.e., (5.2.4)<sub>s</sub> would remain true with  $t$  replaced by any real number, no matter how large. But we do not need this.

since  $r_{i_0} < r_{i_s} < l_s$ . Consequently, for  $n > l_s$ ,

$$\begin{aligned} \ln \left| \frac{g_{i_0}^{(n)}(w)}{g_{i_s}^{(n)}(w)} \right| &= \ln \left( L_s \frac{|a_{i_0}|w^{r_{i_0}}}{|a_{i_s}|w^{r_{i_s}}} \right) + (\ln(l_s - r_{i_0}) - \ln(l_s - r_{i_s})) + \\ &\quad (\ln(l_s + 1 - r_{i_0}) - \ln(l_s + 1 - r_{i_s})) + \cdots + \\ &\quad (\ln(n - 1 - r_{i_0}) - \ln(n - 1 - r_{i_s})). \end{aligned}$$

Applying the Mean-Value Theorem to  $\ln x$ , there exist  $c_j \in (j - r_{i_s}, j - r_{i_0})$  ( $j = l_s, l_s + 1, \dots, n - 1$ ) such that for  $n > l_s$ ,

$$\ln \left| \frac{g_{i_0}^{(n)}(w)}{g_{i_s}^{(n)}(w)} \right| = \ln \left( L_s \frac{|a_{i_0}|w^{r_{i_0}}}{|a_{i_s}|w^{r_{i_s}}} \right) + \frac{r_{i_s} - r_{i_0}}{c_{l_s}} + \frac{r_{i_s} - r_{i_0}}{c_{l_s+1}} + \cdots + \frac{r_{i_s} - r_{i_0}}{c_{n-1}}.$$

Write  $m = \lfloor r_{i_0} \rfloor (< l_s)$ . Then since  $0 < c_j < j - r_{i_0} < j - m$ , we have, for all  $n > l_s$ ,

$$\begin{aligned} \ln \left| \frac{g_{i_s}^{(n)}(w)}{g_{i_0}^{(n)}(w)} \right| & \tag{5.2.7} \\ &> \ln \left( L_s \frac{|a_{i_0}|w^{r_{i_0}}}{|a_{i_s}|w^{r_{i_s}}} \right) + (r_{i_s} - r_{i_0}) \left[ \frac{1}{l_s - m} + \frac{1}{l_s + 1 - m} + \cdots + \frac{1}{n - 1 - m} \right]. \end{aligned}$$

Thus to complete the proof of (5.2.4)<sub>s</sub>, we need an  $M_s > l_s$  such that  $\forall n \geq M_s$  and  $\forall w \in (0, v]$ , the righthand side of (5.2.7) exceeds  $\ln t$ . For this, in turn, it suffices to have  $M_s > l_s$  such that  $\forall n \geq M_s$ ,

$$\ln \left( L_s \frac{|a_{i_0}|v^{r_{i_0}}}{|a_{i_s}|v^{r_{i_s}}} \right) + (r_{i_s} - r_{i_0}) \left[ \frac{1}{l_s - m} + \frac{1}{l_s + 1 - m} + \cdots + \frac{1}{n - 1 - m} \right] > \ln t, \tag{5.2.8}$$

since  $w \leq v$  and  $r_{i_0} - r_{i_s} < 0$ . But (5.2.8) holds for all  $n$  sufficiently large, since (every tail of) the harmonic series diverges; thus we have proved the existence of an  $M_s$  as in (5.2.4)<sub>s</sub>.  $\square$

In Section 7 below, we shall give an upper bound on the  $M_s$  in (5.2.4)<sub>s</sub> (and hence on  $M$  in (5.2.5) and (5.2)).

**Definition 5.3.** For  $f$  as in (1.0.1) and  $0 < v \in \mathbb{R}$ , we define  $M_{f,v}$  as follows:

$$M_{f,v} = \begin{cases} \deg f & \text{if } f \text{ is a polynomial, and} \\ \left[ \begin{array}{l} \text{the least } M \in \mathbb{N} \text{ satisfying} \\ \text{the conclusion of (5.2)} \end{array} \right] & \text{if } f \text{ is not a polynomial.} \end{cases}$$

**Corollary 5.4.** Let  $f$ ,  $v$ , and  $M_{f,v}$  be as in (5.3).

(a) For each fixed  $k \in \mathbb{N}$ , the integer

$$s(f, w, M_{f,v} + k) - s(f, w, M_{f,v}) \tag{5.4.1}$$

is constant for all  $w \in (0, v]$ .

(b) For  $u \in (0, v]$  and for all  $n, m \geq M_{f,v}$ ,

$$s(f, u, n) - s(f, v, n) = s(f, u, m) - s(f, v, m). \quad (5.4.2)$$

*Proof.* Part (a) is trivial if  $f$  is a polynomial, since then (5.4.1) is 0 for all  $w \in (0, v]$ , by (5.3). And if  $f$  is not a polynomial, then (5.4.1) equals  $\max\{0, n - \max\{0, \lceil r_{i_0} \rceil\}\}$  (by (5.2) and (5.1)), which is, again, obviously independent of  $w$ .

As for (b), note that

$$\begin{aligned} s(f, u, n) - s(f, u, M_{f,v}) &= s(f, v, n) - s(f, v, M_{f,v}), \text{ and} \\ s(f, u, m) - s(f, u, M_{f,v}) &= s(f, v, m) - s(f, v, M_{f,v}), \end{aligned}$$

by (a). Therefore

$$\begin{aligned} s(f, u, n) - s(f, v, n) &= s(f, u, M_{f,v}) - s(f, v, M_{f,v}) \\ &= s(f, u, m) - s(f, v, m). \end{aligned} \quad \square$$

**Corollary 5.5.** *Let  $f$ ,  $v$ , and  $M_{f,v}$  be as in (5.3). If  $M_{f,v} > 0$ , then  $M_{f',v} = M_{f,v} - 1$ .*

*Proof.* For each fixed  $w \in (0, v]$ , the sequence (of signs of)

$$f'(w), f''(w), \dots, (f')^{(M_{f,v}-1)}(w), \dots \text{ is the same as (those of)} \quad (5.5.1)$$

$$f(w), f'(w), f''(w), \dots, f^{(M_{f,v})}(w), \dots, \quad (5.5.2)$$

except for dropping the first term. And the sequence (of signs of)

$$(r_{i_0} a_{i_0} x^{r_{i_0}-1})|_{x=w}, \dots, \frac{d^{M_{f,v}-1}}{dx^{M_{f,v}-1}}(r_{i_0} a_{i_0} x^{r_{i_0}-1})|_{x=w}, \dots \quad (5.5.3)$$

is the same as (those of)

$$(a_{i_0} x^{r_{i_0}})|_{x=w}, \frac{d}{dx}(a_{i_0} x^{r_{i_0}})|_{x=w}, \dots, \frac{d^{M_{f,v}}}{dx^{M_{f,v}}}(a_{i_0} x^{r_{i_0}})|_{x=w}, \dots, \quad (5.5.4)$$

except for dropping the first term. Thus the stage at which, for all  $w \in (0, v]$ , the signs of the elements of (5.5.1) begin to agree with those of (5.5.3), is one less than the stage at which, for all  $w \in (0, v]$ , the signs of the elements of (5.5.2) begin to agree with those of (5.5.4).  $\square$

## 6 Proof of the main theorem

We are now ready to prove Theorem 1.1.

Let  $M_{f,v}$  be as in (5.3).

We claim that we may take  $N$  in (1.1) to be  $M_{f,v}$ .

We use induction on  $M_{f,v}$ . Suppose first that  $M_{f,v} = 0$ . Then in case  $f$  is a polynomial,  $f$  is actually a nonzero constant, whence  $z(f, u, v) = 0 = s(f, u, n) - s(f, v, n)$  for all  $n \geq 0$ , so that we may take  $N = 0$  in (1.1), as claimed. And if  $f$  is not a polynomial, then for all  $n \geq 0$ ,  $\text{sgn } f^{(n)}(w)$  is independent of  $w \in [u, v]$  (being equal to  $(-1)^{\max\{0, n - \max\{0, \lceil r_{i_0} \rceil\}} \text{sgn } a_{i_0}$  for  $i_0$  as in (5.2), by (5.2) and (5.1)), which again implies that  $s(f, u, n) - s(f, v, n) = 0$  for all  $n \geq 0$ , and (using the Intermediate-Value theorem) that  $z(f, u, v) = 0$ , again proving our claim.

Next suppose that  $M_f > 0$ . Then  $f' \neq 0$ , and  $M_{f'} = M_f - 1$  by (5.5). We may assume, by the inductive hypothesis, that for all  $n \geq M_{f',v}$ , (1.1.1) and (1.1.2) hold with  $f$  replaced by  $f'$ :

$$z(f', u, v) \leq s(f', u, n) - s(f', v, n), \quad \text{and} \quad (6.0.1)$$

$$z(f', u, v) \equiv s(f', u, n) - s(f', v, n) \pmod{2}. \quad (6.0.2)$$

Write  $l$  and  $l'$  for the multiplicities of  $f$  and  $f'$ , respectively, at  $u$ , and let  $m$  and  $m'$  be those at  $v$ .

*Case 1:* Either

$$\begin{aligned} & [\text{sgn } f^{(l)}(u) = \text{sgn}(f')^{(l')}(u) \quad \text{and} \quad \text{sgn } f^{(m)}(v) = \text{sgn}(f')^{(m')}(v)] \quad \text{or} \\ & [\text{sgn } f^{(l)}(u) \neq \text{sgn}(f')^{(l')}(u) \quad \text{and} \quad \text{sgn } f^{(m)}(v) \neq \text{sgn}(f')^{(m')}(v)]. \end{aligned} \quad (6.0.3)$$

Then, either way, for all  $n > \max\{l', m'\}$ ,

$$s(f, u, n) - s(f, v, n) = s(f', u, n-1) - s(f', v, n-1), \quad (6.0.4)$$

by (4.1). (6.0.4) then actually holds for all  $n \geq M_{f,v}$ , since the lefthand side is constant for all  $n \geq M_{f,v}$  by (5.4.2), and the righthand side is similarly constant whenever  $n-1 \geq M_{f',v}$  ( $= M_{f,v} - 1$ ). Then, mod 2, we have, for all  $n \geq M_{f,v}$ ,

$$\begin{aligned} z(f, u, v) &\equiv z(f', u, v) && \text{by (2.1) and (6.0.3)} \\ &\equiv s(f', u, n-1) - s(f', v, n-1) && \text{by (6.0.2) and (5.5)} \\ &= s(f, u, n) - s(f, v, n) && \text{by (6.0.4),} \end{aligned}$$

proving (1.1.2) in this case. Next, for all  $n \geq M_{f,v}$ ,

$$\begin{aligned} z(f, u, v) &\leq z(f', u, v) && \text{by (6.0.3) and (3.4.1)} \\ &\leq s(f', u, n-1) - s(f', v, n-1) && \text{by (6.0.1) and (5.5)} \\ &= s(f, u, n) - s(f, v, n) && \text{by (6.0.4),} \end{aligned}$$

proving (1.1.1) in this case. This proves that in this case we may take  $N = M_{f,v}$  in (1.1).

*Case 2:*

$$\text{sgn } f^{(l)}(u) = \text{sgn}(f')^{(l')}(u) \quad \text{and} \quad \text{sgn } f^{(m)}(v) \neq \text{sgn}(f')^{(m')}(v). \quad (6.0.5)$$

Then, for all  $n > \max\{l', m'\}$ ,

$$s(f, u, n) - s(f, v, n) = s(f', u, n-1) - s(f', v, n-1) - 1, \quad (6.0.6)$$

by (4.1). (6.0.6) then actually holds for all  $n \geq M_{f,v}$ , for the same reason that (6.0.4) did in Case 1. Then, mod 2, we have, for all  $n \geq M_{f,v}$ ,

$$\begin{aligned} z(f, u, v) &\equiv z(f', u, v) - 1 && \text{by (2.1) and (6.0.5)} \\ &\equiv s(f', u, n-1) - s(f', v, n-1) - 1 && \text{by (6.0.2) and (5.5)} \\ &= s(f, u, n) - s(f, v, n) && \text{by (6.0.6),} \end{aligned}$$

proving (1.1.2) in this case. Next, for all  $n \geq M_{f,v}$ ,

$$\begin{aligned} z(f, u, v) &\leq z(f', u, v) - 1 && \text{by (6.0.5) and (3.4.2)} \\ &\leq s(f', u, n-1) - s(f', v, n-1) - 1 && \text{by (6.0.1) and (5.5)} \\ &= s(f, u, n) - s(f, v, n) && \text{by (6.0.6),} \end{aligned}$$

proving (1.1.1) in this case. This proves that also in this case, we may take  $N = M_{f,v}$  in (1.1).

*Case 3:*

$$\text{sgn } f^{(l)}(u) \neq \text{sgn } (f')^{(l')}(u) \quad \text{and} \quad \text{sgn } f^{(m)}(v) = \text{sgn } (f')^{(m')}(v). \quad (6.0.7)$$

Then, for all  $n > \max\{l', m'\}$ ,

$$s(f, u, n) - s(f, v, n) = 1 + s(f', u, n-1) - s(f', v, n-1), \quad (6.0.8)$$

by (4.1). (6.0.8) then actually holds for all  $n \geq M_{f,v}$ , for the same reason that (6.0.4) did in case 1. Then, mod 2, we have, for all  $n \geq M_{f,v}$ ,

$$\begin{aligned} z(f, u, v) &\equiv 1 + z(f', u, v) && \text{by (2.1) and (6.0.7)} \\ &\equiv 1 + s(f', u, n-1) - s(f', v, n-1) && \text{by (6.0.2) and (5.5)} \\ &= s(f, u, n) - s(f, v, n) && \text{by (6.0.8),} \end{aligned}$$

proving (1.1.2) in this case. Next, for all  $n \geq M_{f,v}$ ,

$$\begin{aligned} z(f, u, v) &\leq 1 + z(f', u, v) && \text{by (3.1)} \\ &\leq 1 + s(f', u, n-1) - s(f', v, n-1) && \text{by (6.0.1) and (5.3)} \\ &= s(f, u, n) - s(f, v, n) && \text{by (6.0.8),} \end{aligned}$$

proving (1.1.1) in this case. This proves that also in this case, we may take  $N = M_{f,v}$  in (1.1), as claimed.  $\square$

## 7 Estimating $M_{f,v}$

Suppose that  $f$  in (1.0.1) is not a polynomial. As in (5.3), write  $M_{f,v}$  for the smallest  $M$  satisfying the conclusion of Lemma 5.2. In this section we give (7.4)

lower bounds for  $L_s$  (for each fixed  $s = 1, 2, \dots, t$ ) in terms of  $l_s, t, r_{i_0}$ , and  $r_{i_s}$ . After that we give (7.5) an upper bound for  $M_{f,v}$  in terms of  $L_1, \dots, L_t$ .

Lower bounds for  $L_s$ :

For  $l \in \mathbb{N}$  and  $r \in \mathbb{R}$ , we write

$$(r)_l := r(r-1)(r-2)\cdots(r-l+1)$$

for the “falling factorial.” Recall that, as long as  $r-l \notin \{-1, -2, \dots\}$ ,

$$(r)_l = \frac{\Gamma(r+1)}{\Gamma(r-l+1)}. \quad (7.0.1)$$

**Lemma 7.1.** For  $a, b \in \mathbb{R}$  with  $a, b > 0$ ,

$$\Gamma(a)\Gamma(b) > \Gamma\left(\frac{a+b}{2}\right)^2. \quad (7.1.1)$$

*Proof.* We may assume that  $a < b$ . Recall that for all  $x > 0$ , the second derivative of  $\ln \Gamma(x)$  is positive; i.e.,  $\ln \Gamma(x)$  is convex [Andrews, et al, 1999, Corollary 1.2.6]. Writing  $c = (a+b)/2$ , the Mean-Value Theorem therefore gives

$$\frac{\ln \Gamma(b) - \ln \Gamma(c)}{b-c} > \frac{\ln \Gamma(c) - \ln \Gamma(a)}{c-a}.$$

Since  $b-c = c-a$ ,

$$\begin{aligned} \ln \Gamma(b) - \ln \Gamma(c) &> \ln \Gamma(c) - \ln \Gamma(a), \text{ so} \\ \ln \Gamma(a) + \ln \Gamma(b) &> 2 \ln \Gamma(c), \quad \text{whence (7.1.1).} \quad \square \end{aligned}$$

**Lemma 7.2.** Suppose  $1 \leq l \in \mathbb{N}$ . Then for all  $r \in \mathbb{R} \setminus \mathbb{N}$  such that  $0 < r < l-1$ ,

$$|(r)_l| \geq \frac{1}{2} \text{dist}(r, \mathbb{Z}) \Gamma\left(\frac{l+1}{2}\right)^2, \quad (7.2.1)$$

where, for any  $r \in \mathbb{R}$ , we write  $\text{dist}$  for distance:

$$\text{dist}(r, \mathbb{Z}) = \min\{\lceil r \rceil - r, r - \lfloor r \rfloor\}.$$

*Proof.* Recall that

$$\forall x \in (1, 2), \quad 0 < \Gamma(x) < 1. \quad (7.2.2)$$

And note that  $r - \lfloor r \rfloor$  is either equal to  $\text{dist}(r, \mathbb{Z})$  or a member of  $[\frac{1}{2}, 1)$ ; and the opposite alternative holds for  $\lceil r \rceil - r$ . Then

$$\begin{aligned} |(r)_l| &= r(r-1)(r-2)\cdots(r-\lfloor r \rfloor+1)(r-\lfloor r \rfloor) \\ &\quad (\lceil r \rceil - r)(\lceil r \rceil + 1 - r)\cdots(l-1-r) \quad (\text{using } 0 < r < l-1) \\ &= \frac{\Gamma(r+1)}{\Gamma(r-\lfloor r \rfloor+1)}(r-\lfloor r \rfloor) \cdot (\lceil r \rceil - r) \frac{\Gamma(l-r)}{\Gamma(\lceil r \rceil+1-r)} \quad (\text{using (7.0.1)}) \\ &> \frac{1}{2} \text{dist}(r, \mathbb{Z}) \Gamma(r+1) \Gamma(l-r) \quad ((7.2.2): r-\lfloor r \rfloor+1 \ \& \ \lceil r \rceil+1-r \in (1, 2)) \\ &\geq \frac{1}{2} \text{dist}(r, \mathbb{Z}) \Gamma\left(\frac{l+1}{2}\right)^2, \quad \text{by (7.1).} \quad \square \end{aligned}$$

**Lemma 7.3** (Stirling's formula). *For all real  $x > 0$ :*

$$\begin{aligned} \left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}} < \Gamma(x) < \left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}} \cdot e^{\frac{1}{12x}}, \quad \text{or, equivalently,} \\ \frac{x^{x-\frac{1}{2}}}{e^x} \sqrt{2\pi} < \Gamma(x) < \frac{x^{x-\frac{1}{2}}}{e^x} \sqrt{2\pi} \cdot e^{\frac{1}{12x}}, \end{aligned} \quad (7.3.1)$$

where  $e$  is the base of the natural logarithm.

*Proof.* [Feller, 1950, pp. 43–4] stated and proved (7.3.1) only for *integer*  $x > 0$ ; more precisely, he proved that for integral  $x > 0$ ,

$$(x/e)^x \sqrt{2\pi x} < x! < (x/e)^x \sqrt{2\pi x} \cdot e^{1/(12x)}.$$

Upon replacing  $x!$  by  $x \Gamma(x)$ , his proof works for all *real*  $x > 0$  just as well as for all integral  $x > 0$ .  $\square$

Recall the definition of  $L_s$  in (5.2.6):

$$L_s = \begin{cases} 1 & \text{if } l_s = 0 \\ \left| \frac{r_{i_0}(r_{i_0}-1)(r_{i_0}-2)\cdots(r_{i_0}-l_s+1)}{r_{i_s}(r_{i_s}-1)(r_{i_s}-2)\cdots(r_{i_s}-l_s+1)} \right| & \text{if } l_s \geq 1, \end{cases} \quad (5.2.6)$$

where  $r_{i_0}, r_{i_s} \in \mathbb{R} \setminus \mathbb{N}$ ,  $r_{i_0} < r_{i_s}$ , and  $l_s = \max\{0, \lceil r_{i_s} \rceil\}$ .

**Lemma 7.4.** *For  $L_s$  as in (5.2.6),*

$$L_s = 1 \quad \text{if } (r_{i_0} <) r_{i_s} < 0, \quad (7.4.1)$$

$$L_s \geq \frac{0.85(l_s - r_{i_0})^{l_s - r_{i_0} - \frac{1}{2}} \cdot e^{\frac{1}{12r_{i_0}}}}{(-r_{i_0})^{-r_{i_0} - \frac{1}{2}} (r_{i_s} + 1)^{r_{i_s} + \frac{1}{2}}} \quad \text{if } r_{i_0} < 0 < r_{i_s}, \quad (7.4.2)$$

$$L_s > \frac{\text{dist}(r_{i_0}, \mathbb{Z})}{2^{l_s+1} \sqrt{(r_{i_s} - l_s + 1)(l_s + 1)}} \quad \text{if } 0 < r_{i_0} < l_s - 1, \quad (7.4.3)$$

$$L_s \geq \frac{r_{i_0} - l_s + 1}{r_{i_s}} \quad \text{if } 0 \leq l_s - 1 < r_{i_0}. \quad (7.4.4)$$

*Proof.* (7.4.1) is obvious from (5.2.6), since the hypothesis  $r_{i_s} < 0$  implies  $l_s = 0$ .  
To prove (7.4.2), first observe that, according to Mathematica,

$$\min_{0 < x \leq 1} x^{x-\frac{1}{2}} = .930174 \dots ; \quad (7.4.5)$$

this minimum is achieved at  $x = 0.729845 \dots$ . Next, suppose  $r_{i_0} < 0 < r_{i_s}$ .  
Then

$$\begin{aligned} L_s &= \frac{(0 - r_{i_0})(1 - r_{i_0}) \cdots ((l_s - 1) - r_{i_0})}{(r_{i_s} - 0)(r_{i_s} - 1) \cdots (r_{i_s} - (l_s - 1))} \\ &= \frac{\Gamma(l_s - r_{i_0})}{\Gamma(-r_{i_0})} \cdot \frac{\Gamma(r_{i_s} - l_s + 1)}{\Gamma(r_{i_s} + 1)} \quad \text{by (7.0.1),} \\ &> \frac{\frac{(l_s - r_{i_0})^{l_s - r_{i_0} - \frac{1}{2}}}{e^{l_s - r_{i_0}}} \cdot \sqrt{2\pi}}{\frac{(-r_{i_0})^{-r_{i_0} - \frac{1}{2}}}{e^{-r_{i_0}}} \cdot \sqrt{2\pi} \cdot e^{-\frac{1}{12r_{i_0}}}} \cdot \frac{\frac{(r_{i_s} - l_s + 1)^{r_{i_s} - l_s + \frac{1}{2}}}{e^{r_{i_s} - l_s + 1}} \sqrt{2\pi}}{\frac{(r_{i_s} + 1)^{r_{i_s} + \frac{1}{2}}}{e^{r_{i_s} + 1}} \sqrt{2\pi} \cdot e^{\frac{1}{12(r_{i_s} + 1)}}}} \quad \text{by (7.3.1),} \\ &= \frac{(l_s - r_{i_0})^{l_s - r_{i_0} - \frac{1}{2}} \cdot (r_{i_s} - l_s + 1)^{r_{i_s} - l_s + \frac{1}{2}}}{(-r_{i_0})^{-r_{i_0} - \frac{1}{2}} \cdot (r_{i_s} + 1)^{r_{i_s} + \frac{1}{2}}} \cdot e^{\frac{1}{12r_{i_0}}} \cdot e^{-\frac{1}{12(r_{i_s} + 1)}} \\ &> \frac{(l_s - r_{i_0})^{l_s - r_{i_0} - \frac{1}{2}} \cdot (0.93)}{(-r_{i_0})^{-r_{i_0} - \frac{1}{2}} (r_{i_s} + 1)^{r_{i_s} + \frac{1}{2}}} \cdot e^{\frac{1}{12r_{i_0}}} \cdot e^{-\frac{1}{12}} \quad \text{by (7.4.5),} \\ &> \frac{(0.93)(0.92)(l_s - r_{i_0})^{l_s - r_{i_0} - \frac{1}{2}} \cdot e^{\frac{1}{12r_{i_0}}}}{(-r_{i_0})^{-r_{i_0} - \frac{1}{2}} (r_{i_s} + 1)^{r_{i_s} + \frac{1}{2}}}, \quad \text{since } e^{-\frac{1}{12}} = 0.920044 \dots, \\ &> \frac{0.85(l_s - r_{i_0})^{l_s - r_{i_0} - \frac{1}{2}} \cdot e^{\frac{1}{12r_{i_0}}}}{(-r_{i_0})^{-r_{i_0} - \frac{1}{2}} (r_{i_s} + 1)^{r_{i_s} + \frac{1}{2}}}. \end{aligned}$$

To prove (7.4.3), first observe that

$$\min_{x > 0} x^x = (e^{-1})^{e^{-1}} = 0.692201 \dots \quad (7.4.6)$$

by calculus, and that for  $l_s \geq 1$ ,

$$\frac{\sqrt{l_s}}{l_s + 1} \geq \frac{1}{\sqrt{2(l_s + 1)}}. \quad (7.4.7)$$

Next, assume that  $0 < r_{i_0} < l_s - 1$ . Then

$$\begin{aligned}
L_s &= |r_{i_0}(r_{i_0} - 1)(r_{i_0} - 2) \cdots (r_{i_0} - l_s + 1)| \cdot (r_{i_s}(r_{i_s} - 1) \cdots (r_{i_s} - l_s + 1))^{-1} \\
&\geq \frac{1}{2} \operatorname{dist}(r_{i_0}, \mathbb{Z}) \Gamma\left(\frac{l_s + 1}{2}\right)^2 \cdot \frac{\Gamma(r_{i_s} - l_s + 1)}{\Gamma(r_{i_s} + 1)} \quad \text{by (7.2) and (7.0.1),} \\
&\geq \frac{1}{2} \operatorname{dist}(r_{i_0}, \mathbb{Z}) \left(\frac{l_s + 1}{2e}\right)^{l_s + 1} \frac{4\pi}{l_s + 1} \cdot \frac{\left(\frac{r_{i_s} - l_s + 1}{e}\right)^{r_{i_s} - l_s + 1} \sqrt{\frac{2\pi}{r_{i_s} - l_s + 1}}}{\left(\frac{r_{i_s} + 1}{e}\right)^{r_{i_s} + 1} \sqrt{\frac{2\pi}{r_{i_s} + 1}} \cdot e^{\frac{1}{12(r_{i_s} + 1)}}} \\
&= \frac{1}{2} \operatorname{dist}(r_{i_0}, \mathbb{Z}) \frac{4\pi}{2^{l_s + 1} e} \cdot \frac{(l_s + 1)^{l_s + 1}}{(r_{i_s} + 1)^{r_{i_s} + 1}} \cdot \frac{(r_{i_s} - l_s + 1)^{r_{i_s} - l_s + 1}}{(l_s + 1) e^{\frac{1}{12(r_{i_s} + 1)}}} \cdot \sqrt{\frac{r_{i_s} + 1}{r_{i_s} - l_s + 1}} \\
&> \operatorname{dist}(r_{i_0}, \mathbb{Z}) \frac{\pi}{2^{l_s} e} \cdot 1 \cdot \frac{0.69}{e^{\frac{1}{12}}} \cdot \frac{\sqrt{l_s}}{l_s + 1} \cdot \frac{1}{\sqrt{r_{i_s} - l_s + 1}} \quad \text{using } l_s > r_{i_s} \text{ and (7.4.6)} \\
&> \frac{\operatorname{dist}(r_{i_0}, \mathbb{Z})}{\sqrt{r_{i_s} - l_s + 1}} \cdot \frac{\pi}{2^{l_s} e} \cdot \frac{.69}{1.09} \cdot \frac{1}{\sqrt{2(l_s + 1)}}, \quad \text{using (7.4.7)} \\
&> \frac{\operatorname{dist}(r_{i_0}, \mathbb{Z})}{\sqrt{r_{i_s} - l_s + 1}} \cdot \frac{.73}{2^{l_s} \sqrt{l_s + 1}} > \frac{\operatorname{dist}(r_{i_0}, \mathbb{Z})}{2^{l_s + 1} \sqrt{(r_{i_s} - l_s + 1)(l_s + 1)}}.
\end{aligned}$$

Finally, to prove (7.4.4), assume  $0 \leq l_s - 1 < r_{i_0}$ . Then

$$\begin{aligned}
L_s &= \frac{r_{i_0}(r_{i_0} - 1) \cdots (r_{i_0} - l_s + 1)}{r_{i_s}(r_{i_s} - 1) \cdots (r_{i_s} - l_s + 1)} \quad \text{using } r_{i_0} - l_s + 1 > 0, \\
&= \frac{1}{r_{i_s}} \cdot \frac{r_{i_0}}{r_{i_s} - 1} \cdot \frac{r_{i_0} - 1}{r_{i_s} - 2} \cdots \frac{r_{i_0} - l_s + 2}{r_{i_s} - l_s + 1} \cdot (r_{i_0} - l_s + 1) \\
&\geq \frac{1}{r_{i_s}} \cdot 1 \cdot 1 \cdots 1 \cdot (r_{i_0} - l_s + 1), \quad \text{since } r_{i_s} - 1 < l_s - 1 < r_{i_0}, \\
&= \frac{r_{i_0} - l_s + 1}{r_{i_s}}. \quad \square
\end{aligned}$$

**Theorem 7.5.** *Suppose that  $f$  in (1.0.1) is not a polynomial. For  $M_{f,v}$  as in (5.3) and  $t$  as in (5.2.2),  $M_{f,v} \leq 1 + \lceil r_k \rceil$  if  $t = 0$ , and*

$$M_{f,v} \leq \max \left\{ 1 + \lceil r_k \rceil, \max_{1 \leq s \leq t} \left\{ \lfloor r_{i_0} \rfloor + (l_s - \lfloor r_{i_0} \rfloor) v \left( \frac{t}{L_s} \left| \frac{a_{i_s}}{a_{i_0}} \right| \right)^{\frac{1}{r_{i_s} - r_{i_0}}} \right\} \right\}$$

if  $t \geq 1$ .

*Proof.* Recall that  $M_{f,v} \leq \max\{1 + \lceil r_k \rceil, M_1, \dots, M_t\}$ , by (5.2.5) and (5.3), where  $t$  is as in (5.2.2), and where each  $M_s$  is as in (5.2.4)<sub>s</sub>. And recall (5.2.8) that for any fixed  $s = 1, 2, \dots, t$ , we took  $M_s$  to be any integer larger than  $l_s$  and large enough so that  $\forall n \geq M_s$ ,

$$\ln \left( L_s \frac{|a_{i_0}| v^{r_{i_0}}}{|a_{i_s}| v^{r_{i_s}}} \right) + (r_{i_s} - r_{i_0}) \left[ \frac{1}{l_s - m} + \frac{1}{l_s + 1 - m} + \cdots + \frac{1}{n - 1 - m} \right] > \ln t, \quad (5.2.8)$$

where  $i_0 < i_1 < \dots < i_t$  are as in (5.2.2),  $m = \lfloor r_{i_0} \rfloor$ , and the  $a_0, \dots, a_k$  are as in (1.0.1). (5.2.8) is equivalent to

$$\frac{1}{l_s - m} + \frac{1}{l_s + 1 - m} + \dots + \frac{1}{n - 1 - m} > \frac{1}{r_{i_s} - r_{i_0}} \ln \left( \frac{t |a_{i_s}| v^{r_{i_s}}}{L_s |a_{i_0}| v^{r_{i_0}}} \right). \quad (7.5.1)$$

The left hand side of (7.5.1) exceeds

$$\int_{l_s - m}^{n - m} \frac{1}{z} dz;$$

thus (7.5.1) will hold if

$$\ln \frac{n - m}{l_s - m} > \frac{1}{r_{i_s} - r_{i_0}} \ln \left( \frac{t |a_{i_s}| v^{r_{i_s}}}{L_s |a_{i_0}| v^{r_{i_0}}} \right),$$

which is equivalent to

$$\frac{n - m}{l_s - m} > \left( \frac{t |a_{i_s}| v^{r_{i_s}}}{L_s |a_{i_0}| v^{r_{i_0}}} \right)^{\frac{1}{r_{i_s} - r_{i_0}}} = v \left( \frac{t}{L_s} \left| \frac{a_{i_s}}{a_{i_0}} \right| \right)^{\frac{1}{r_{i_s} - r_{i_0}}}.$$

And this is equivalent to

$$\begin{aligned} n &> m + (l_s - m) v \left( \frac{t}{L_s} \left| \frac{a_{i_s}}{a_{i_0}} \right| \right)^{\frac{1}{r_{i_s} - r_{i_0}}} \\ &= \lfloor r_{i_0} \rfloor + (l_s - \lfloor r_{i_0} \rfloor) v \left( \frac{t}{L_s} \left| \frac{a_{i_s}}{a_{i_0}} \right| \right)^{\frac{1}{r_{i_s} - r_{i_0}}}. \quad \square \end{aligned}$$

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