

SUPPORT PROPERTIES AND HOLMGREN'S UNIQUENESS THEOREM FOR DIFFERENTIAL OPERATORS WITH HYPERPLANE SINGULARITIES

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ABSTRACT. Let W be a finite Coxeter group acting linearly on \mathbb{R}^n . In this article we study support properties of W -invariant partial differential operator D on \mathbb{R}^n with real analytic coefficients. Our assumption is that the principal symbol of D has a special form, related to the root system corresponding to W . In particular the zeros of the principal symbol are supposed to be located on hyperplanes fixed by reflections in W . We show that $\text{conv}(\text{supp } Df) = \text{conv}(\text{supp } f)$ holds for all compactly supported smooth functions f so that $\text{conv}(\text{supp } f)$ is W -invariant. The main tools in the proof are Holmgren's uniqueness theorem and some elementary convex geometry. Several examples and applications linked to the theory of special functions associated with root systems are presented.

INTRODUCTION

Let D be a linear partial differential operator on \mathbb{R}^n with constant coefficients. Then a classical theorem of Lions and Titchmarsh states that, for every distribution u on \mathbb{R}^n with compact support, the convex hulls of the supports of Du and u are equal:

$$\text{conv}(\text{supp } Du) = \text{conv}(\text{supp } u), \quad u \in \mathcal{E}'(\mathbb{R}^n). \quad (1)$$

By regularization, this is equivalent to stating that for every compactly supported smooth function f on \mathbb{R}^n , the convex hulls of the supports of Df and f are equal:

$$\text{conv}(\text{supp } Df) = \text{conv}(\text{supp } f), \quad f \in C_c^\infty(\mathbb{R}^n). \quad (2)$$

In fact, Lions [Lio51] proved a more general version of the support theorem, namely

$$\text{conv}(\text{supp}(v * u)) = \text{conv}(\text{supp } v) + \text{conv}(\text{supp } u), \quad \forall v, u \in \mathcal{E}'(\mathbb{R}^n). \quad (3)$$

We refer to [Boe85] for an elementary proof of this theorem. The first version (1) obviously follows from the third one (3) by taking $u = \delta_0$, the delta distribution at the origin.

The comparison of the supports of f and Df plays an important role at several places in analysis. One typical situation is the study of solvability of differential operators. Recall

2000 *Mathematics Subject Classification.* Primary 33C67, 43A90; Secondary 43A85.

Key words and phrases. Support theorem, Holmgren's uniqueness theorem, invariant singular partial differential operators, finite reflection groups, invariant differential operators, Heckman-Opdam hypergeometric system, shift operators, symmetric spaces.

The first author was supported by NSF grants DMS-0070607, DMS-0139783 and DMS-0402068. Part of this research was conducted when the second author visited LSU in February 2003. She gratefully acknowledge financial support from NSF and the Louisiana Board of Regents grant *Visiting Experts in Mathematics*. The final version of the article was prepared while both authors were visiting the Lorentz Center at Leiden University. They would like to thank E. Opdam, M. de Jeu, S. Hille, E. Koelink, W. Kosters, M. Pevzner and F. Bakker for their invitation.

that a linear partial differential operator

$$D = \sum_{|I| \leq m} a_I(x) \partial^I \quad (4)$$

with smooth coefficients a_I is said to be *solvable in \mathbb{R}^n* , provided $DC^\infty(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$, that is, if for every $g \in C^\infty(\mathbb{R}^n)$ the differential equation $Df = g$ has a solution $f \in C^\infty(\mathbb{R}^n)$. The following theorem gives a necessary and sufficient condition for the solvability of D (see [Tre75], Theorem 3.3).

Theorem 0.1. *Let D be a linear partial differential operator with smooth coefficients in \mathbb{R}^n . Then D is solvable if and only if the following two conditions are satisfied:*

- (a) (semi-global solvability) *For every compact subset K of \mathbb{R}^n and for every $g \in C^\infty(\mathbb{R}^n)$ there is a function $f \in C^\infty(\mathbb{R}^n)$ so that $Df = g$ on K .*
- (b) (D -convexity of \mathbb{R}^n) *For every compact subset K of \mathbb{R}^n there is a compact set K' so that for every $f \in C_c^\infty(\mathbb{R}^n)$ the inclusion $\text{supp}(D^t f) \subseteq K$ implies $\text{supp} f \subseteq K'$. Here D^t denotes the formal transpose of D .*

The support theorem of Lions and Titchmarsh implies that, for every linear partial differential operator D with constant coefficients, \mathbb{R}^n is D -convex, and that we can in fact take $K' = \text{conv} K$. Observe that, in this case, condition (b) also implies that the operator D^t is injective on $C^\infty(\mathbb{R}^n)$.

The main result of this paper is Theorem 2.1, which provides an extension of the support theorem of Lions and Titchmarsh to specific (but very large) classes of invariant singular linear partial differential operators with real analytic coefficients and to distributions having a compact support with invariant convex hull. The invariance considered here is with respect to certain finite groups of orthogonal transformations generated by reflections. In the 2-dimensional case, examples of such groups are the groups of symmetries of regular n -agons. The principal symbols of the examined differential operators are allowed to vanish, but only in a precise way, along the reflecting hyperplanes. See formula (10) below.

Some restrictions in generalizing the theorem of Titchmarsh and Lions are of course needed. The following easy example shows that the theorem cannot hold for arbitrary linear differential operators with variable coefficients, even in the one-dimensional case and with real analytic coefficients.

Example 0.2. Consider the differential operator $D = x \frac{d}{dx}$ on \mathbb{R} . Let $u = \chi_{[0,1]}$ be the characteristic function of the interval $[0, 1]$. Then $Du = -\delta_1$, where δ_1 denotes the delta measure at 1. Therefore $\text{supp}(Du) = \{1\}$ and $\text{supp} u = [0, 1]$ are convex and different.

In the one-dimensional case, the differential operator $D = x \frac{d}{dx}$ of Example 0.2 belongs to the class of differential operators to which our support theorem applies. Observe that D is an even differential operator which is singular at $x = 0$. For this very special differential operator, our theorem states that $\text{conv}(\text{supp} Du) = \text{conv}(\text{supp} u)$ if $u \in \mathcal{E}'(\mathbb{R})$ satisfies one of the following conditions:

- (1) $\text{supp} u \subset]0, +\infty[$,
- (2) $\text{supp} u \subset]-\infty, 0[$,
- (3) $\text{conv}(\text{supp} u)$ is symmetric with respect to the origin 0.

Of course the distribution $\chi_{[0,1]}$ from Example 0.2 does not fulfill any of these conditions.

The core of the proof of Theorem 2.1 is to show that the considered situation allows us to apply Holmgren's uniqueness theorem for comparing the size of the supports of f and Df when f is a compactly supported smooth function with the property that $\text{conv}(\text{supp } f)$ is invariant. The employ of Holmgren's theorem is the reason for imposing to the coefficients of the considered differential operators to be real analytic.

Several other authors employed Holmgren's uniqueness theorem for proving D -convexity properties (see e.g. [RW76], [Cha79], [vdBS93]). Remarkable is nevertheless, that the proof of our theorem is very elementary. It requires only basic facts from convex geometry and an application of equation (3).

The article consists of two parts. The first part contains the proof of Theorem 2.1, which does not require any knowledge of symmetric spaces. In the second part, we discuss several applications of our main results, including hypergeometric differential operators, Bessel differential operators, shift operators, Hamiltonian systems, and invariant differential operators on symmetric spaces.

The solvability of G -invariant differential operators is one of the fundamental problems in the analysis on a symmetric space G/H (see [Hel84], p. 275). Recall that Theorem 0.1 holds more generally if \mathbb{R}^n is replaced with a 2nd countable smooth manifold (see [Tre75], p. 14). Since G/H is a second countable smooth manifold, one obtains the equivalence between global and semi-global solvability, provided one can prove that G/H is D -convex. The solvability of invariant differential operators on Riemannian symmetric space was proved by Helgason [He73]. In the general pseudo-Riemannian symmetric case, van den Ban and Schlichtkrull [vdBS93] determined a sufficient condition for a G -invariant differential operator D ensuring that G/H is D -convex. This condition involves the degree of the polynomial which is the image of D under the Harish-Chandra isomorphism. It is for instance always satisfied when G/H is split, i.e., has a vectorial Cartan subspace. The Riemannian symmetric spaces of the noncompact type are examples of split symmetric spaces and so are the K_ϵ space of Oshima and Sekiguchi, [OS80].

As a first application of Theorem 2.1, we deduce in Section 4 the D -convexity of Riemannian symmetric spaces G/K of noncompact type when D is a G -invariant differential operator on G/K . Our method, which is based on taking the radial component of D along the Cartan subgroup, is different from those of used in [He73] and [vdBS93].

Support theorems play also an important role in harmonic analysis, in particular in Paley-Wiener type theorems. These theorems characterize the space of functions which are image, under a suitable generalization of the Fourier transform, of the compactly supported smooth functions. Applications in this setting appeared first in the work of van den Ban and Schlichtkrull on Fourier transforms on pseudo-Riemannian symmetric spaces [vdBS97]. The basic idea, which we shall outline more precisely in Section 4, is to cancel the singularities appearing in a wave packet f by applying a suitable differential operator D . The problem is to compare the size of the support of Df , which can be easily determined, with the –hard to determine– size of the support of the original wave packet f . In fact, our need for support properties like those stated in the present paper turned up in the proof of a Paley-Wiener type theorem for the Θ -hypergeometric transform, which is a Fourier type transform related to the theory of hypergeometric functions associated with root systems. See [ÓP04]. However, we point out that Theorem 2.1 is stated in a very general setting and applies to many

different situations. See Example 1.3, and the Subsections 3.1, 3.2 and 3.3 for several special cases.

Our paper is organized as follows. In Section 1 we introduce the general setting in which our extension of the theorem of Lions and Titchmarsh will be proved. The main results, Theorems 2.1 and 2.2, will be stated and proved in Section 2. Section 3 presents several concrete situations where our support theorem applies. The presented examples are related to the theory of special functions associated with root systems. The last section is devoted to applications. We deduce the D -convexity of Riemannian symmetric spaces of noncompact type when D is an invariant differential operator. Moreover, we describe how to employ Theorem 2.2 for proving of Paley-Wiener type theorems in the harmonic analysis on symmetric spaces and on root systems.

1. NOTATION AND SETTING

1.1. Finite Coxeter Groups. In this section we introduce the notation and set up that will be used in this article. In particular we introduce the class of differential operators that will be considered in this article and give few examples.

In the following \mathfrak{a} stands for an real Euclidean vector space of dimension n , i.e., $\mathfrak{a} \simeq \mathbb{R}^n$. Furthermore D will stand for a differential operator on \mathfrak{a} with analytic coefficients. We denote by $\langle \cdot, \cdot \rangle$ a (positive definite) inner product on \mathfrak{a} . Set $|x| = \sqrt{\langle x, x \rangle}$. For $\varepsilon > 0$ we denote by $B_\varepsilon := \{x \in \mathfrak{a} : |x| \leq \varepsilon\}$ the closed Euclidean ball in \mathfrak{a} with center 0 and radius ε .

Let \mathfrak{a}^* denote the real dual of \mathfrak{a} . For each $\alpha \in \mathfrak{a}^* \setminus \{0\}$ we denote by y_α the unique element of \mathfrak{a} satisfying $\alpha(x) = \langle x, y_\alpha \rangle$ for all $x \in \mathfrak{a}$. We set

$$x_\alpha := 2y_\alpha / \langle y_\alpha, y_\alpha \rangle \tag{5}$$

and notice that x_α is independent of $\langle \cdot, \cdot \rangle$. With each $\alpha \in \mathfrak{a}^* \setminus \{0\}$ we associate the reflection r_α of \mathfrak{a} across the hyperplane $\mathcal{H}_\alpha := \ker \alpha$. Thus

$$r_\alpha(x) = x - \alpha(x)x_\alpha, \quad x \in \mathfrak{a}. \tag{6}$$

A finite set $\Delta \subset \mathfrak{a}^* \setminus \{0\}$ is called a (*reduced*) *root system* if the following conditions holds for Δ :

- (R1) If $\alpha \in \Delta$, then $\Delta \cap \mathbb{R}\alpha = \{\pm\alpha\}$;
- (R2) If $\alpha, \beta \in \Delta$, then $r_\alpha(\beta) \in \Delta$.

The elements of Δ are called *roots*. Observe that *we are not requiring* that Δ contains a basis of \mathfrak{a}^* . In particular, our definition allows Δ to be the empty set.

A subset Π of Δ is called a *simple system* if Π is linearly independent and if any root in Δ can be written as a linear combination of elements in Π in which all non-zero coefficients are either all positive or all negative. If $\Delta = \emptyset$, then we set $\Pi = \emptyset$.

Let a simple system Π of Δ be fixed. Set $\Delta^+ := \mathbb{R}_{\geq 0}\Pi \cap \Delta$, where $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$. The elements in Δ^+ are said to be positive. Observe that $-\alpha = r_\alpha(\alpha) \in \Delta$ for all $\alpha \in \Delta$. Hence $\Delta = \Delta^+ \dot{\cup} (-\Delta^+)$.

Let $W \subset \text{GL}(\mathfrak{a})$ be the group of orthogonal transformations of \mathfrak{a} generated by the reflections $\{r_\alpha : \alpha \in \Delta\}$. It coincides with the group generated by $\{r_\alpha : \alpha \in \Pi\}$. We set $W = \{\text{id}\}$, if Δ is empty. If $\dim \mathfrak{a} = 1$ and $\Delta \neq \emptyset$, then $W = \{\pm \text{id}\}$. The group W is a *finite Coxeter group*. Conversely, every finite Coxeter group originates from a root system as above. See e.g. [GP00], Chapter 1. Among the finite Coxeter groups we find the *Weyl*

groups (for instance the finite groups of permutations) and the *dihedral groups* (that is, the groups of symmetries of the regular n -gons).

The group W acts on functions $f : \mathfrak{a} \rightarrow \mathbb{C}$ according to

$$(w \cdot f)(x) := f(w^{-1}x), \quad w \in W, x \in \mathfrak{a}. \quad (7)$$

It also acts on compactly supported distributions $u \in \mathcal{E}'(\mathfrak{a})$ and on differential operators D on \mathfrak{a} by:

$$\langle (w \cdot u), f \rangle := \langle u, w^{-1} \cdot f \rangle, \quad w \in W, f \in C^\infty(\mathfrak{a}), \quad (8)$$

$$(w \cdot D)f := w \cdot D(w^{-1} \cdot f), \quad w \in W, f \in C^\infty(\mathfrak{a}), \quad (9)$$

where we have written the pairing between distributions and functions by $\langle u, f \rangle := u(f)$. The function f (resp. the compactly supported distribution u or the differential operator D) is said to be *W-invariant* provided $w \cdot f = f$ for all $w \in W$ (resp. $w \cdot u = u$ or $w \cdot D = D$ for all $w \in W$). For instance, if $\dim \mathfrak{a} = 1$ and $\Delta \neq \emptyset$, then *W-invariant* means even. More generally, let χ be a character of W . Then f is said to *transform under W according to χ* if $w \cdot f = \chi(w)f$ for all $w \in W$. This definition extends similarly to distributions and differential operators. Notice that if f transforms under W according to a character χ , then the support of f is *W-invariant*.

1.2. X -elliptic polynomials and X -elliptic differential operators. We will be studying differential operators with leading symbol of a specific form. We will therefore need the following definition:

Definition 1.1. Let X be a non-empty W -invariant convex open subset of \mathfrak{a} , and $P : \mathfrak{a} \times \mathfrak{a}^* \rightarrow \mathbb{C}$ a polynomial function. We say that P is a *homogeneous X -elliptic polynomial* if the following holds:

- (P1) P is a homogeneous polynomial in $\lambda \in \mathfrak{a}^*$ with real analytic coefficients on \mathfrak{a} , i.e. there is $m \in \mathbb{N}_0$ so that

$$P(x, \lambda) = \sum_{|I|=m} a_I(x) \lambda^I$$

where $a_I(x)$ is real analytic, and $|I| := \sum_{k=1}^n i_k$, if $I = (i_1, \dots, i_n) \in \mathbb{N}_0^n$ is a multi-index.

- (P2) If $(x, \lambda) \in X \times \mathfrak{a}^*$ and $\lambda \neq 0$, then $P(x, \lambda) \neq 0$.

Let D be a W -invariant linear partial differential operator on \mathfrak{a} with real analytic coefficients. We say that D is *X -elliptic* if its principal symbol is of the form

$$\sigma(D)(x, \lambda) = \left(p(\lambda) \prod_{\alpha \in \Delta} [\alpha(x)]^{n(\alpha)} \right) P(x, \lambda), \quad (x, \lambda) \in \mathfrak{a} \times \mathfrak{a}^*, \quad (10)$$

where $p(\lambda)$ is a homogeneous polynomial, $n(\alpha) \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$ for all $\alpha \in \Delta$, and $P(x, \lambda)$ is a homogeneous X -elliptic polynomial.

In this paper we consider X -elliptic linear partial differential operators as operators acting on functions or distributions on X . Note that the decomposition of $\sigma(D)(x, \lambda)$ in (10) is in general not unique: if $\alpha(x) \neq 0$ for all $x \in X$, then we can replace P by $\alpha(x)^k P(x, \lambda)$, $0 < k \leq n(\alpha)$, and replace $n(\alpha)$ by $n(\alpha) - k$.

The class of X -elliptic contains all elliptic partial differential operators on \mathfrak{a} by taking $X = \mathfrak{a}$ and $\Delta = \emptyset$. But, more generally, the principal symbols of the considered operators are allowed to vanish at the zeros of $p(\lambda)$ as well as along the hyperplanes \mathcal{H}_α with $\alpha \in \Delta$. Observe that the condition in (10) imposes a restriction only on the principal part of the differential operators. In particular, suppose D_1, D_2 are W -invariant linear partial differential operators with real analytic coefficients so that $\deg D_1 > \deg D_2$. If D_1 satisfies condition (10), then the same is true for $D_1 + D_2$.

Example 1.2. Any partial differential operator with constant coefficients $p(D)$ satisfies (10) when we choose $X = \mathfrak{a}$ and $\Delta = \emptyset$. Indeed, in this case we do not impose any symmetry condition, and $\sigma(p(D))(\lambda) = p(\lambda)$ is of the form (10).

Example 1.3 (The one-dimensional case). Suppose $\dim \mathfrak{a} = 1$ and $\Delta \neq \emptyset$. We shall identify $\mathfrak{a} \equiv \mathfrak{a}^*$ with \mathbb{R} . The possible subsets X are the open intervals of the form $] - R, R[$ with $0 < R \leq +\infty$. Then the differential operators considered are the even ordinary differential operators with real analytic coefficients and with principal symbol

$$\sigma(D)(x, \lambda) = x^n p(\lambda) P(x, \lambda).$$

Here, $n \in \mathbb{N}_0$, the polynomial $p(\lambda)$ is homogeneous, and $P(x, \lambda)$ is homogeneous X -elliptic. For $X = \mathbb{R}$, examples are $x \frac{d}{dx}$ and the Euler operator

$$E = x^2 \frac{d^2}{dx^2} + ax \frac{d}{dx} + b,$$

where $a, b \in \mathbb{C}$. Other examples, which are relevant in harmonic analysis, are constructed from the Jacobi and the Bessel differential operators on $]0, +\infty[$. Let L and L_0 respectively denote the Jacobi and the Bessel differential operators, i.e.

$$L = \frac{d^2}{dx^2} + \left[a \coth(x) + b \coth(2x) \right] \frac{d}{dx},$$

$$L_0 = \frac{d^2}{dx^2} + \left[a \frac{1}{x} + b \frac{1}{2x} \right] \frac{d}{dx}.$$

Then, for specific positive integral values of the constants a and b , the operator L agrees with the radial part of the Laplace-Beltrami operator of a rank-one Riemannian symmetric spaces G/K of noncompact type, whereas L_0 is the radial part of the Laplace-Beltrami operator of the corresponding rank-one Riemannian symmetric space G_0/K of Euclidean type. Examples of such spaces G/K are the (real, complex or quaternionic) hyperboloids; examples of spaces G_0/K are given by \mathbb{R}^n regarded as homogeneous space $M(n)/O(n)$. We refer to [Hel84], Ch. II, Propositions 3.9 and 3.13, for more information on radial parts of the Laplace-Beltrami operators.

The operators $D := \sinh^2 x \cdot L$ and $D_0 := x^2 \cdot L_0$ satisfy our assumptions since they are even operators on \mathbb{R} with real analytic coefficients, and $\sigma(D)(x, \lambda) = x^2 \lambda^2 \left(\frac{\sinh x}{x} \right)^2$ and $\sigma(D_0)(x, \lambda) = x^2 \lambda^2$. Generalizations in more variables of these examples will be treated in Section 3. Note that, in the first case, we can take $P(x, \lambda) = \left(\frac{\sinh x}{x} \right)^2 \lambda^2$ or $P(x, \lambda) = \left(\frac{\sinh x}{x} \right)^2$.

Example 1.4 (The Calogero model). The Calogero model is a non-relativistic quantum mechanical system of $n + 1$ identical particles on a line interacting pairwise. Such a system

is described by the Hamiltonian

$$H_{\text{Cal}}(x) = -\frac{1}{2} \sum_{j=1}^{n+1} p_j^2 + g^2 \sum_{1 \leq i < j \leq n+1} \frac{1}{(x_i - x_j)^2}, \quad (p_1, \dots, p_{n+1}), (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1},$$

where the positive constant g^2 is the coupling coefficient. The associated Schrödinger operator is

$$S_{\text{Cal}} = -\frac{1}{2} \sum_{j=1}^{n+1} \partial_j^2 + g^2 \sum_{1 \leq i < j \leq n+1} \frac{1}{(x_i - x_j)^2}.$$

See e.g. [Pe90], (3.1.1) and (3.1.14), I. See also [Ca71]. Let $\{e_1, \dots, e_{n+1}\}$ be the standard basis of \mathbb{R}^{n+1} and let $\langle \cdot, \cdot \rangle$ be the standard inner product on \mathbb{R}^n . For $1 \leq i, j \leq n+1$ with $i \neq j$ set $\alpha_{i,j} := e_i - e_j$. Then $\Delta := \{\alpha_{i,j} : 1 \leq i, j \leq n+1, i \neq j\}$ is a root system of type A_n . We take $\Delta^+ := \{\alpha_{i,j} : 1 \leq i < j \leq n+1\}$ as a set of positive roots. The corresponding system of simple roots is $\Pi = \{\alpha_{j,j+1} : j = 1, \dots, n\}$. The finite Coxeter group W associated to A_n is the group \mathfrak{S}_n of permutations of the set $\{e_1, \dots, e_{n+1}\}$. We can now write

$$S_{\text{Cal}}(x) = -\frac{1}{2} L_{\mathfrak{a}} + g^2 \sum_{\alpha \in \Delta^+} \frac{1}{\langle \alpha, x \rangle^2}$$

where $L_{\mathfrak{a}}$ is the Euclidean Laplace operator on \mathfrak{a} . It follows that S_{Cal} is W -invariant. If $\pi(x) := \prod_{\alpha \in \Delta} \langle \alpha, x \rangle$, then the differential operator $\pi(x) S_{\text{Cal}}(x)$ satisfies our requirements in (10). This example will be generalized in Section 3.

Remark 1.5. Multiplication by $\sinh^2 x$ and x^2 in Example 1.3 (resp. by π in Example 1.4), which is used to obtain W -invariant differential operators with analytic coefficients in \mathfrak{a} , is in fact inessential when dealing with even (resp. W -invariant) functions). Indeed, despite the singularity of their coefficients, the operators L and L_0 map even smooth functions on \mathbb{R} into smooth functions. Likewise, the operator S_{Cal} maps smooth W -invariant functions on \mathfrak{a} into smooth functions. This remark will be stated, in a more general form, in Section 3.

1.3. Function spaces. We now introduce the class of functions that will be considered in this paper. As above, let X be a fixed W -invariant open convex subset of \mathfrak{a} . Let $\mathcal{E}'(X; W)$ denote the space consisting of distributions u on \mathfrak{a} so that $\text{conv}(\text{supp } u)$ is a W -invariant compact subset of X , and let $C_c^\infty(X; W)$ be the subspace of $\mathcal{E}'(X; W)$ consisting of C^∞ functions. Important subspaces of $C_c^\infty(X; W)$ are the spaces $C_c^\infty(X; \chi)$, formed by the smooth compactly supported functions $f : X \rightarrow \mathbb{C}$ which transform under W according to the character χ of W . For instance, if χ is the trivial character, then we obtain the space $C_c^\infty(X)^W$ of W -invariant smooth functions on X with compact support. Similarly, inside $\mathcal{E}'(X; W)$ we find the spaces $\mathcal{E}'(X; \chi)$ of compactly supported distributions on X which transform according to χ , and the space $\mathcal{E}'(X)^W$ of W -invariant distributions on W with compact support.

Example 1.6. If $\dim \mathfrak{a} = 1$ and $X = \mathfrak{a}$, then we can identify $X = \mathfrak{a} \equiv \mathfrak{a}^*$ with \mathbb{R} . Suppose first $\Delta = \emptyset$. Then W is trivial, and $C_c^\infty(\mathfrak{a}; W)$ (resp. $\mathcal{E}'(\mathfrak{a}; W)$) reduces to the space of C^∞ functions (resp. distributions) on \mathbb{R} with compact support. If $\Delta \neq \emptyset$, then $W = \{\pm \text{id}\}$. In this case, $C_c^\infty(\mathfrak{a}; W)$ (resp. $\mathcal{E}'(\mathfrak{a}; W)$) is the space of smooth functions (resp. distributions) f on \mathbb{R} so that $\text{conv}(\text{supp } f)$ is a bounded interval of the form $[-R, R]$ for some $R > 0$. The subspace $C_c^\infty(\mathfrak{a})^W$ consists of the compactly supported even smooth functions on \mathbb{R} ; with χ

equal to the sign character, the subspace $C_c^\infty(\mathfrak{a}; \chi)$ consists of the compactly supported odd smooth functions on \mathbb{R} .

2. THE SUPPORT THEOREM

In this section we proof the main result of this article. This is the following version of the the support theorem of Lions and Titchmarsh:

Theorem 2.1 (The W -Invariant Support Theorem). *Let $\emptyset \neq X \subseteq \mathfrak{a}$ be open, convex and W -invariant. Suppose D is a W -invariant linear partial differential operator on \mathfrak{a} with real analytic coefficients and with principal symbol of the form (10). Then*

$$\text{conv}(\text{supp } Du) = \text{conv}(\text{supp } u)$$

for each $u \in \mathcal{E}'(X; W)$.

We shall prove below that, by regularization, Theorem 2.1 is equivalent to the following smooth version:

Theorem 2.2. *Let X and D be as in Theorem 2.1. Then*

$$\text{conv}(\text{supp } Df) = \text{conv}(\text{supp } f)$$

for each $f \in C_c^\infty(X; W)$.

Equivalently, for each $f \in C_c^\infty(X; W)$ and for every compact convex W -invariant subset $C \subset X$ we have

$$\text{supp } Df \subseteq C \quad \iff \quad \text{supp } f \subseteq C.$$

Proof of the equivalence of Theorems 2.1 and 2.2. It is clear that Theorem 2.1 implies Theorem 2.2. For the other direction, let $u \in \mathcal{E}'(X; W)$. Let $\{\psi_\varepsilon : \varepsilon > 0\}$ be an approximate identity with $\text{supp } \psi_\varepsilon \subseteq B_\varepsilon$ for all $\varepsilon > 0$. Since X is a W -invariant, we can, if necessary, replace ψ_ε by $\frac{1}{|W|} \sum_{w \in W} (w \cdot \psi_\varepsilon)$. We can therefore suppose that ψ_ε is W -invariant. It follows, in particular, that $\text{supp } \psi_\varepsilon$ and hence $\text{conv}(\text{supp } \psi_\varepsilon)$ are W -invariant. Thus, by (3),

$$\text{conv}(\text{supp}(u * \psi_\varepsilon)) = \text{conv}(\text{supp } u) + \text{conv}(\text{supp } \psi_\varepsilon)$$

is a W -invariant compact subset of \mathfrak{a} . Since X is open, we can choose $\varepsilon_0 > 0$ so that $u * \psi_\varepsilon \in C_c^\infty(X; W)$ for $0 < \varepsilon < \varepsilon_0$. The functions $u * \psi_\varepsilon$ converge to u in $\mathcal{E}'(\mathfrak{a})$ as $\varepsilon \rightarrow 0$, and $D(u * \psi_\varepsilon) = Du * \psi_\varepsilon$. It is easy to check (without invoking deeper support theorems) that there exists $\delta(\varepsilon) > 0$, converging to 0 as $\varepsilon \rightarrow 0$, so that

$$\text{supp } u \subseteq \text{supp}(u * \psi_\varepsilon) + B_{\delta(\varepsilon)}. \tag{11}$$

Replacing u with Du , we obtain

$$\begin{aligned} \text{supp}(Du * \psi_\varepsilon) &\subseteq \text{supp } Du + B_\varepsilon, \\ \text{supp } Du &\subseteq \text{supp}(Du * \psi_\varepsilon) + B_{\delta(\varepsilon)}. \end{aligned}$$

The same inclusions are preserved for the convex hulls. By Theorem 2.2, we have

$$\text{supp}(u * \psi_\varepsilon) = \text{supp}(D(u * \psi_\varepsilon)).$$

Hence,

$$\begin{aligned} \text{conv}(\text{supp } u) &\subseteq \text{conv}(\text{supp}(u * \psi_\varepsilon)) + B_{\delta(\varepsilon)} \\ &= \text{conv}(\text{supp}(Du * \psi_\varepsilon)) + B_{\delta(\varepsilon)} \\ &\subseteq \text{conv}(\text{supp } Du) + B_{\varepsilon + \delta(\varepsilon)}, \end{aligned}$$

which implies that $\text{conv}(\text{supp } u) \subseteq \text{conv}(\text{supp } Du)$. Similarly, $\text{conv}(\text{supp } Du) \subseteq \text{conv}(\text{supp } u)$. This proves Theorem 2.1. \square

Before going into the details of the proof of Theorem 2.2, let us briefly explain the main ideas involved. Since $\text{supp } Df \subseteq \text{supp } f$, it suffices to show that if C is a compact convex W -invariant subset of X , then $\text{supp } Df \subseteq C$ implies that $\text{supp } f \subseteq C$. This will be proved by contradiction. The main tool will be Holmgren's uniqueness theorem.

Theorem 2.3 (Holmgren's uniqueness theorem). *Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be open, and let φ be a real valued function in $C^1(\Omega)$. Let D be a linear partial differential operator with analytic coefficients defined in Ω . Let $\sigma(D)$ denote the principal symbol of D .*

Suppose that x_0 is a point in Ω such that

$$\sigma(D)(x_0, d\varphi(x_0)) \neq 0. \quad (12)$$

Then there exists a neighborhood $\Omega' \subseteq \Omega$ of x_0 with the following property: If the distribution $u \in \mathcal{D}'(\Omega)$ is annihilated by D , i.e., $Du = 0$, and u vanishes on the set $\{x \in \Omega : \varphi(x) > \varphi(x_0)\}$, then u must also vanish in Ω' .

Proof. This is Theorem 5.3.1 in [Hoer64]. \square

For non-elliptic differential operators the delicate matter is to choose points x_0 for which condition (12) is fulfilled. This is exactly the kind of difficulty one encounters in the proof of support theorems. Coming back to the streamline ideas of the proof of Theorem 2.2, let C be as above. Set $S := \text{supp } f$. Then $S \subseteq X$. To reach a contradiction, we assume that $S \not\subseteq C$. Let $x_0 \in S \setminus C$. As C is convex and compact, there exists a hyperplane strictly separating x_0 and C , i.e., we can find $\lambda_0 \in \mathfrak{a}^*$ such that

$$\max_{y \in C} \lambda_0(y) < \lambda_0(x_0).$$

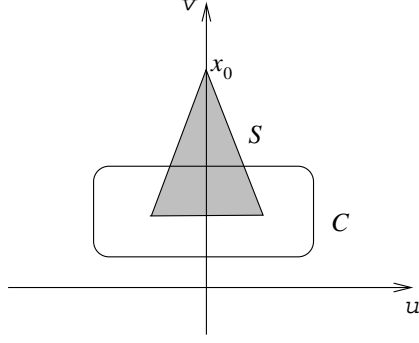
Without loss of generality we can also assume that

$$\lambda_0(x_0) = \max_{x \in S} \lambda_0(x).$$

Otherwise we translate the hyperplane to the boundary of S in the direction opposite to C . In this way, the entire set S lies inside a closed half-space supported by the hyperplane

$$\mathcal{H}_0 := \{x \in \mathfrak{a} : \lambda_0(x) = \lambda_0(x_0)\}.$$

Our plan is to apply Holmgren's uniqueness theorem to $\Omega = X \setminus C$, $\varphi = \lambda_0$ and $u = f$. Note that in this case $d\varphi = \lambda_0$ is constant and non-zero. Observe also that $Df \equiv 0$ on Ω and $f \equiv 0$ on the side of \mathcal{H}_0 not containing C (which is described by the equation $\varphi(x) > \varphi(x_0)$). If the principal symbol of D were not zero at (x_0, λ_0) , then all assumptions would be satisfied, and we could conclude that $f \equiv 0$ in a neighborhood of x_0 . This would yield the required contradiction because $x_0 \in S = \text{supp } f$.



$$\begin{aligned} \mathfrak{a} &= X = \mathbb{R}^2, \\ \Delta &= \{\pm(1, 0)\} \text{ of type } A_1, \\ \Sigma^+ &= \Pi = \{(1, 0)\} \end{aligned}$$

FIGURE 1

Since $\lambda_0 \neq 0$, the condition $\sigma(D)(x_0, \lambda_0) \neq 0$ is equivalent to

$$p(\lambda_0) \prod_{\alpha \in \Delta} [\alpha(x_0)]^{n(\alpha)} \neq 0. \quad (13)$$

This might not be satisfied by the chosen pair (x_0, λ_0) . It is even possible to have a situation where there is no choice of λ_0 for which the above procedure could guarantee that $\alpha(x_0) \neq 0$. Figure 1 sketches an example in which this problem arises because of the “corner” at the boundary of S . Note that the set S in this example is also convex and invariant with respect to the group $W = \{\text{id}, r_v\}$, where r_v denotes the reflection with respect to the v -axis.

In case $p(\lambda_0) \prod_{\alpha \in \Delta} [\alpha(x_0)]^{n(\alpha)} = 0$, the above procedure must be modified. The first step is to show that it suffices to consider the case of smooth functions f with the property that the convex hull C' of $S := \text{supp } f$ is W -invariant and has C^1 boundary. The point is that the x_0 , selected as above, will always belong to the boundary $\partial(C')$ of C' , and that \mathcal{H}_0 is a supporting hyperplane for C' . For the modified procedure we need some preparations.

For $\varepsilon > 0$ let $B_\varepsilon \subset \mathfrak{a}$ denote the closed ball centered at the origin and with radius ε . Let $B_\varepsilon(C) := C + B_\varepsilon$ be the (closed) ε -neighborhood of C .

Lemma 2.4. *Let C be a proper convex subset of $\mathfrak{a} \equiv \mathbb{R}^n$ with nonempty interior, and let $\varepsilon > 0$. Then $B_\varepsilon(C)$ is an n -dimensional convex subset of \mathfrak{a} and its boundary $\partial B_\varepsilon(C)$ is a C^1 $(n - 1)$ -dimensional submanifold of \mathfrak{a} .*

Proof. This is Satz 17.6 in [NGL88]. □

Lemma 2.5. *Let D be a differential operator on \mathfrak{a} . Suppose there exists a smooth compactly supported function \tilde{f} on \mathfrak{a} and a compact convex subset $\tilde{C} \subset \mathfrak{a}$ with nonempty interior such that*

$$\text{supp } D\tilde{f} \subseteq \tilde{C} \quad \text{but} \quad \text{supp } \tilde{f} \not\subseteq \tilde{C}.$$

Then there exists a function $f \in C_c^\infty(\mathfrak{a})$ and a compact convex subset $C \subset \mathfrak{a}$ with nonempty interior such that

$$\text{supp } Df \subset C \quad \text{and} \quad \text{supp } f \not\subseteq C.$$

Moreover, the boundary $\partial(\text{conv}(\text{supp } f))$ of the convex hull of $\text{supp } f$ is a C^1 -manifold.

If $\tilde{f} \in C_c^\infty(X; W)$, where X is a W -invariant open convex subset of \mathfrak{a} , and $\tilde{C} \subseteq X$ is W -invariant. Then we can choose $f \in C_c^\infty(X; W)$ and $C \subseteq X$ to be W -invariant.

Proof. Let $\{\psi_\varepsilon : \varepsilon > 0\}$ be an approximate identity with $\text{supp } \psi_\varepsilon = B_\varepsilon$ for all ε . Then $\tilde{f} * \psi_\varepsilon \in C^\infty$ and

$$\text{supp } (D(f * \psi_\varepsilon)) = \text{supp } (Df * \psi_\varepsilon) \subseteq \text{supp } Df + B_\varepsilon \subseteq B_\varepsilon(\tilde{C}).$$

Notice that (3) applied to compactly supported smooth functions implies that for all $g \in C_c^\infty$ we have

$$\text{supp } g \subseteq B_\varepsilon(\text{conv}(\text{supp } g)) = \text{conv}(\text{supp } g) + B_\varepsilon = \text{conv}(\text{supp}(g * \psi_\varepsilon)). \quad (14)$$

Hence, there exists $\varepsilon_0 > 0$ so that $\text{supp}(\tilde{f} * \psi_{\varepsilon_0}) \not\subseteq B_{\varepsilon_0}(\tilde{C})$. Otherwise (14) would imply that $\text{supp } \tilde{f} \subseteq \text{conv}(\text{supp}(\tilde{f} * \psi_\varepsilon)) \subseteq B_\varepsilon(\tilde{C})$ for all $\varepsilon > 0$, and hence $\text{supp } \tilde{f} \subseteq \tilde{C}$. As in (14), we have

$$\text{conv}(\text{supp}(\tilde{f} * \psi_{\varepsilon_0})) = B_{\varepsilon_0}(\text{conv}(\text{supp } \tilde{f})).$$

Therefore, by Lemma 2.4, $\text{conv}(\text{supp}(\tilde{f} * \psi_{\varepsilon_0}))$ has C^1 -boundary. We can thus select $f = \tilde{f} * \psi_{\varepsilon_0}$ and $C = B_{\varepsilon_0}(\tilde{C})$.

Finally, suppose that $\text{conv}(\text{supp } \tilde{f})$ and \tilde{C} are W -invariants subset of X . Since X is open, then, by choosing a sufficiently small $\varepsilon_0 > 0$, so are also $\text{conv}(\text{supp}(\tilde{f} * \psi_{\varepsilon_0})) = B_{\varepsilon_0}(\text{conv}(\text{supp } \tilde{f}))$ and $B_{\varepsilon_0}(\tilde{C})$. \square

In the following we shall suppose that f and C are chosen as in Lemma 2.5. As before, we set $S := \text{supp } f$ and $C' = \text{conv}(\text{supp } f)$. We also fix $y_0 \in S \setminus C$. We now proceed to the selection of the pair (x_0, λ_0) .

Lemma 2.6. *Let $y_0 \in S \setminus C$ be fixed, and let*

$$U_0 := \{\lambda \in \mathfrak{a}^* : \max_{y \in C} \lambda(y) < \lambda(y_0)\}.$$

Then U_0 is a nonempty open subset of \mathfrak{a}^ .*

Proof. The separation properties of compact convex sets ensure that $U_0 \neq \emptyset$. Observe that

$$\begin{aligned} U_0 &= \{\lambda \in \mathfrak{a}^* : \max_{y \in C} \lambda(y - y_0) < 0\} \\ &= \{\lambda \in \mathfrak{a}^* : \min_{y \in y_0 - C} \lambda(y) > 0\}. \end{aligned}$$

Hence $U_0 = h^{-1}(]0, +\infty[)$ where $h(\lambda) := \min_{y \in y_0 - C} \lambda(y)$. This proves the lemma because h is proper and convex, hence continuous (see [Rock70], Theorems 5.3 and 10.1). \square

Since U_0 is open and nonempty we can choose $\lambda_0 \in U_0$ with the following properties:

- (1) $p(\lambda_0) \neq 0$,
- (2) $\langle \lambda_0, \alpha \rangle \neq 0$ for all $\alpha \in \Delta$ (i.e. $\lambda_0(x_\alpha) \neq 0$ for all $\alpha \in \Delta$).

Since $C' = \text{conv } S$,

Lemma 2.7 (Choice of (x_0, λ_0)). *Let λ_0 be as above. Then there exists $x_0 \in S$ such that $\lambda_0(x_0) = \max_{x \in C'} \lambda_0(x)$. Furthermore $x_0 \in \partial S \cap \partial C'$ and $x_0 \in X$. Finally*

$$\lambda_0(x_0) = \max_{x \in S} \lambda_0(x) \geq \lambda_0(y_0) > \max_{y \in C} \lambda_0(y).$$

Proof. This follow as C' is the convex hull of S . \square

When $\Delta \neq \emptyset$, the W -invariance of the situation plays a role because of the following lemma.

Lemma 2.8. *Suppose C' is a W -invariant compact convex subset of \mathfrak{a} with nonempty interior and C^1 -boundary. Let $\alpha \in \Delta$ and $x \in \partial C'$ such that $\alpha(x) = 0$. Let $T_x(\partial C')$ denote the tangent space to $\partial C'$ at x (regarded as subspace of \mathfrak{a}) and let x_α be as in (5). Then $x_\alpha \in T_x(\partial C')$.*

Proof. The reflection r_α of \mathfrak{a} across the hyperplane $\ker \alpha$ maps C' onto itself and fixes x . Hence it maps $(C')^0$ (resp. $\partial C'$) onto itself. The derived involution r_α is therefore an automorphism of $T_x(\partial C')$. Hence $r_\alpha(N) = \pm N$, where N is the outer normal vector to $\partial C'$ at x . Since $r_\alpha N \neq -N$ by invariance of C' under r_α , we conclude that $r_\alpha N = N$, i.e. $N \in \ker \alpha = x_\alpha^\perp$. Thus $x_\alpha \in x_\alpha^{\perp\perp} \subseteq N^\perp = T_x(\partial C')$. \square

Proposition 2.9. *Let x_0 and λ_0 be as in Lemma 2.7. Then (x_0, λ_0) satisfies (13).*

Proof. The element λ_0 has been chosen so that $p(\lambda_0) \neq 0$. This suffices to prove (10) when $\Delta = \emptyset$. If $\Delta \neq \emptyset$, then it remains to show that $\alpha(x_0) \neq 0$ for all $\alpha \in \Delta$. Since $\lambda_0(x_0) = \max_{x \in C'} \lambda_0(x)$, the hyperplane $\mathcal{H}_0 := \{x \in \mathfrak{a} : \lambda_0(x) = \lambda_0(x_0)\}$ is the supporting hyperplane for $C' := \text{conv}(\text{supp } f)$ at x_0 . The set C' is W -invariant and its boundary $\partial C'$ is C^1 . Hence $\mathcal{H} = x_0 + T_{x_0} \partial C'$. If $\alpha(x_0) = 0$ for some $\alpha \in \Delta$, then $x_\alpha \in T_{x_0} \partial C'$ by Lemma 2.8, i.e. $x_0 + x_\alpha \in \mathcal{H}_0$. Thus $\lambda_0(x_0) + \lambda_0(x_\alpha) = \lambda_0(x_0 + x_\alpha) = \lambda_0(x_0)$, which implies $\lambda_0(x_\alpha) = 0$, against our choice of λ_0 . Thus $\alpha(x_0) \neq 0$ for all $\alpha \in \Delta$. \square

Proof of Theorem 2.2. Arguing by contradiction, we assume that there exists $f \in C_c^\infty(X; W)$ and a W -invariant subset C of X as in Lemma 2.5. We select (x_0, λ_0) as in Proposition 2.9. Set $\Omega = X \setminus C$ and $\varphi = \lambda_0$. Hence $\mathcal{H}_0 := \{x \in \Omega : \lambda_0(x) = \lambda_0(x_0)\}$ is a supporting hyperplane for $C' := \text{conv}(\text{supp } f)$, and $d\varphi = \lambda_0$ is constant and non-zero. Moreover, $Df = 0$ on Ω and $f \equiv 0$ on the side of \mathcal{H}_0 not containing C (which is described by the equation $\lambda_0(x) > \lambda_0(x_0)$). Proposition 2.9 ensures that $\sigma(D)(x_0, \lambda_0) \neq 0$. Holmgren's Uniqueness Theorem then implies that $f \equiv 0$ in a neighborhood of x_0 . This gives the the required contradiction because $x_0 \in \text{supp } f$. \square

Before concluding this section we prove some immediate consequences of Theorem 2.1. Recall that the transpose of the partial differential operator D given by (4) is

$$D = \sum_{|I| \leq m} (-1)^{|I|} a_I(x) \partial^I. \quad (15)$$

Hence D^t belongs to the class of invariant differential operators considered in this papers if so does D . In fact, the principal symbols of D and D^t are linked by the relation

$$\sigma(D^t)(x, \lambda) = (-1)^m \sigma(D)(x, \lambda), \quad (x, \lambda) \in \mathfrak{a} \times \mathfrak{a}^*.$$

Corollary 2.10. *Let D be a W -invariant differential operators D on \mathfrak{a} with real analytic coefficients and with principal symbol of the form (10). Then the following properties are true.*

- (a) D is injective on $C_c^\infty(X; W)$.
- (b) For all $f \in C_c^\infty(X; W)$ we have

$$\text{conv}(\text{supp } D^t f) = \text{conv}(\text{supp } f).$$

3. APPLICATIONS TO SOME SPECIAL DIFFERENTIAL OPERATORS

In this section we present some examples linked to the theory of special functions associated with root systems. In these examples the group W is a parabolic subgroup of a fixed Coxeter group \widetilde{W} acting on $\mathfrak{a} \simeq \mathbb{R}^n$, and the differential operator D is in fact invariant under the larger finite Coxeter group \widetilde{W} . The general situation corresponds to a (not necessarily reduced) root systems Σ . A *root system* is a finite set $\Sigma \subset \mathfrak{a}^* \setminus \{0\}$ satisfying condition (R2) of Section 1. In this section we will also assume that Σ satisfies the following additional conditions:

- (R0) Σ spans \mathfrak{a}^* ;
- (R2) Σ is *crystallographic*, that is

$$2 \frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$$

for all $\alpha, \beta \in \Sigma$.

Crystallographic root systems arise naturally in several places in algebra and analysis. In particular, they are relevant in the theory of real Lie algebras, Lie groups, and symmetric spaces.

If Σ is a root system, then $\Delta := \{\alpha \in \Sigma : 2\alpha \notin \Sigma\}$ is a reduced root system according to the definition of Section 1. The finite Coxeter group associated to Σ is the finite Coxeter group \widetilde{W} associated to Δ . It is also called the *Weyl group* of Σ . A *multiplicity function* is a \widetilde{W} -invariant function $m : \Sigma \rightarrow \mathbb{C}$. For $\alpha \in \Sigma$, we adopt the common notation m_α to denote $m(\alpha)$.

The set Σ^+ of positive roots and the set Π of positive simple roots can be defined as in the case of reduced root systems. Because of (R0), Π is a basis of \mathfrak{a}^* . Fix a Π set of simple roots in Σ^+ . For each subset Θ of Π we define $\langle \Theta \rangle$ as the set of elements of Σ which are linear combinations of elements from Θ , i.e., $\langle \Theta \rangle := \mathbb{Z}\Theta \cap \Sigma$. It is itself a root system, but in general it does not satisfy (R0). We denote the corresponding finite Coxeter group by W_Θ . Note that $W_\Theta \subseteq \widetilde{W}$ is generated by the reflections r_α with $\alpha \in \Theta$. For instance, $W_\Pi = \widetilde{W}$ and $W_\emptyset = \text{id}$. We also set $\langle \Theta \rangle^+ := \langle \Theta \rangle \cap \Sigma^+$ for the set of positive roots inside $\langle \Theta \rangle$. The Coxeter group W_Θ will play the role of the group W of the previous sections. Recall that a subgroup W of \widetilde{W} is called a *parabolic subgroup* if it is of the form W_Θ for some $\Theta \subseteq \Pi$. The parabolic subgroups can also be characterized as those subgroups of \widetilde{W} that stabilize a subspace of \mathfrak{a} . Thus $W \subseteq \widetilde{W}$ is a parabolic subgroup if and only if there exists a subspace $\mathfrak{b} \subseteq \mathfrak{a}$ such that

$$W = \{w \in \widetilde{W} : w(\mathfrak{b}) = \mathfrak{b}\}.$$

A *chamber* in \mathfrak{a} a connected component of $\mathfrak{a} \setminus \cup_{\alpha \in \Sigma} \mathcal{H}_\alpha$ (see [Bou02], Ch. V, §3). From now on we fix the chamber $\mathfrak{a}^+ := \{x \in \mathfrak{a} : \alpha(x) > 0 \text{ for all } \alpha \in \Pi\}$. It is an open simplicial cone with vertex 0 (*loc. cit.*, Ch. V, §3.9, Proposition 7 (iii)), and its closure $\overline{\mathfrak{a}^+} := \{x \in \mathfrak{a} : \alpha(x) \geq 0 \text{ for all } \alpha \in \Pi\}$ is a fundamental domain for the action of \widetilde{W} on \mathfrak{a} (*loc. cit.*, Ch. V, §3.3, Theorem 2).

We define

$$\mathfrak{a}_\Theta := W_\Theta(\overline{\mathfrak{a}^+})^0, \tag{16}$$

where 0 denotes the interior. For instance, $\mathfrak{a}_\emptyset = \mathfrak{a}^+$ and $\mathfrak{a}_\Pi = \mathfrak{a}$. In the following examples, the sets \mathfrak{a}_Θ will play the role of the set X appearing in Theorem 2.1.

The set $\overline{\mathfrak{a}_\Theta} = W_\Theta(\overline{\mathfrak{a}^+})$ is the smallest cone in \mathfrak{a} which is closed, W_Θ -invariant and contains $\overline{\mathfrak{a}^+}$. It is a union of closed chambers. The open polyhedral cone in \mathfrak{a}^*

$$C_\Theta := \sum_{\alpha \in \Sigma^+ \setminus \langle \Theta \rangle^+} \mathbb{R}^+ x_\alpha$$

has the dual cone

$$\begin{aligned} C_\Theta^* &:= \{x \in \mathfrak{a} : \langle x, y \rangle \geq 0 \text{ for all } y \in C_\Theta\} \\ &= \{x \in \mathfrak{a} : \alpha(x) \geq 0 \text{ for all } \alpha \in \Sigma^+ \setminus \langle \Theta \rangle^+\}. \end{aligned}$$

C_Θ^* is a closed convex cone. It is the intersection of the closed hyperplanes defined by roots, and hence a union of closed chambers in \mathfrak{a} .

Lemma 3.1. *We have*

$$\overline{\mathfrak{a}_\Theta} = C_\Theta^*.$$

Consequently, $\overline{\mathfrak{a}_\Theta}$ is a closed W_Θ -invariant convex cone in \mathfrak{a} and also its interior \mathfrak{a}_Θ is convex. Moreover,

$$\mathfrak{a}_\Theta = \{x \in \mathfrak{a} : \alpha(x) > 0 \text{ for all } \alpha \in \Sigma^+ \setminus \langle \Theta \rangle^+\}. \quad (17)$$

Proof. This was proven in [Pa04], Lemma 3.4, when W is a Weyl group. The same proof applies also to the more general case of finite Coxeter groups. \square

Specializing the notation of Section 1 to this context, we consider the space $C_c^\infty(\mathfrak{a}_\Theta; W_\Theta)$ of smooth functions $f : \mathfrak{a} \rightarrow \mathbb{C}$ with the property that $\text{conv}(\text{supp } f)$ is a W_Θ -invariant compact subset of \mathfrak{a}_Θ . Its subspace of W_Θ -invariant functions on \mathfrak{a}_Θ with compact support is $C_c^\infty(\mathfrak{a}_\Theta)^{W_\Theta}$. Furthermore, $\mathcal{E}'(\mathfrak{a}_\Theta; W_\Theta)$ is the space consisting of distributions u on \mathfrak{a} so that $\text{conv}(\text{supp } u)$ is a W_Θ -invariant compact subset of \mathfrak{a}_Θ .

3.1. Hypergeometric differential operators. As before let $\mathfrak{a} \simeq \mathbb{R}^n$ be a finite dimensional Euclidean space and let Σ be a (non-necessarily reduced) root system in \mathfrak{a}^* . Further, let \widetilde{W} be the corresponding Weyl group and let m be a multiplicity function on Σ . Heckman and Opdam associated with such a triple $(\mathfrak{a}, \Sigma, m)$ a commutative family of \widetilde{W} -invariant differential operators on \mathfrak{a} , the *hypergeometric differential operators*, having meromorphic coefficients of a specific type. More precisely, their coefficients are meromorphic functions on the complexification $\mathfrak{a}_\mathbb{C}$ of \mathfrak{a} and their singularities are cancelled by multiplication by a suitable power of the Weyl denominator

$$\delta(x) := \prod_{\alpha \in \Sigma^+} \sinh \alpha(x). \quad (18)$$

We refer to [Hec97], [Opd00] and [HS94] for more information on Heckman-Opdam's theory of hypergeometric differential operators.

For special values of m , the triple $(\mathfrak{a}, \Sigma, m)$ arises from a Riemannian symmetric space G/K of the noncompact type, i.e., G is a noncompact connected semisimple Lie group with finite center and $K \subset G$ is a maximal compact subgroup, see Section 4. In this case, the hypergeometric differential operators coincide with the radial parts (with respect to the K -action) on \mathfrak{a}^+ of the G -invariant differential operators on G/K . Here we identify \mathfrak{a}^+ with its diffeomorphic image in G/K under the exponential map (usually denoted A^+ in the

literature on analysis on symmetric spaces). For instance, the hypergeometric differential operator

$$L := L_{\mathfrak{a}} + \sum_{\alpha \in \Sigma^+} m_{\alpha} \coth \alpha \partial(y_{\alpha}) \quad (19)$$

coincides with the radial part of the Laplace-Beltrami operator of G/K . Notice that L is singular on the hyperplanes \mathcal{H}_{α} , $\alpha \in \Sigma$. Furthermore, L is the multivariable analog of the Jacobi differential operator of Example 1.3. In (19), $\partial(y)$ denotes the directional derivative in the direction of $y \in \mathfrak{a}$, and $L_{\mathfrak{a}}$ is the Laplace operator on the Euclidean vector space \mathfrak{a} , that is, $L_{\mathfrak{a}} = \sum_{j=1}^n \partial(x_j)^2$ where $\{x_j\}_{j=1}^n$ is an orthonormal basis of \mathfrak{a} .

Set

$$\begin{aligned} \pi_{\Theta}(x) &:= \prod_{\alpha \in \langle \Theta \rangle^+} \alpha(x), \\ \delta_{\Theta}(x) &:= \prod_{\alpha \in \langle \Theta \rangle^+} \sinh \alpha(x), \\ \delta_{\Theta}^c(x) &:= \prod_{\alpha \in \Sigma^+ \setminus \langle \Theta \rangle^+} \sinh \alpha(x), \end{aligned}$$

with the usual convention that empty products are equal to 1. We write $\pi(x)$ instead of $\pi_{\Pi}(x)$. Finally, define $D := \delta^2 \cdot L$. The principal symbol of D is

$$\sigma(D)(x, \lambda) = \delta(x)^2 \cdot \sigma(L_{\mathfrak{a}})(x, \lambda) = \langle \lambda, \lambda \rangle \pi_{\Theta}(x)^2 P(\lambda, x)$$

where

$$P(\lambda, x) := [\delta_{\Theta}^c(x)]^2 \left[\frac{\delta_{\Theta}(x)}{\pi_{\Theta}(x)} \right]^2. \quad (20)$$

Because of Lemma 3.1, each $\alpha \in \Sigma^+ \setminus \langle \Theta \rangle^+$ is positive on \mathfrak{a}_{Θ} . Therefore $P(\lambda, x)$ is a homogeneous \mathfrak{a}_{Θ} -elliptic polynomial on $\mathfrak{a}^* \times \mathfrak{a}$ (of degree 0 in $\lambda \in \mathfrak{a}^*$). More generally, for each hypergeometric differential operator D_0 there is $k \in \mathbb{N}$ so that the linear partial differential operator $D := \delta(x)^{2k} D_0$ is a \widetilde{W} -invariant differential operator with real analytic coefficients and with principal symbol of the form (10) where $P(\lambda, x)$ is a homogeneous \mathfrak{a}_{Θ} -elliptic polynomial. In this case, we say that $D = \delta(x)^k D_0$ is a *regularization* of D_0 . The nature of the principal symbol of D can in fact be deduced by the explicit representation of the hypergeometric differential operators in terms of Cherednik operators (see e.g. [Hec97]). According to this representation, each hypergeometric differential operator can be constructed as the “differential part” of the differential-reflection operator associated with a \widetilde{W} -invariant polynomial in the symmetric algebra over $\mathfrak{a}_{\mathbb{C}}$ by means of Cherednik operators. If D_0 is associated with the \widetilde{W} -invariant polynomial p , then the principal part of the regularized operator $D = \delta(x)^{2k} \cdot D_0$ is

$$\begin{aligned} \sigma(D)(x, \lambda) &= p_h(\lambda) \pi(x)^{2k} \left[\frac{\delta(x)}{\pi(x)} \right]^{2k} \\ &= p_h(\lambda) \pi_{\Theta}(x)^{2k} P(\lambda, x) \end{aligned}$$

where $p_h(\lambda)$ is the highest homogeneous part of $p(\lambda)$ and

$$P(\lambda, x) := [\delta_{\Theta}^c(x)]^{2k} \left[\frac{\delta_{\Theta}(x)}{\pi_{\Theta}(x)} \right]^{2k}.$$

In this setting, Theorems 2.1 and 2.2 yield the following result.

Theorem 3.2. *Let the notation be as above. Let $\Theta \subseteq \Pi$. For a hypergeometric differential operator D_0 let $D = \delta(x)^{2k} \cdot D_0$ be a regularization of D_0 . Then for every $u \in \mathcal{E}'(\mathfrak{a}_\Theta; W_\Theta)$ we have*

$$\text{conv}(\text{supp } Du) = \text{conv}(\text{supp } u).$$

Equivalently, for every $f \in C_c^\infty(\mathfrak{a}_\Theta; W_\Theta)$, we have

$$\text{conv}(\text{supp } Df) = \text{conv}(\text{supp } f).$$

In the case of W_Θ -invariant functions, Corollary 3.2 can be stated directly for the hypergeometric operators.

Theorem 3.3. *Let the notation be as above. Let D_0 be a hypergeometric differential operator and $f \in C_c^\infty(\mathfrak{a}_\Theta)^{W_\Theta}$. Then $D_0 f \in C_c^\infty(\mathfrak{a}_\Theta)^{W_\Theta}$ for all $f \in C_c^\infty(\mathfrak{a}_\Theta)^{W_\Theta}$ and*

$$\text{conv}(\text{supp } D_0 f) = \text{conv}(\text{supp } f).$$

Proof. As

$$\mathfrak{a} = \bigcup_{w \in \widetilde{W}} \overline{w(\mathfrak{a}^+)} = \bigcup_{w \in \widetilde{W}/W_\Theta} \overline{w(\mathfrak{a}_\Theta)}$$

and $\text{supp}(f) \subset \mathfrak{a}_\Theta$ is compact and W_Θ -invariant, it follows that there exists a unique $\tilde{f} \in C_c^\infty(\mathfrak{a})^{\widetilde{W}}$ so that $\tilde{f}|_{\mathfrak{a}_\Theta} = f$. Let $D = \delta^{2k} \cdot D_0$ be a regularization of D_0 . Then

$$(D\tilde{f})|_{\mathfrak{a}_\Theta} = D(\tilde{f}|_{\mathfrak{a}_\Theta}) = Df. \quad (21)$$

Suppose D_0 is the differential part of the Cherednik operator T_0 (see e.g. [Hec97]). Then $D_0 \tilde{f} = T_0 \tilde{f}$ by \widetilde{W} -invariance. Cherednik operators map \widetilde{W} -invariant smooth functions into \widetilde{W} -invariant smooth functions. It follows that $D_0 \tilde{f}$ is smooth and \widetilde{W} -invariant. From (21) we therefore deduce that

$$D_0 f = \frac{(\delta^{2k} \cdot D_0 \tilde{f})|_{\mathfrak{a}_\Theta}}{\delta^{2k}}$$

extends to be smooth and \widetilde{W} -invariant on \mathfrak{a}_Θ . If g is a continuous function, then $\delta^{2k} \cdot g$ and g have the same support. The Theorem therefore follows. \square

3.2. Bessel differential operators. Let G be a connected noncompact semisimple Lie group with finite center and $K \subset G$ a maximal compact subgroup. Then $K = G^\theta$ for some Cartan involution θ . The Lie algebra \mathfrak{g} decomposes into eigenspaces of the derived homomorphism $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

where \mathfrak{k} is the +1 eigenspace and \mathfrak{p} is the -1 eigenspace. Consider \mathfrak{p} as a abelian Lie group. The group K acts linearly on \mathfrak{p} by $k \cdot X = \text{Ad}(k)X$. We can therefore consider the semi-direct product

$$G_0 = \mathfrak{p} \times_{\text{Ad}} K.$$

Let $(\mathfrak{a}, \Sigma, m)$ be as in Example 3.1. The *Bessel differential operators* associated with $(\mathfrak{a}, \Sigma, m)$ are the ‘‘rational’’ analogs of the hypergeometric differential operators. They are

W -invariant and the singularities of their meromorphic coefficients are cancelled by multiplication by a power of the polynomial

$$\pi(x) := \prod_{\alpha \in \Sigma^+} \alpha(x). \quad (22)$$

When the triple $(\mathfrak{a}, \Sigma, m)$ arises from the structure of Riemannian symmetric space G/K of the noncompact type, the Bessel differential operators coincide with the radial parts on \mathfrak{a}^+ of the G_0 -invariant differential operators on the corresponding Riemannian symmetric space G_0/K of Euclidean type. For instance, the Bessel differential operator

$$L_0 := L_{\mathfrak{a}} + \sum_{\alpha \in \Sigma^+} m_{\alpha} \frac{1}{\alpha} \partial(y_{\alpha}) \quad (23)$$

coincides with the radial part of the Laplace-Beltrami operator on G_0/K . The operator L_0 is the multivariable analog of the Bessel differential operator of Example 1.3. The Bessel differential operators can be constructed as “differential parts” of differential-reflection operators associated with W -invariant polynomial functions p on $\mathfrak{a}_{\mathbb{C}}^*$ by means of the Dunkl operators [dJ04]. Suppose D_0 is the Bessel differential operator associated with p . Let $k \in \mathbb{N}$ be chosen so that $D := \pi(x)^{2k} D_0$ has real analytic coefficients. Then the principal symbol of D is

$$\sigma(D)(x, \lambda) = p_h(\lambda) \pi(x)^{2k} = p_h(\lambda) \pi_{\Theta}(x)^{2k} P(\lambda, x),$$

where $p_h(\lambda)$ is the highest homogeneous part of $p(\lambda)$ and $P(\lambda, x) = \prod_{\alpha \in \Sigma^+ \setminus \langle \Theta \rangle^+} \alpha(x)^{2k}$ is a homogeneous \mathfrak{a}_{Θ} -elliptic polynomial of degree 0 in λ . As for the hypergeometric differential operators, we obtain the following corollary of Theorems 2.1 and 2.2.

Theorem 3.4. *Let the notation be as above. Let Θ be a fixed set of positive simple roots in Σ . For a Bessel differential operator D_0 let $D = \pi(x)^{2k} \cdot D_0$ be a regularization of D_0 . Then, for $u \in \mathcal{E}'(\mathfrak{a}_{\Theta}; W_{\Theta})$ we have*

$$\text{conv}(\text{supp } Du) = \text{conv}(\text{supp } u).$$

Equivalently, for every $f \in C_c^{\infty}(\mathfrak{a}_{\Theta}; W_{\Theta})$, we have

$$\text{conv}(\text{supp } Df) = \text{conv}(\text{supp } f).$$

If $f \in C_c^{\infty}(\mathfrak{a}_{\Theta})^{W_{\Theta}}$, then $D_0 f \in C_c^{\infty}(\mathfrak{a}_{\Theta})^{W_{\Theta}}$ and

$$\text{conv}(\text{supp } D_0 f) = \text{conv}(\text{supp } f).$$

3.3. Shift operators. For the study of hypergeometric differential operators corresponding to different multiplicity functions (see Example 3.1), Opdam introduced certain W -invariant differential operators, called *shift operators*. For simplicity, we treat here only the case of a reduced root system Σ and we refer the reader to [HS94], Part I, Ch. 3 for the general case. As in the case of hypergeometric differential operators, Opdam’s shift operators have meromorphic coefficients with singularities along the root hyperplanes, and their singularities are cancelled by multiplication by powers of Δ . A *shift* is a multiplicity function l which is even (i.e. such that $l_{\alpha} \in 2\mathbb{Z}$ for all $\alpha \in \Sigma$). Opdam’s shift operators exist for any shift l . They can be considered as generalizations of the hypergeometric differential operators, which form the space $\mathcal{S}(0)$ of shift operators of shift 0. For an arbitrary nonzero shift l , the space $\mathcal{S}(l)$ of shift operators of shift l is a free rank-one (right) $\mathcal{S}(0)$ -module generated by a certain shift operator $G(l)$. See *loc. cit.*, Theorem 3.3.7. Suppose furthermore that either

$l_\alpha > 0$ for all α or $l_\alpha < 0$ for all α . Then $G(l)$ is obtained by composition of *fundamental shift operators* $G_{S,\pm}$, shifting of ± 2 the multiplicities along each Weyl group orbit (*loc. cit.*, p. 42). The principal symbol of each fundamental shift operator is

$$\sigma(G_{S,\pm})(x, \lambda) = c \cdot \Delta_S(x)^{\pm 1} \prod_{\alpha \in S^+} \lambda(x_\alpha),$$

where $S^+ = \Sigma^+ \cap S$ and $\Delta_S(x) = \prod_{\alpha \in S^+} \sinh \alpha(x)$. See *loc. cit.*, Remark 3.3.8. The analysis done in Example 3.1 for the hypergeometric differential operators, in particular Corollary 3.2, easily extends also to the fundamental shift operators, and hence to all shift operators.

A similar argument can be also applied to the shift operators associated with the Bessel differential operators. For more information on the latter we refer to [Hec97], where they are studied in the general case in which W is an arbitrary finite Coxeter group.

3.4. Hamiltonian systems. A wide class of integrable Hamiltonian systems associated with root systems were introduced by Olshanetsky and Perelemov in [OPe76]. Let $(\mathfrak{a}, \Sigma, g)$ be a triple consisting of a finite dimensional Euclidean space \mathfrak{a} , a root system Σ in \mathfrak{a}^* , and a real-valued multiplicity function g on Σ . These integrable systems are described by a Hamiltonian of the form

$$H = -\frac{1}{2}\langle p, p \rangle + U(q), \quad p, q \in \mathfrak{a},$$

with potential energy

$$U(q) := \sum_{\alpha \in \Sigma^+} g_\alpha^2 v(\langle q, \alpha \rangle)$$

where the function v has five possible forms. Here we consider the cases I, II and V as listed in [Pe90], (3.1.14) and (3.8.3):

$$\begin{aligned} \text{(I)} \quad & v(\xi) = \xi^{-2}, \\ \text{(II)} \quad & v(\xi) = \sinh^{-2} \xi, \\ \text{(V)} \quad & v(\xi) = \xi^{-2} + \omega^2 \xi^2. \end{aligned}$$

The Calogero Hamiltonian of Example 1.4 is a special instance of case I. The associated Schrödinger operators are the linear partial differential operators

$$S = -\frac{1}{2}L_{\mathfrak{a}} + U(x), \quad x \in \mathfrak{a}.$$

They are invariant with respect to the Weyl group \widetilde{W} of Σ . The operator S can be regularized by multiplication by the polynomial $\pi(x)$ of (22) in the cases I and V, and by multiplication by δ as in (18). The possibility of applying Theorems 2.1 and 2.2 to these regularized differential operators can be proven as in Examples 3.1 and 3.2. In fact, there is a close relation between the operator S of case II, resp. of case I, and the operator L of (19), resp. to the operator L_0 of (23). See e.g. [HS94], Theorem 2.1.1, and [OPe76].

For an overview on the role of the Hamiltonian systems treated in this example in different areas of theoretical physics and mathematics, we refer the reader to [DV00].

4. APPLICATIONS TO SYMMETRIC SPACES

In this section we apply Theorem 2.2 to differential operators on symmetric spaces. In particular we give a new proof of the D -convexity of the Riemannian symmetric space G/K , c.f. [He73, vdBS93]. Our proof uses Theorem 2.2, applied to the radial part of invariant differential operators and is, as far as we know, new. Our notation is the same as in Subsection 3.2.

Let us start by recalling the notation from Subsection 3.2. Here G is a connected non-compact semisimple Lie group with finite center. Furthermore $\theta : G \rightarrow G$ is a Cartan involution and $K = G^\theta$ the corresponding maximal compact subgroup. Denote by $\kappa : G \rightarrow G/K$, the natural projection $g \mapsto gK$. We denote also by θ the derived involution on \mathfrak{g} . Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where \mathfrak{k} is the $+1$ -eigenspace of θ , and \mathfrak{p} the -1 -eigenspace. Then \mathfrak{k} is the Lie algebra of K . Fix a Cartan subspace \mathfrak{a} , that is a maximal abelian subspace of \mathfrak{p} . The Killing form on \mathfrak{g} defines an Euclidean inner product on \mathfrak{a} . For $\alpha \in \mathfrak{a}^*$ set $\mathfrak{g}^\alpha = \{y \in \mathfrak{g} : [x, y] = \alpha(x)y \text{ for all } x \in \mathfrak{a}\}$. Then the set Σ consisting of all $\alpha \in \mathfrak{a}^* \setminus \{0\}$ for which $\mathfrak{g}^\alpha \neq \{0\}$ is a (generally non-reduced) root system as defined in Section 3. It is called the *(restricted) root system* of $(\mathfrak{g}, \mathfrak{a})$. The multiplicity m_α of $\alpha \in \Sigma$ is defined as the dimension of \mathfrak{g}^α . The map m given by $m(\alpha) := m_\alpha$ is a multiplicity function on Σ . This construction associates with the Riemannian symmetric space G/K a triple $(\mathfrak{a}, \Sigma, m)$. As before we denote by \widetilde{W} the corresponding Weyl group.

Let Σ^+ be a choice of positive roots and let $\mathfrak{a}^+ := \{x \in \mathfrak{a} : \alpha(x) > 0 \text{ for all } \alpha \in \Sigma^+\}$ be the corresponding positive chamber. Denote by $\exp : \mathfrak{g} \rightarrow G$ the exponential map. Then $A := \exp \mathfrak{a}$ is an abelian subgroup of G diffeomorphic to \mathfrak{a} . We set $A^+ := \exp \mathfrak{a}^+$. The map $K \times A \times K \ni (k_1, a, k_2) \mapsto k_1 a k_2 \in G$ is surjective and the A -component is unique up to conjugation by an element of \widetilde{W} . Hence every K -bi-invariant subset of G is of the form $K(\exp B)K$ where B is a \widetilde{W} -invariant subset of \mathfrak{a} . Moreover, B is compact if and only if $\kappa(B) \subset G/K$ is. Let $C_c^\infty(G/K)$ denote the space of compactly supported smooth functions on G/K . Using the map $C_c^\infty(G/K) \rightarrow C_c^\infty(G/K)^K$, $f \mapsto f \circ \kappa$, we will often identify smooth functions on G/K with the corresponding right K -invariant function on G .

The decomposition $G = KAK$ yields the following lemma.

Lemma 4.1. *Let $f \in C_c^\infty(G/K)$ and suppose that $\text{supp } f$ is K -bi-invariant. Then $\text{supp } f = K(\text{supp } f|_A)K$ where $f|_A$ denotes the restriction of f to A . Moreover, $\text{supp } f|_A$ is a \widetilde{W} -invariant compact subset of A .*

Denote by $\mathbb{D}(G/K)$ the (commutative) algebra of G -invariant differential operators on G/K . We identify A^+ can be identified with the submanifold $\kappa(A^+) \subset G/K$. Then, for every $D \in \mathbb{D}(G/K)$, there is a unique \widetilde{W} -invariant differential operator, $\omega(D)$ on A^+ , called the *radial part of D* , so that for all $f \in C^\infty(G/K)$ one has

$$(Df)|_{A^+} = \omega(D)(f|_{A^+}) \tag{24}$$

See [Hel84], p. 259.

Define a differential operator on \mathfrak{a} , also denoted by $\omega(D)$, by

$$\omega(D)g := \omega(D)(g \circ \exp^{-1}) \circ \exp, \quad g \in C^\infty(\mathfrak{a}).$$

In this way, we can consider $\omega(D)$ as a (singular) \widetilde{W} -invariant differential operator on the Euclidean space \mathfrak{a} . To simplify our notation, we shall adopt the identification $A \equiv \mathfrak{a}$ using

the exponential map. Then $\exp := \text{id}$. We then write $f|_{\mathfrak{a}}$ instead of $f|_A$, and the above mentioned decomposition of a K -bi-invariant subset of G will be written as KBK instead of $K(\exp B)K$. With these identifications, the operator $\omega(D)$ turns out to be a hypergeometric differential operator associated with the triple $(\mathfrak{a}, \Sigma, m)$ as considered in Subsection 3.1. When $f \in C_c^\infty(G/K)$ is K -invariant, we can moreover extend (24) by \widetilde{W} -invariance to obtain

$$(Df)|_{\mathfrak{a}} = \omega(D)(f|_{\mathfrak{a}}) \quad (25)$$

Lemma 4.2. *Let G/K be a Riemannian symmetric space of the noncompact type. Let $D \in \mathbb{D}(G/K)$ and let B be a compact, convex and \widetilde{W} -invariant subset of \mathfrak{a} . Set $X_B := KBK$. Then for all $f \in C_c^\infty(G/K)$, we have*

$$\text{supp } f \subseteq X_B \quad \text{if and only if} \quad \text{supp}(Df) \subseteq X_B.$$

Proof. We need to prove that $\text{supp}(Df) \subseteq X_B$ implies $\text{supp } f \subseteq X_B$. The first step is, as in [vdBS93], to reduce to the case in which the support of f is left- K -invariant. The function f can in fact be expanded as a sum of K -finite functions. Since X_B is left- K -invariant, we will obtain $\text{supp } f \subseteq X_B$ if the support of each K -finite summand is contained in X_B . We can therefore assume that f , and hence Df , are K -finite. As the support of a K -finite function is left K -invariant, we obtain from Lemma 4.1 that $\text{supp } f = K(\text{supp } f|_{\mathfrak{a}})K$ and $\text{supp}(Df) = K(\text{supp}(Df)|_{\mathfrak{a}})K$ where $\text{supp } f|_{\mathfrak{a}}$ and $\text{supp}(Df)|_{\mathfrak{a}}$ are \widetilde{W} -invariant and compact. Since $\text{supp}(Df) \subseteq X_B$, we then conclude that $\text{supp}(Df)|_{\mathfrak{a}} \subseteq B$. Since $\omega(D)$ is a hypergeometric differential operator, Corollary 3.3 with $\Theta = \Pi$ yields that $\text{conv}(\text{supp } f|_{\mathfrak{a}}) = \text{conv}(\omega(D) \text{supp } f|_{\mathfrak{a}}) = \text{conv}(\text{supp}(Df)|_{\mathfrak{a}}) \subseteq B$. This proves the required inclusion. \square

Theorem 4.3. *Let G/K and $D \in \mathbb{D}(G/K)$ be as in Lemma 4.2. Then G/K is D -convex, that is for every compact subset S of G/K there is a compact set S' so that for every $f \in C_c^\infty(G/K)$ the inclusion $\text{supp}(D^t f) \subseteq S$ implies $\text{supp } f \subseteq S'$. Here D^t denotes the formal transpose of D .*

Proof. Choose $S_1 \subseteq G$ so that $S_1/K = S$. The set KS_1H is K -bi-invariant, hence $KS_1K = X_B := KBK$ for some \widetilde{W} -invariant compact subset B of $A \equiv \mathfrak{a}$. Since $D^t \in \mathbb{D}(G/K)$ we can apply Lemma 4.1 to it. So $\text{supp}(f) \subseteq X_{\text{conv } B}$. We can then select $S' := X_{\text{conv } B}/K$. \square

We conclude this section by a short discussion of the application of the theorem of our support theorem in harmonic analysis corresponding to the Θ -hypergeometric transform, c.f. [ÓP04]. Because of the technical nature of this application, a detailed exposition would require a certain amount of notation and of background information. Instead, we prefer here just to outline the main ideas involved. We refer the interested reader to [ÓP04] for further information.

As already remarked in the introduction, the support theorem plays a role in the study of the *Paley-Wiener space*. This is the set of images under a suitable generalization \mathcal{F} of the Fourier transform, of the compactly supported smooth functions. The Paley-Wiener space generally consists of entire or meromorphic functions with exponential growth and possibly satisfying additional symmetry conditions. The size of the support of the original function is linked to the exponential growth of its Fourier transform. The thrust of Paley-Wiener type theorems is usually to prove that, if a function g in the Paley-Wiener space has a given exponential growth, then the support of the associated “wave packet” $\mathcal{I}g$ (obtained by formal

application to g of the inverse Fourier transform \mathcal{I} of \mathcal{F}) is compact and has the correct size. In the classical situation of the Fourier transform on \mathbb{R}^n , this is proven by a suitable “shift” of contour of integration by means of Cauchy’s theorem. The shift is allowed because $\mathcal{I}g$ is given by integration of an entire function of exponential type and rapidly decreasing. A suitable generalization of this argument was applied also to the spherical Fourier transform on Riemannian symmetric spaces. See [Hel84], Ch. IV, §7.2, or [GV88], §6.6. For pseudo-Riemannian symmetric spaces the situation is more complicated. Here the wave packages are integrals of meromorphic functions and the required shift of integration would generally require addition of certain “residues”. For some kinds of Fourier transforms, like the spherical Fourier-Laplace transform on noncompactly causal symmetric spaces with even multiplicities [ÓP04], the singularities of the integrand are cancelled by multiplication by a certain polynomial function p . Hence, for every function g in the Paley-Wiener space the wave-packet $\mathcal{I}(pg)$ is given by integration of an entire function. Furthermore, there is an invariant differential operator D so that $D(\mathcal{I}g) = \mathcal{I}(pg)$. The possibility of shifting the contour of integration in $\mathcal{I}(pg)$ allows us then to determine the size of the support of $D(\mathcal{I}g)$. The generalization of the support theorem of Lions and Titchmarsh presented in this papers allows us finally to recover the size of the support of $\mathcal{I}g$. A concrete application of this procedure using Theorem 2.2 can be found in [ÓP04] in the context of the Θ -hypergeometric transform. The latter transform, stated in a setting of transforms associated with root systems, is a common generalization of Opdam’s hypergeometrical transform (hence of Harish-Chandra’s spherical transform on Riemannian symmetric spaces of noncompact type) and of the Fourier-Laplace transform on noncompactly causal symmetric spaces [HÓ96]. A similar application would also be needed for determining the Paley-Wiener space for the transform associated with the Θ -Bessel functions of [BSO04].

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