Generalized Gross heat equation with noises

Abdessatar Barhoumi
Department of Mathematics
University of Tunis El Manar Tunisia
E-mail: abdessatar.barhoumi@ipein.rnu.tn

Hui-Hsiung Kuo
Department of Mathematics
Louisiana State University
Baton Rouge, LA, USA
E-mail: kuo@math.lsu.edu

Habib Ouerdiane
Department of Mathematics
University of Tunis El Manar Tunisia
E-mail: habib.ouerdiane@fst.rnu.tn

August 11, 2004

Abstract

In this paper we study the Gross heat equation perturbed by noises with the initial condition being a generalized function. The noises are given by either a white noise or a space-time white noise. The main technique we use is the representation of the Gross Laplacian as a convolution operator. It enables us to apply the convolution calculus on a suitable distribution space to obtain the explicit solutions of the generalized Gross heat equations.

1 Introduction

First we review from the paper [4] basic concepts, notations, and some results which will be needed in the present paper. Independent development for similar results can be found in the paper [1].

Let X be a real nuclear Frechét space with topology given by an increasing family $\{|\cdot|_p;\, p \in I\!N_0\}$ of Hilbertian norms, $I\!N_0$ being the set of nonnegative integers. Then X is represented as

$$X = \bigcap_{p \in I\!N_0} X_p,$$

where X_p is the completion of X with respect to the norm $|\cdot|_p$. We use X_{-p} to denote the dual space of X_p. Then the dual space X' of X can be represented as

$$X' = \bigcup_{p \in I\!N_0} X_{-p}$$

and is equipped with the inductive limit topology.

Let $N = X + iX$ and $N_p = X_p + iX_p$, $p \in \mathbf{Z}$, be the complexifications of X and X_p, respectively. For $n \in I\!N_0$ we denote by $N^{\widehat{\otimes} n}$ the n-fold symmetric tensor product of N equipped with the π-topology and by $N_p^{\widehat{\otimes} n}$ the n-fold symmetric Hilbertian tensor product of N_p. We will preserve the notation $|\cdot|_p$ and $|\cdot|_{-p}$ for the norms on $N_p^{\widehat{\otimes} n}$ and $N_{-p}^{\widehat{\otimes} n}$, respectively.

Let θ be a Young function, i.e., it is a continuous, convex, and increasing function defined on $I\!R_+$ and satisfies the condition $\lim_{x\to\infty} \theta(x)/x = \infty$, see [10]. We define the conjugate function θ^* of θ by

$$\theta^*(x) = \sup_{t \ge 0} (tx - \theta(t)), \quad x \ge 0.$$

For a Young function θ, we denote by $\mathcal{F}_\theta(N')$ the space of holomorphic functions on N' with exponential growth of order θ and of minimal type. Similarly, let $\mathcal{G}_\theta(N)$ denote the space of holomorphic functions on N with exponential growth of order θ and of arbitrary type. Moreover, for each $p \in \mathbf{Z}$ and $m > 0$, define $\mathrm{Exp}(N_p, \theta, m)$ to be the space of entire functions f on N_p satisfying the condition:

$$\|f\|_{\theta,p,m} = \sup_{x \in N_p} |f(x)| e^{-\theta(m|x|_p)} < \infty.$$

Then the spaces $\mathcal{F}_\theta(N')$ and $\mathcal{G}_\theta(N)$ can be represented as

$$\mathcal{F}_\theta(N') = \bigcap_{p\in I\!N_0,\, m>0} \mathrm{Exp}(N_{-p}, \theta, m),$$

$$\mathcal{G}_\theta(N) = \bigcup_{p\in I\!N_0,\, m>0} \mathrm{Exp}(N_{p}, \theta, m),$$

and are equipped with the projective limit topology and the inductive limit topology, respectively. The space $\mathcal{F}_\theta(N')$ is called the space of *test functions* on N'. Its dual space $\mathcal{F}'_\theta(N')$, equipped with the strong topology, is called the space of *distributions* on N'.

For $p \in I\!N_0$ and $m > 0$, we define the Hilbert spaces

$$F_{\theta,m}(N_p) = \Big\{\vec{\varphi} = (\varphi_n)_{n=0}^\infty\ ;\ \varphi_n \in N_p^{\widehat{\otimes} n},\ \sum_{n\geq 0} \theta_n^{-2} m^{-n} |\varphi_n|_p^2 < \infty\Big\},$$

$$G_{\theta,m}(N_{-p}) = \Big\{\vec{\Phi} = (\Phi_n)_{n=0}^\infty\ ;\ \Phi_n \in N_{-p}^{\widehat{\otimes} n},\ \sum_{n\geq 0} (n!\theta_n)^2 m^{n} |\Phi_n|_{-p}^2 < \infty\Big\},$$

where $\theta_n = \inf_{r>0} e^{\theta(r)}/r^n$, $n \in I\!N_0$. Put

$$F_\theta(N) = \bigcap_{p\in I\!N_0, m>0} F_{\theta,m}(N_p),$$

$$G_\theta(N') = \bigcup_{p\in I\!N_0, m>0} G_{\theta,m}(N_{-p}).$$

The space $F_\theta(N)$ equipped with the projective limit topology is a nuclear Frechét space [4]. Its space $G_\theta(N')$ carries the dual topology of $F_\theta(N)$ with respect to the $\mathbb{C}$-bilinear pairing given by

$$\langle\!\langle \vec{\Phi}, \vec{\varphi}\, \rangle\!\rangle = \sum_{n\geq 0} n! \langle \Phi_n, \varphi_n\rangle, \tag{1}$$

where $\vec{\Phi} = (\Phi_n)_{n=0}^\infty \in G_\theta(N')$ and $\vec{\varphi} = (\varphi_n)_{n=0}^\infty \in F_\theta(N)$.

It was proved in [4] that the *Taylor map* defined by

$$T\colon\ \varphi \longmapsto \Big(\frac{1}{n!}\varphi^{(n)}(0)\Big)_{n=0}^\infty$$

is a topological isomorphism from $\mathcal{F}_\theta(N')$ onto $F_\theta(N)$. The Taylor map T is also a topological isomorphism from $\mathcal{G}_{\theta^*}(N))$ onto $G_\theta(N'))$. The action of a distribution $\Phi \in \mathcal{F}'_\theta(N')$ on a test function $\varphi \in \mathcal{F}_\theta(N')$ can be expressed in terms of the Taylor map as follows:

$$\langle\!\langle \Phi, \varphi \rangle\!\rangle = \langle\!\langle \vec{\Phi}, \vec{\varphi} \rangle\!\rangle,$$

where $\vec{\Phi} = (T^*)^{-1}\Phi$ and $\vec{\varphi} = T\varphi$.

It is easy to see that for each $\xi \in N$, the exponential function

$$e_\xi(z) = e^{\langle z, \xi \rangle}, \quad z \in N',$$

is a test function in the space $\mathcal{F}_\theta(N')$ for any Young function θ. Thus we can define the *Laplace transform* of a distribution $\Phi \in \mathcal{F}'_\theta(N')$ by

$$\widehat{\Phi}(\xi) = \langle\!\langle \Phi, e_\xi \rangle\!\rangle, \quad \xi \in N. \tag{2}$$

From the paper [4], we have the duality theorem which says that the Laplace transform is a topological isomorphism from $\mathcal{F}'_\theta(N')$ onto $\mathcal{G}_{\theta^*}(N)$.

For $\varphi \in \mathcal{F}_\theta(N')$, the *translation* $\mathrm{t}_x\varphi$ of φ by $x \in N'$ is defined by

$$\mathrm{t}_x\varphi(y) = \varphi(y - x), \quad y \in N'.$$

It is easy to check that t_x is a continuous linear operator from $\mathcal{F}_\theta(N')$ into itself for any $x \in N'$. By a *convolution operator* on the test space $\mathcal{F}_\theta(N')$ we mean a continuous linear operator from $\mathcal{F}_\theta(N')$ into itself which commutes with all translation operators t_x, $x \in N'$.

We define the *convolution* $\Phi * \varphi$ of a distribution $\Phi \in \mathcal{F}'_\theta(N')$ and a test function $\varphi \in \mathcal{F}_\theta(N')$ to be the function

$$(\Phi * \varphi)(x) = \langle\!\langle \Phi, \mathrm{t}_{-x}\varphi \rangle\!\rangle, \quad x \in N'.$$

Direct calculations show that $\Phi * \varphi \in \mathcal{F}_\theta(N')$ for any $\varphi \in \mathcal{F}_\theta(N')$ and that the mapping T_Φ defined by

$$T_\Phi\colon\ \varphi \longmapsto \Phi * \varphi, \quad \varphi \in \mathcal{F}_\theta(N'),$$

is a convolution operator on $\mathcal{F}_\theta(N')$. Conversely, it was proved in [3] that all convolution operators on $\mathcal{F}_\theta(N')$ occur this way, i.e., if T is a convolution operator on $\mathcal{F}_\theta(N')$, then there exists a unique $\Phi \in \mathcal{F}'_\theta(N')$ such that $T = T_\Phi$, or equivalently,

$$T(\varphi) = T_\Phi(\varphi) = \Phi * \varphi, \quad \varphi \in \mathcal{F}_\theta(N'). \tag{3}$$

Suppose $\Phi_1, \Phi_2 \in \mathcal{F}'_\theta(N')$. Let T_{Φ_1} and T_{Φ_2} be the convolution operators given by Φ_1 and Φ_2, respectively, as in Equation (3). It is clear that the composition $T_{\Phi_1} \circ T_{\Phi_2}$ is also a convolution operator on $\mathcal{F}_\theta(N')$. Hence there exists a unique distribution, denoted by $\Phi_1 * \Phi_2$, in $\mathcal{F}'_\theta(N')$ such that

$$T_{\Phi_1} \circ T_{\Phi_2} = T_{\Phi_1 * \Phi_2}. \tag{4}$$

The distribution $\Phi_1 * \Phi_2$ in Equation (4) is called the *convolution* of Φ_1 and Φ_2. From Proposition 1 of the paper [2] we have the following equality for any $\Phi_1, \Phi_2 \in \mathcal{F}'_\theta(N')$:

$$(\Phi_1 * \Phi_2)\widehat{\ } = \widehat{\Phi}_1 \, \widehat{\Phi}_2. \tag{5}$$

Let γ be a continuous, convex, and increasing function on $\mathbb{R}^+$. Suppose f is function in $\mathrm{Exp}(\mathbb{C}, \gamma, m)$ for some $m > 0$. For each distribution Φ in $\mathcal{F}'_\theta(N')$, we define the $*$-*composition* $f^*(\Phi)$ of f and Φ by

$$(f^*(\Phi))\widehat{\ } = f(\widehat{\Phi}). \tag{6}$$

It was proved in [2] that $f^*(\Phi)$ belongs to $\mathcal{F}'_\lambda(N')$ with $\lambda = (\gamma \circ e^{\theta^*})^*$.

In particular, when $\gamma(x) = x$, $x \in \mathbb{R}_+$, and $f(z) = e^z$, $z \in \mathbb{C}$, we get a distribution $e^{*\Phi}$ in the space $\mathcal{F}'_{(e^{\theta^*})^*}(N')$ for each $\Phi \in \mathcal{F}'_\theta(N')$. Moreover, by Equation (6), we have

$$(e^{*\Phi})\widehat{\ } = e^{\widehat{\Phi}}. \tag{7}$$

The distribution $e^{*\Phi}$ has the following series expansion

$$e^{*\Phi} = \sum_{n=0}^{\infty} \frac{1}{n!} \Phi^{*n},$$

where $\Phi^{*n} = \Phi * \Phi * \cdots * \Phi$ (n times) and the convergence is in $\mathcal{F}'_{(e^{\theta^*})^*}(N')$ with respect to the strong topology.

2 Generalized Gross heat equation

Let $I \subset \mathbb{R}$ be an interval containing the origin. Consider a family $\{\Phi_t \,;\, t \in I\}$ of distributions in $\mathcal{F}'_\theta(N')$. We assume that the function $t \mapsto \Phi_t$ is continuous from I into $\mathcal{F}'_\theta(N')$. Then the function $t \mapsto \widehat{\Phi_t}$ is continuous from I into

$\mathcal{G}_{\theta^*}(N)$. Thus for each $t \in I$, the set $\{\widehat{\Phi_s}\,;\, s \in [0,t]\}$ is a compact subset of $\mathcal{G}_{\theta^*}(N)$. In particular, it is bounded in $\mathcal{G}_{\theta^*}(N)$. Hence there exist constants $p \in \mathbb{N}_0$, $m > 0$, and $C_t > 0$ such that

$$|\widehat{\Phi_s}(\xi)| \leq C_t\, e^{\theta^*(m|\xi|_p)}, \quad \forall\, s \in [0,t] \text{ and } \xi \in N_p.$$

This inequality shows that the function $\xi \mapsto \int_0^t \widehat{\Phi_s}(\xi)\, ds$ belongs to the space $\mathcal{G}_{\theta^*}(N)$. Hence there exists a unique distribution, denoted by $\int_0^t \Phi_s\, ds$, in $\mathcal{F}'_\theta(N')$ satisfying

$$\left(\int_0^t \Phi_s\, ds\right)^{\widehat{}}\,(\xi) \;=\int_0^t \widehat{\Phi_s}(\xi)\, ds, \quad \xi \in N.$$

Moreover, the process $E_t = \int_0^t \Phi_s\, ds$, $t \in I$, is differentiable in $\mathcal{F}'_\theta(N')$ and satisfies the equation

$$\frac{\partial}{\partial t} E_t = \Phi_t.$$

Let $\{\Phi_t\}$ and $\{M_t\}$ be two continuous $\mathcal{F}'_\theta(N')$-processes. Consider the initial value problem

$$\frac{dX_t}{dt} = \Phi_t * X_t + M_t\,, \quad X_0 = F \in \mathcal{F}'_\theta(N'). \tag{8}$$

The next theorem is from Theorem 4 of the paper [2].

Theorem 1 *The stochastic differential equation (8) has a unique solution in $\mathcal{F}'_{(e^{\theta^*}-1)^*}(N')$ given by*

$$X_t = F * e^{*\int_0^t \Phi_s\, ds} + \int_0^t e^{*\int_s^t \Phi_u\, du} * M_s\, ds.$$

We can apply Theorem 1 to study a generalized Gross heat equation perturbed by a white noise with initial condition being a generalized function. Let $\varphi \in \mathcal{F}_\theta(N')$ be represented by

$$\varphi(x) = \sum_{n\geq 0} \langle x^{\otimes n}, \varphi^{(n)}\rangle.$$

The Gross Laplacian $\Delta_G\varphi(x)$ of φ at $x \in N'$ is given by

$$\Delta_G\varphi(x) = \sum_{n\geq 0} (n+2)(n+1)\langle x^{\otimes n}, \langle \tau, \varphi^{(n+2)}\rangle\rangle,$$

where τ is the trace operator defined by

$$\langle \tau, \xi \otimes \eta \rangle = \langle \xi, \eta \rangle, \quad \xi, \eta \in N.$$

For more information on the Gross Laplacian, see [5][7][8][12].

In fact, the Gross Laplacian Δ_G is a convolution operator given by

$$\Delta_G(\Psi) = \mathcal{T} * \Psi, \quad \Psi \in \mathcal{F}'_\theta(N'), \tag{9}$$

where $\mathcal{T}$ is the distribution in $\mathcal{F}'_\theta(N')$ associated with $\vec{\mathcal{T}} = (0, 0, \tau, 0, \cdots) \in G_\theta(N')$ as in Equation (1).

Now, consider the case when X is the Schwartz space $\mathcal{S}$ of real-valued rapidly decreasing functions on the real line $I\!R$. In this case, we have a white noise process $\dot{W}(t)$ taking values in the space $\mathcal{F}'_\theta(N')$ arising from $\mathcal{S}$, i.e., its Laplace transform is given by

$$\widehat{\dot{W}_t}(\xi) = \xi(t).$$

Then we can apply Theorem 1 to get the next theorem.

Theorem 2 *Let θ be a Young function such that $\lim_{r\to\infty} \theta(r)/r^2 < \infty$. Let $F \in \mathcal{F}'_\theta(N')$. Then the following generalized Gross heat equation perturbed by the white noise $\dot{W}_t$*

$$\frac{\partial U_t}{\partial t} = \Delta_G U_t + \alpha \dot{W}_t, \quad U_0 = F, \;\; \alpha \in I\!R, \tag{10}$$

has a unique solution in $\mathcal{F}'_\theta(N')$ given by

$$U_t = F * e^{*t\mathcal{T}} + \alpha \int_0^t e^{*(t-s)\mathcal{T}} * \dot{W}_s \, ds, \tag{11}$$

where $\mathcal{T}$ is the generalized function given by Equation (9).

Proof By using Equation (9), we can rewrite Equation (10) as

$$\frac{\partial U_t}{\partial t} = \mathcal{T} * U_t + \alpha \dot{W}_t, \quad U_0 = F.$$

Therefore, we can apply Theorem 1 to this equation to get the unique solution in Equation (11). □

We can further rewrite the solution in Equation (11) in another way. Let μ be the standard Gaussian measure on the space X' in Section 1, i.e., its characteristic function is given by

$$\int_{X'} e^{i\langle y,\xi\rangle}\, d\mu(y) = e^{-|\xi|_0^2/2}, \quad \xi \in X,$$

where $|\cdot|_0$ is the norm $|\cdot|_p$ on X for $p = 0$ (see Section 1.)

For $t > 0$, define $\mu_t(\cdot) = \mu(\cdot/\sqrt{t})$. Then the probability μ_t induces a positive distribution $\widetilde{\mu}_t$ in $\mathcal{F}'_\theta(N')$ given by

$$\langle\!\langle \widetilde{\mu}_t, \varphi \rangle\!\rangle = \int_{X'} \varphi(x)\, d\mu_t(x) = \int_{X'} \varphi(\sqrt{t}\, x)\, d\mu(x). \tag{12}$$

For details, see the book [12]. It can be easily checked that the following equality holds:

$$e^{*t\mathcal{T}} = \widetilde{\mu}_t.$$

Therefore, the solution U_t in Equation (11) can be rewritten as

$$U_t = F * \widetilde{\mu}_t + \alpha \int_0^t \widetilde{\mu}_{t-s} * \dot{W}_s\, ds.$$

3 Space-time white noise

In this section we consider the generalized Gross heat equation perturbed by a space-time white noise. By replacing X in Section 1 with $\mathbb{R} \times X$, we can use the same procedure to introduce the spaces $\mathcal{F}_\theta(\mathbb{C} \times N')$ and $\mathcal{F}'_\theta(\mathbb{C} \times N')$. The *space-time white noise* is defined to be

$$Z(t,x) = \langle \cdot, \delta_t \otimes \widetilde{\delta_x} \rangle, \quad t \in \mathbb{R},\ x \in N',$$

where δ_t is the Dirac delta function at t and $\widetilde{\delta_x}$ is the Kubo-Yokoi delta function at x (see the book [12].) Equivalently, the space-time white noise $Z(t,x)$ is given by the following Laplace transform

$$\widehat{Z(t,x)}(\varphi) = \varphi(t,x), \quad \varphi \in \mathcal{F}_\theta(\mathbb{C} \times N').$$

For each fixed t, the function $Z(t,\cdot)$, i.e., $Z(t,x)$ as a function of $x \in N'$, belongs to $\mathcal{F}'_\theta(N')$. For a space-time white noise $W(t,x)$ with $t \in \mathbb{R}$ and $x \in [0,1]$, see the papers [19][20] by Saitô.

For the generalized Gross heat equation perturbed by the space-time white noise $Z(t,x)$, we have the following theorem which can be proved by the similar argument as in Theorem 2.

Theorem 3 *Let* θ *be a Young function such that* $\lim_{r\to\infty}\theta(r)/r^2<\infty$. *Let* $F\in\mathcal{F}'_\theta(N')$. *Then the generalized Gross heat equation perturbed by the space-time white noise* $Z(t,x)$

$$\frac{\partial U_t}{\partial t}=\Delta_G U_t+\alpha Z(t,x),\quad U_0=F,\ \ \alpha\in\mathbb{R},$$

has a unique solution in $\mathcal{F}'_\theta(N')$ *given by*

$$U_t=F*e^{*t\mathcal{T}}+\alpha\int_0^t e^{*(t-s)\mathcal{T}}*Z(s,\cdot)\,ds,$$

where $\mathcal{T}$ *is the generalized function given by Equation (9). The solution* U_t *can also be expressed as*

$$U_t=F*\widetilde{\mu}_t+\alpha\int_0^t\widetilde{\mu}_{t-s}*Z(s,\cdot)\,ds,$$

where $\widetilde{\mu}_t$ *is given by Equation (12).*

Acknowledgements. Kuo would like to thank Professor H. Ouerdiane for the invitation to visit the University of Tunis El Manar, May 16-30, 2004. He is grateful for the hospitality of Professor Ouerdiane and his research group.

References

[1] **N. Asai, I. Kubo, and H.-H. Kuo:** General characterization theorems and intrinsic topologies in white noise analysis; *Hiroshima Math. J.* **31** (2001) 299–330

[2] **M. Ben Chrouda, M. El Oued, and H. Ouerdiane:** Convolution calculus and applications to stochastic differentntial equations; *Soochow J. of Math.* **28** (2002) 375–388

[3] **R. Gannoun, R. Hachaichi, P. Krée, and H. Ouerdiane:** Division de fonctions holomorphes a croissance θ-exponentielle; *BiBoS Preprint*, E 00-01-04 (2000)

[4] **R. Gannoun, R. Hachaichi , H. Ouerdiane, and A. Rezgui:** Un théorème de dualité entre espace de fonctions Holomorphes à croissance exponentielle; *J. Funct. Analysis* **171** (2000) 1–14

[5] **L. Gross and P. Malliavin:** Hall's transform and the Segal-Bargmann map; in: "*Itô's Stochastic Calculus and Probability Theory*" 73–116, Springer-Verlag, Tokyo, 1996

[6] **T. Hida:** *Brownian Motion.* Springer-Verlag, New York, 1980

[7] **T. Hida, H. Kuo, J. Potthoff, and L. Streit:** *White Noise: An Infinite Dimensional Calculus.* Kluwer Academic Publishers, Dordrecht, 1993

[8] **H. Holden , B. Øksendal, J. Ubøe, and T. Zhang:** *Stochastic Partial Differential Equations.* Birkhäuser, 1996

[9] **Y. G. Kondratiev and L. Streit:** Spaces of white noise distributions: Constructions, Descriptions, Applications. I; *Reports on Math. Phys.* **33** (1993) 341–366

[10] **M-A. Krasnosel'ski and Ya-B. Ritickili:** *Convex Functions and Orliez Spaces.* P. Novdhoff, Itd, Groningen, The Nethelands, 1961

[11] **I. Kubo and Y. Yokoi:** A remark on the space of testing random variables in the white noise calculus; *Nagoya Math. J.***115** (1989) 139–149

[12] **H.-H. Kuo:** *White Noise Distribution Theory.* CRC Press, Boca Raton, 1996

[13] **N. Obata:** *White Noise Calculus and Fock Space.* Lecture Notes in Math. **1577**, Springer-Verlag, 1994

[14] **N. Obata:** Wick product of white noise operators and quantum stochastic differential equations; *J. Math. Soc. Japan* **51** (1999) 613–641

[15] **H. Ouerdiane:** Fonctionnelles analytiques avec conditions de croissance et application à l'analyse gaussienne; *Japanese Journal of Math.* **20** (1994) 187–198

[16] **H. Ouerdiane:** Algèbres nucléaires et équations aux dérivées partielles stochastiques; *Nagoya Math. Journal* **151** (1998) 107–127

[17] **H. Ouerdiane:** Infinite dimensional entire functions and applications to stochastic differential equations; *Notices of the South African Mathematical Society* **35** (2004) 23-45

[18] **H. Ouerdiane and A. Rezgui:** Representation intégrales de fonctionnelles analytiques; *Stochastic Processes, Physics and Geometry: New Interplays.* A volume in honor of S. Albeverio, Canadian Math. Society. Conference Proceedings Series, 1999

[19] **K. Saitô:** Itô's formula and Lévy's Laplacian; *Nagoya Mathematical Journal* **108** (1987) 67–76

[20] **K. Saitô:** Itô's formula and Lévy's Laplacian II; *Nagoya Mathematical Journal* **123** (1991) 153–169

[21] **F. Trèves:** *Topological Vector Spaces, Distributions, and Kernels.* Academic Press, New York and London, 1967