

GENERATING FUNCTION METHOD FOR ORTHOGONAL POLYNOMIALS AND JACOBI-SZEGÖ PARAMETERS

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Let μ be a probability measure on \mathbb{R} with finite moments of all orders. Suppose μ is not supported by a finite set of points. Then there exists a unique sequence $\{P_n(x)\}_{n=0}^{\infty}$ of orthogonal polynomials such that $P_n(x)$ is a polynomial of degree n with leading coefficient 1 and the equality $(x - \alpha_n)P_n(x) = P_{n+1}(x) + \omega_n P_{n-1}(x)$ holds. The numbers $\{\alpha_n, \omega_n\}_{n=0}^{\infty}$ are called the Jacobi-Szegő parameters of μ . The family $\{P_n(x), \alpha_n, \omega_n\}_{n=0}^{\infty}$ determines the interacting Fock space of μ . In this paper we use the concept of generating function to give several methods for computing the orthogonal polynomials $P_n(x)$ and the Jacobi-Szegő parameters α_n and ω_n . We also describe how to identify the orthogonal polynomials in terms of differential or difference operators.

1. Accardi–Bożejko unitary isomorphism

Let μ be a probability measure on \mathbb{R} with finite moments of all orders. Assume that μ is not supported by a finite set of points and that the linear span of the monomials $\{x^n; n \geq 0\}$ is dense in the complex Hilbert space $L^2(\mu)$. Then we can apply the Gram-Schmidt orthogonalization procedure

to the monomials $\{1, x, x^2, \dots, x^n, \dots\}$, in this order, to get orthogonal polynomials $\{P_0(x), P_1(x), \dots, P_n(x), \dots\}$. Here $P_n(x)$ is a polynomial of degree n with leading coefficient 1. It is well-known (see, e.g., the books by Chihara⁵ and by Szegő⁷) that these orthogonal polynomials satisfy the recursion formula

$$(x - \alpha_n)P_n(x) = P_{n+1}(x) + \omega_n P_{n-1}(x), \quad n \geq 0, \quad (1)$$

where $\alpha_n \in \mathbb{R}$, $\omega_n > 0$ and by convention $\omega_0 = 1$, $P_{-1} = 0$. The numbers α_n and ω_n are called the Jacobi-Szegő parameters of μ .

Define a sequence $\{\lambda_n\}_{n=0}^\infty$ associated with the measure μ by

$$\lambda_n = \omega_0 \omega_1 \cdots \omega_n, \quad n \geq 0. \quad (2)$$

It can be easily checked that $\lambda_n = \int_{\mathbb{R}} |P_n(x)|^2 d\mu(x)$. Assume that the sequence satisfies the condition that $\inf_{n \geq 0} \lambda_n^{1/n} > 0$. Define a complex Hilbert space Γ_μ by

$$\Gamma_\mu = \left\{ (c_0, c_1, \dots, c_n, \dots) \mid c_n \in \mathbb{C}, \sum_{n=0}^\infty \lambda_n |c_n|^2 < \infty \right\}$$

with norm $\|\cdot\|$ given by

$$\|(c_0, c_1, \dots, c_n, \dots)\| = \left(\sum_{n=0}^\infty \lambda_n |c_n|^2 \right)^{1/2}.$$

Let $\Phi_n = (0, \dots, 0, 1, 0, \dots)$ with 1 in the $(n+1)$ st component. Define the *creation*, *annihilation*, and *neutral operators* a^+ , a^- , and a^0 acting on Γ_μ , respectively, by

$$a^+ \Phi_n = \Phi_{n+1}, \quad a^- \Phi_n = \omega_n \Phi_{n-1}, \quad a^0 \Phi_n = \alpha_n \Phi_n, \quad n \geq 0,$$

where $\Phi_{-1} = 0$ by convention and α_n 's and ω_n 's are the Jacobi-Szegő parameters of μ . It can be easily shown that the operators a^+ and a^- are adjoint to each other.

The Hilbert space Γ_μ together with the operators $\{a^+, a^-, a^0\}$ is called the *interacting Fock space* associated with the measure μ . It has been shown by Accardi and Bożejko¹ that there exists a unitary isomorphism $U : \Gamma_\mu \rightarrow L^2(\mu)$ satisfying the conditions:

- (1) $U\Phi_0 = 1$,
- (2) $Ua^+U^*P_n = P_{n+1}$,
- (3) $Ua^-U^*P_n = \omega_n P_{n-1}$,
- (4) $U(a^+ + a^- + a^0)U^* = X$,

where the polynomials $P_n(x)$'s are given in Equation (1) and X is the multiplication operator by x .

Note that the Hilbert space is determined only by the numbers ω_n 's, while the numbers α_n 's and the polynomials $P_n(x)$'s are related to the unitary operator U . It is natural to ask the following question:

Question: Given a probability measure μ on \mathbb{R} , how to compute the associated orthogonal polynomials and the Jacobi-Szegö parameters $\{P_n, \alpha_n, \omega_n\}$?

In Section 2 we will explain the generating function method to derive the orthogonal polynomials. In Section 3 we will describe two ways for the computation of the Jacobi-Szegö parameters. In Section 4 we will discuss the computation of the orthogonal polynomials by differential and difference operators. In Section 5 we will list some important classical examples from the viewpoint of generating functions.

2. Pre-generating and generating functions

Let μ be a probability measure on \mathbb{R} satisfying the conditions mentioned in Section 1. In a series of papers^{2,3,4} we have introduced the generating function method to derive the associated orthogonal polynomials $\{P_n(x)\}$ and the Jacobi-Szegö parameters $\{\alpha_n, \omega_n\}$.

A *pre-generating function* is a function $\varphi(t, x)$ which admits a power series expansion in t as follows:

$$\varphi(t, x) = \sum_{n=0}^{\infty} g_n(x)t^n,$$

where $g_n(x)$ is a polynomial of degree n and $\limsup_{n \rightarrow \infty} \|g_n\|_{L^2(\mu)}^{1/n} < \infty$. A *generating function* for μ is a pre-generating function of the form

$$\psi(t, x) = \sum_{n=0}^{\infty} a_n P_n(x)t^n, \quad (3)$$

where $P_n(x)$'s are the orthogonal polynomials associated with μ as given in Equation (1). Note that a generating function for μ is not unique because we can always replace t in Equation (3) with ct for a nonzero constant $c \neq 1$ to get a different function $\psi(t, x)$. However, it is possible to have two essentially different generating functions for the same measure. For

example, the following functions

$$\psi(t, x) = \frac{1}{1 - 2tx + t^2}$$

$$\psi(t, x) = \left(\frac{2}{(1 - 2tx + t^2)(1 - tx + \sqrt{1 - 2tx + t^2})} \right)^{1/2}$$

are generating functions for the measure $d\mu(x) = \frac{2}{\pi} \sqrt{1 - x^2} dx, |x| \leq 1$.

Suppose $\varphi(t, x)$ is a pre-generating function. Consider its multiplicative renormalization defined by

$$\psi(t, x) = \frac{\varphi(t, x)}{E_\mu \varphi(t, \cdot)}.$$

Theorem 2.1. *The multiplicative renormalization $\psi(t, x)$ is a generating function for μ if and only if $E_\mu[\psi(t, \cdot)\psi(t, \cdot)]$ is a function of ts .*

If we can check that $E_\mu[\psi(t, \cdot)\psi(t, \cdot)]$ is a function of ts , then by the above theorem $\psi(t, x)$ is a generating function. We can expand $\psi(t, x)$ as a power series in t to get

$$\psi(t, x) = \sum_{n=0}^{\infty} Q_n(x)t^n,$$

where $Q_n(x)$ is a polynomial of degree n . Let a_n be the leading coefficient of $Q_n(x)$ and let $P_n(x) = Q_n(x)/a_n$. Then the polynomials $\{P_n(x)\}$ are the orthogonal polynomials satisfying Equation (1) for the measure μ .

In the papers^{2,3,4} we have applied the generating function method to pre-generating functions of the form

$$\varphi(t, x) = h(\rho(t)x),$$

where $h(x) = e^x$ or $h(x) = (1 - x)^c$ and $\rho(t)$ is a function to be derived so that the condition in Theorem 2.1 is satisfied.

Case 1: $h(x) = e^x$

measure	polynomials
Gaussian	Hermite
Poisson	Charlier
gamma	Laguerre
negative binomial	Meixner

Case 2: $h(x) = (1 - x)^c$

measure	polynomials
uniform	Legendre
arcsine	Chebyshev of 1st kind
semi-circle	Chebyshev of 2nd kind
beta-type	Gegenbauer

The above polynomials are derived from the power series expansion of the resulting generating functions. Consequently they are expressed in terms of sums of monomials. In Section 4 we will use differential and difference operators to identify these polynomials.

Recently all measures of exponential type have been derived in the paper⁶. In particular, the probability law of the Lévy stochastic area is in this class.

3. Computation of orthogonal polynomials and Jacobi-Szegö parameters

Suppose $\psi(t, x)$ is a generating function for μ . The object is to compute the orthogonal polynomials $\{P_n(x)\}$ and the Jacobi-Szegö parameters $\{\alpha_n, \omega_n\}$ from $\psi(t, x)$. Recall that $\lambda_n = \omega_0\omega_1 \cdots \omega_n$, $n \geq 0$, as defined by Equation (2). The following theorem has been proved in our paper³.

Theorem 3.1. *Let $\psi(t, x) = \sum_{n=0}^{\infty} a_n P_n(x) t^n$ be a generating function for μ . Then we have*

$$\lim_{t \rightarrow 0} \psi\left(t, \frac{x}{t}\right) = \sum_{n=0}^{\infty} a_n x^n, \tag{4}$$

$$E_{\mu}[\psi(t, \cdot)^2] = \sum_{n=0}^{\infty} a_n^2 \lambda_n t^{2n}, \tag{5}$$

$$E_{\mu}[x\psi(t, \cdot)^2] = \sum_{n=0}^{\infty} \left(a_n^2 \lambda_n \alpha_n t^{2n} + 2a_n a_{n-1} \lambda_n t^{2n-1} \right), \tag{6}$$

where $a_{-1} = 0$ by convention.

Thus once we have found a generating function for μ , then we can compute $\{P_n, \alpha_n, \omega_n\}$ as follows:

$$\psi(t, x) \bullet \longrightarrow \{a_n, P_n\} \bullet \longrightarrow \{\lambda_n\} \bullet \longrightarrow \{\alpha_n, \omega_n\}$$

Namely, first expand $\psi(t, x)$ as a power series in t to get a_n and $P_n(x)$, which is expressed as a sum of monomials. If we are not interested in finding P_n , then we do not have to expand $\psi(t, x)$ as a power series in t . We can simply use Equation (4) to find a_n . Then we use Equation (5) to find λ_n , which can be used in turn to find ω_n since $\omega_n = \lambda_n/\lambda_{n-1}$, $n \geq 1$, $\omega_0 = 1$. Finally we can use Equation (6) to derive α_n .

In our paper³ we have used this method to compute $\{P_n, \alpha_n, \omega_n\}$ for those measures listed in Section 2. However, the computation is somewhat complicated due to the following difficulties:

Difficulties: There are several difficulties in applying Theorem 3.1.

- (1) It might be difficult to find the series expansion of $\psi(t, x)$ in t .
- (2) The computation of $E_\mu[\psi(t, \cdot)^2]$ in Equation (5) might be very complicated.
- (3) The computation of $E_\mu[x\psi(t, \cdot)^2]$ in Equation (6) is even more involved.
- (4) The orthogonal polynomials are expressed as sums of monomials. How to find the close forms for them?

Resolution: Here are some ideas to overcome the above difficulties:

- (a) Find a computation method without having to use the power series expansion of $\psi(t, x)$ in t .
- (b) Find a system of linear equations for the Jacobi-Szegő parameters.
- (c) Determine $P_n(x)$ by a differential or difference operator.
- (d) Find a differential equation satisfied by $P_n(x)$. This equation is determined by the measure μ .
- (e) Determine $\{\alpha_n, \omega_n\}$ from the eigenvalues of a differential operator.

We have made some progress regarding to Items (a), (b), and (c). The key idea is to use the series expansions of $\psi(t, 0)$ and $\partial_x \psi(t, x)|_{x=0}$ in t . Then we can avoid the difficulty (1).

So, suppose $\varphi(t, x)$ is a pre-generating function and assume that its multiplicative renormalization $\psi(t, x) = \varphi(t, x)/E_\mu \varphi(t, \cdot)$ is a generating function for μ . Define three functions $A(x)$, $B(t)$, and $C(t)$ with their respective power series expansions by

$$A(x) = \lim_{t \rightarrow 0} \psi\left(t, \frac{x}{t}\right) = \sum_{n=0}^{\infty} a_n x^n, \quad (7)$$

$$B(t) = \psi(t, 0) = \sum_{n=0}^{\infty} b_n t^n, \quad (8)$$

$$C(t) = \frac{\partial}{\partial x} \psi(t, x) \Big|_{x=0} = \sum_{n=0}^{\infty} c_n t^n. \quad (9)$$

The next theorem is from our paper⁴.

Theorem 3.2. *Suppose $\psi(t, x) = \varphi(t, x)/E_\mu \varphi(t, \cdot)$ is a generating function for μ . Let a_n, b_n, c_n be the numbers defined in Equations (7)–(9). Assume that $b_n c_{n-1} \neq b_{n-1} c_n$. Then the Jacobi-Szegö parameters $\{\alpha_n, \omega_n\}$ are the unique solution of the system of the linear equations:*

$$\begin{cases} \frac{b_n}{a_n} \alpha_n + \frac{b_{n-1}}{a_{n-1}} \omega_n = -\frac{b_{n+1}}{a_{n+1}}, \\ \frac{c_n}{a_n} \alpha_n + \frac{c_{n-1}}{a_{n-1}} \omega_n = -\frac{c_{n+1}}{a_{n+1}} + \frac{b_n}{a_n}. \end{cases} \quad (10)$$

Thus we can compute the Jacobi-Szegö parameters as follows:

$$\psi(t, x) \bullet \longrightarrow \{A(t), B(t), C(t)\} \bullet \longrightarrow \{a_n, b_n, c_n\} \bullet \longrightarrow \{\alpha_n, \omega_n\}$$

Now, consider the special case when the pre-generating function $\varphi(t, x)$ is of the form

$$\varphi(t, x) = h(\rho(t)x), \quad (11)$$

where $\rho(t) = \sum_{n=1}^{\infty} \rho_n t^n$ is an analytic function near $x = 0$ with $\rho_1 \neq 0$ and $h(x) = \sum_{n=0}^{\infty} h_n x^n$ is an analytic function near $x = 0$ with $h_0 = 1, h_n \neq 0$ for all $n \geq 1$ and there exists $t_1 > 0$ such that $\rho(tx)$ is analytic in x on the support of μ for all $|t| < t_1$, and $\limsup_{n \rightarrow \infty} (|h_n| \|x^n\|_{L^2(\mu)})^{1/n} < \infty$. In this case the function $A(x)$ in Equation (7) is given by

$$A(x) = \lim_{t \rightarrow 0} \frac{h\left(\rho(t) \frac{x}{t}\right)}{E_\mu h(\rho(t) \cdot)} = \frac{h(\rho_1 x)}{h_0} = h(\rho_1 x).$$

Therefore, $a_n = h_n \rho_1^n$ and so Equation (10) becomes

$$\begin{cases} \frac{\rho_1 b_n}{h_n} \alpha_n + \frac{\rho_1^2 b_{n-1}}{h_{n-1}} \omega_n = -\frac{b_{n+1}}{h_{n+1}}, \\ \frac{\rho_1 c_n}{h_n} \alpha_n + \frac{\rho_1^2 c_{n-1}}{h_{n-1}} \omega_n = -\frac{c_{n+1}}{h_{n+1}} + \frac{\rho_1 b_n}{h_n}. \end{cases} \quad (12)$$

In particular, when μ is symmetric, we have $\alpha_n = 0$ for all n . Then Equation (12) yields the values of ω_n 's as follows:

$$\begin{cases} \omega_{2m+1} = -\frac{b_{2m+2} h_{2m}}{\rho_1^2 b_{2m} h_{2m+2}}, \\ \omega_{2m} = -\frac{c_{2m+1} h_{2m-1}}{\rho_1^2 c_{2m-1} h_{2m+1}} + \frac{b_{2m} h_{2m-1}}{\rho_1 c_{2m-1} h_{2m}}. \end{cases} \quad (13)$$

4. Orthogonal polynomials in terms of differential or difference operators

Suppose $\psi(t, x)$ is a generating function for μ . Then we can expand $\psi(t, x)$ as a power series in t

$$\psi(t, x) = \sum_{n=0}^{\infty} a_n P_n(x) t^n,$$

where $P_n(x)$'s are the orthogonal polynomials associated with μ . As pointed out in Section 3, these polynomials are expressed as sums of monomials. Since it might be difficult to compute the power series expansion of $\psi(t, x)$, it is desirable to determine or identify these polynomials in some other ways, namely, we raise the following

Question: How to determine or identify the orthogonal polynomials $\{P_n(x)\}$ without using the power series expansion of $\psi(t, x)$ in t ?

For absolutely continuous measures we have the next two theorems from our paper⁴. Recall that $A(x) = \sum_{n=0}^{\infty} a_n x^n$ from Equation (7) and that $\lambda_n = \omega_0 \omega_1 \cdots \omega_n$.

Theorem 4.1. *Let μ be a measure on (a, b) with a smooth density function $\theta(x)$ and assume that $\psi(t, x)$ is a generating function for μ . Suppose $q_n(x)$ is a smooth function satisfying the conditions:*

(a) $\frac{1}{\theta(x)} D_x^n [q_n(x) \theta(x)] \in L^2(\mu).$

(b) $D_x^k [q_n(x)\theta(x)] = 0$ at $x = a, b$ for all $0 \leq k < n$.

Then the orthogonal polynomial $P_n(x)$ associated with μ is given by

$$P_n(x) = \frac{1}{k_n \theta(x)} D_x^n [q_n(x)\theta(x)]$$

with some constant k_n if and only if

$$\int_a^b [D_x^n \psi(t, x)] q_n(x) d\mu(x) = d_n t^n$$

with some constant d_n . In this case, d_n and k_n are related by

$$d_n = (-1)^n \lambda_n a_n k_n.$$

For the special case when $\varphi(t, x)$ is of the form in Equation (11), the generating function is given by

$$\psi(t, x) = \frac{\varphi(t, x)}{E_\mu \varphi(t, \cdot)}.$$

In this case, we can make use of the function $B(t)$ defined in Equation (8) and take $q_n(x) = \theta_n(x)/\theta(x)$ in Theorem 4.1 to get the next theorem.

Theorem 4.2. Let μ be a measure on (a, b) with a smooth density function $\theta(x)$ and assume that

$$\psi(t, x) = \frac{h(\rho(t)x)}{E_\mu [h(\rho(t)\cdot)]}$$

is a generating function for μ . Suppose $\theta_n(x)$ is a smooth function with support in $[a, b]$ satisfying the conditions:

(a) $\frac{1}{\theta(x)} D_x^n [\theta_n(x)] \in L^2(\mu)$.

(b) $D_x^k [\theta_n(x)] = 0$ at $x = a, b$ for all $0 \leq k < n$.

Then the orthogonal polynomial $P_n(x)$ associated with μ is given by

$$P_n(x) = \frac{1}{k_n \theta(x)} D_x^n [\theta_n(x)]$$

with some constant k_n if and only if

$$\int_a^b h^{(n)}(\rho(t)x) \theta_n(x) dx = \frac{d_n}{B(t)} \left(\frac{t}{\rho(t)} \right)^n$$

with some constant d_n . In this case, d_n and k_n are given by

$$d_n = n! \rho_1^n h_n, \quad k_n = (-1)^n \frac{n!}{\lambda_n}.$$

For discrete measures on the set $\mathbb{N}_0 = \{0, 1, 2, \dots, n, \dots\}$ we have the analogous results in the following two theorems from our paper⁴. Define the right- and left-hand difference operators Δ_{x+} and Δ_{x-} acting on functions defined on \mathbb{N}_0 by

$$\Delta_{x+}f(x) = f(x + 1) - f(x),$$

$$\Delta_{x-}f(x) = f(x) - f(x - 1),$$

where $f(-1) = 0$ by convention. We have $\Delta_{x+}^n f(x - n) = \Delta_{x-}^n f(x)$.

Theorem 4.3. *Let μ be a measure on \mathbb{N}_0 and let $\theta(x) = \mu(\{x\})$, $x \in \mathbb{N}_0$. Assume that $\psi(t, x)$ is a generating function for μ . Suppose $q_n(x)$ is a function on \mathbb{N}_0 satisfying the conditions:*

$$(a) \frac{1}{\theta(x)} \Delta_{x-}^n [q_n(x)\theta(x)] \in L^2(\mu).$$

$$(b) \Delta_{x+}^k [q_n(x)\theta(x)] = 0 \text{ at } x = 0, \infty \text{ for all } 0 \leq k < n.$$

Then the orthogonal polynomial $P_n(x)$ associated with μ is given by

$$P_n(x) = \frac{1}{k_n \theta(x)} \Delta_{x-}^n [q_n(x)\theta(x)]$$

with some constant k_n if and only if

$$\sum_{x=0}^{\infty} [\Delta_{x+}^n \psi(t, x)] q_n(x)\theta(x) = d_n t^n$$

with some constant d_n . In this case, d_n and k_n are related by

$$d_n = (-1)^n \lambda_n a_n k_n.$$

The next theorem is analogous to Theorem 4.2. It is a special case of Theorem 4.3 for the multiplicative renormalization of the pre-generating function $\varphi(t, x) = e^{\rho(t)x}$.

Theorem 4.4. *Let μ be a measure on \mathbb{N}_0 and let $\theta(x) = \mu(\{x\})$, $x \in \mathbb{N}_0$. Assume that the multiplicative renormalization*

$$\psi(t, x) = B(t)e^{\rho(t)x} \left(B(t) = \frac{1}{E_{\mu}[e^{\rho(t)\cdot}]} \right)$$

is a generating function for μ , Here $\rho(t) = \sum_{n=1}^{\infty} \rho_n t^n$ is analytic with $\rho_1 \neq 0$. Suppose $\theta_n(x)$ is a probability mass function on \mathbb{N}_0 for each n satisfying the conditions:

- (a) $\frac{1}{\theta(x)} \Delta_{x-}^n [\theta_n(x)] \in L^2(\mu)$.
- (b) $\Delta_{x+}^k [\theta_n(x)] = 0$ at $x = 0, \infty$ for all $0 \leq k < n$.

Then the orthogonal polynomial $P_n(x)$ associated with μ is given by

$$P_n(x) = \frac{1}{k_n \theta(x)} \Delta_{x-}^n [\theta_n(x)]$$

with some constant k_n if and only if

$$\sum_{x=0}^{\infty} e^{\rho(t)x} \theta_n(x) = \frac{d_n t^n}{B(t)(e^{\rho(t)} - 1)^n}$$

with some constant d_n . In this case, d_n and k_n are given by

$$d_n = \rho_1^n, \quad k_n = (-1)^n \frac{n!}{\lambda_n}.$$

5. Classical examples

In our paper³ we have used our method to derive generating functions $\psi(t, x)$ and $\{P_n(x), \alpha_n, \omega_n\}$ for those measures listed in Section 2. But some computations, for example in identifying the polynomials $P_n(x)$, are rather complicated. In the paper⁴ the computations are simplified by using Equations (12) and (13) and Theorems 4.2 and 4.4. Recall that $\omega_0 = 1$.

Example 5.1. (*Gaussian measure*) $\theta(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}$, $x \in \mathbb{R}$.

$$\begin{aligned} \psi(t, x) &= e^{tx - \sigma^2 t^2/2}, \\ P_n(x) &= (-\sigma^2)^n e^{x^2/2\sigma^2} D_x^n [e^{-x^2/2\sigma^2}] \quad (\text{Hermite polynomial}), \\ \alpha_n &= 0, \quad n \geq 0, \\ \omega_n &= \sigma^2 n, \quad n \geq 1. \end{aligned}$$

Example 5.2. (*Gamma distribution*) $\theta(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x}$, $x > 0$. Here $\alpha > 0$.

$$\begin{aligned} \psi(t, x) &= (1+t)^{-\alpha} e^{\frac{t}{1+t}x}, \\ P_n(x) &= (-1)^n x^{-\alpha+1} e^x D_x^n [x^{n+\alpha-1} e^{-x}] \quad (\text{Laguerre polynomial}), \\ \alpha_n &= 2n + \alpha, \quad n \geq 0, \\ \omega_n &= n(n + \alpha - 1), \quad n \geq 1. \end{aligned}$$

Example 5.3. (*Beta-type distribution*)

$$\theta(x) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\beta + 1)}{\Gamma(\beta + \frac{1}{2})} (1 - x^2)^{\beta - 1/2}, \quad |x| < 1,$$

where the parameter $\beta > -1/2$, $\beta \neq 0$.

$$\psi(t, x) = \frac{1}{(1 - 2tx + t^2)^\beta}, \tag{14}$$

$$P_n(x) = (-1)^n \frac{\Gamma(n + 2\beta)}{\Gamma(2n + 2\beta)} (1 - x^2)^{-\beta + 1/2} D_x^n [(1 - x^2)^{n + \beta - 1/2}]$$

(Gegenbauer polynomial),

$$\alpha_n = 0, \quad n \geq 0,$$

$$\omega_n = \frac{n(n - 1 + 2\beta)}{4(n + \beta)(n - 1 + \beta)}, \quad n \geq 1.$$

We have two special cases. When $\beta = 1/2$, the measure μ is the *uniform measure* on $[-1, 1]$ and $P_n(x)$ is the Legendre polynomial. When $\beta = 1$, the measure μ is the *semi-circle distribution* on $[-1, 1]$ and $P_n(x)$ is the Chebyshev polynomial of the second kind, which can be verified to equal

$$P_n(x) = \frac{1}{2^n} \frac{\sin [(n + 1) \cos^{-1} x]}{\sin [\cos^{-1} x]}, \quad n \geq 0.$$

Note that we have to exclude $\beta = 0$ in this example since the function in Equation (14) is not a generating function when $\beta = 0$. The next example takes care of this case.

Example 5.4. (*Arcsine distribution*) $\theta(x) = \frac{1}{\pi} \frac{1}{\sqrt{1 - x^2}}$, $|x| < 1$.

$$\psi(t, x) = \frac{1 - t^2}{(1 - 2tx + t^2)},$$

$$P_n(x) = \begin{cases} 1, & \text{if } n = 0, \\ \frac{1}{2^{n-1}} \cos [n \cos^{-1} x], & \text{if } n \geq 1. \end{cases}$$

(Chebyshev polynomial of the first kind),

$$\alpha_n = 0, \quad n \geq 0,$$

$$\omega_n = \begin{cases} 1/2, & \text{if } n = 1, \\ 1/4, & \text{if } n \geq 2. \end{cases}$$

Example 5.5. (*Poisson measure*) $\theta(x) = e^{-\lambda} \frac{\lambda^x}{x!}$, $x \in \mathbb{N}_0$, $\lambda > 0$.

$$\begin{aligned}\psi(t, x) &= e^{-\lambda t} (1+t)^x, \\ P_n(x) &= (-1)^n \lambda^{-x} \Gamma(x+1) \Delta_{x-}^n \left[\frac{\lambda^{x+n}}{\Gamma(x+1)} \right] \\ &= (-1)^n \lambda^{-x} \Gamma(x+1) \Delta_{x+}^n \left[\frac{\lambda^x}{\Gamma(x-n+1)} \right] \\ &\quad \text{(Charlier polynomial),} \\ \alpha_n &= n + \lambda, \quad n \geq 0, \\ \omega_n &= \lambda n, \quad n \geq 1.\end{aligned}$$

Example 5.6. (*Negative binomial measure*)

$$\theta(x) = p^r \binom{-r}{x} (-1)^x (1-p)^x, \quad x \in \mathbb{N}_0,$$

where $r > 0$ and $0 < p < 1$.

$$\begin{aligned}\psi(t, x) &= (1+t)^x (1+(1-p)t)^{-x-r}, \\ P_n(x) &= (-1)^n \frac{1}{p^n} \frac{\Gamma(x+1)}{\Gamma(x+r)} (1-p)^{-x} \Delta_{x+}^n \left[\frac{\Gamma(x+r)}{\Gamma(x-n+1)} (1-p)^x \right] \\ &\quad \text{(Meixner polynomial),} \\ \alpha_n &= \frac{(2-p)n + r(1-p)}{p}, \quad n \geq 0, \\ \omega_n &= \frac{n(n+r-1)(1-p)}{p^2}, \quad n \geq 1.\end{aligned}$$

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