

THE IMAGE OF THE HEAT KERNEL TRANSFORM ON RIEMANNIAN SYMMETRIC SPACES OF THE NONCOMPACT TYPE

BERNHARD KRÖTZ, GESTUR ÓLAFSSON AND ROBERT J. STANTON

ABSTRACT. The heat kernel transform on G/K , a Riemannian symmetric space of noncompact type, maps an L^2 -function on G/K to a holomorphic function on the complex crown. In this article we determine the image of this transform on L^2 .

1. INTRODUCTION

The heat equation, and the associated heat kernel transform (a.k.a. the Bargmann-Segal transform), is a natural counterpart of a Riemannian metric. For the homogeneous spaces \mathbb{R}^n or a compact symmetric space endowed with the usual metric, many properties of the heat kernel transform are known [2, 7, 16]. The relevant property for this note is the fundamental observation made by Bargmann and Segal that the image of the heat kernel transform for \mathbb{R}^n consists of functions with holomorphic extension to \mathbb{C}^n . More generally, let X denote one of the aforementioned Riemannian manifolds and k_t the heat kernel. The isometry group of X acts transitively, consequently X is diffeomorphic and isometric to G/K , K the isotropy group of a basepoint. One knows that the Riemannian manifold X has a natural complexification, denoted here by Ξ , which is G -diffeomorphic to the tangent bundle TX , and on which k_t has a holomorphic extension, say \tilde{k}_t . For a function f on X , $k_t * f$, the solution of the heat equation with initial data f , has holomorphic extension to Ξ given by $\tilde{k}_t * f$. For f in the Hilbert space $L^2(X)$ one obtains thus a G -equivariant linear map, the *heat kernel transform*, $H_t : L^2(X) \rightarrow \mathcal{O}(\Xi)$. For these spaces X it is known that the image of H_t is a weighted Bergman space on Ξ , and that H_t is a G -equivariant unitary map between these Hilbert spaces.

That this might not be a general phenomenon was observed first in [13] for the Heisenberg group with a left invariant Laplace operator. For this Riemannian space the image of the heat kernel transform is no longer one weighted Bergman space but a sum of two, each of which corresponds to a nonpositive weight function with strong oscillatory behaviour. Surprisingly, neither the Heisenberg group nor the other examples suggest the form of the result for symmetric spaces of noncompact type.

One problem to be resolved had been to determine the *natural* complexification of $X = G/K$. Introduced in [1] for this purpose, the *Akhiezer-Gindikin domain* Ξ , or as it is sometimes called the *complex crown* of G/K , is a G -invariant domain in $X_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}}$ containing X , and biholomorphic to an open complex subdomain in TX endowed with the adapted complex structure. The G -orbit structure of Ξ is nicely given by $\Xi = G \exp(i\Omega) \cdot x_o$, where the basepoint $x_o = eK_{\mathbb{C}} \in X_{\mathbb{C}}$, X is identified with the G -orbit through x_o , and Ω is a specific polyhedral subset of a maximal flat subspace $\mathfrak{a} \subset T_{x_o}X$.

In [11, 12] it was shown that Ξ is a maximal domain of holomorphy for eigenfunctions of $\mathbb{D}(X)$, the algebra of invariant differential operators on X , in particular for the Laplace operator. Consequently, this provides a canonical domain on which to investigate the image of the heat kernel transform on X .

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Such a study was initiated in [12] where a G -equivariant holomorphic heat kernel transform for X was introduced. There it was shown that the heat kernel k_t extends to a holomorphic function \tilde{k}_t on Ξ , and that, similarly, one has a G -equivariant heat kernel transform $H_t : L^2(X) \rightarrow \mathcal{O}(\Xi)$

$$H_t(f)(z) = \int_X \tilde{k}_t(g^{-1}z) f(gK) d\mu_X(gK).$$

The purpose of this note is to give an explicit determination of the image of the heat kernel transform for the noncompact Riemannian symmetric spaces. We show that, unlike the previously known cases, the image can not be a weighted Bergman space, i.e., there is no measure μ_t on Ξ such that the norm on $\text{im} H_t$ is given by $\int_{\Xi} |F(z)|^2 d\mu_t(z)$.

Some of the techniques we use were developed in [11] and [6]. The main new tool, however, is the G -orbital integral

$$\mathcal{O}_h(iY) = \int_G h(g \exp(\frac{i}{2}Y) \cdot x_o) dg,$$

h a function on Ξ suitably decreasing at the boundary and $Y \in 2\Omega$. In Theorem 3.3 we show that there exists a pseudo-differential shift operator D such that for $f \in L^2(X)$ the function $D\mathcal{O}_{|H_t f|^2}$, initially defined on 2Ω , has a natural holomorphic extension to $\mathfrak{a}_{\mathbb{C}}$, and that there is an equality of norms

$$\|f\|^2 = \int_{\mathfrak{a}} (D\mathcal{O}_{|H_t f|^2})(iY) w_t(Y) dY,$$

the weight function w_t being given by

$$w_t(Y) = \frac{1}{|\mathcal{W}|} \cdot \frac{e^{2t|\rho|^2}}{(2\pi t)^{n/2}} \cdot e^{-\frac{|Y|^2}{2t}} \quad (Y \in \mathfrak{a}),$$

\mathcal{W} the usual Weyl group and $n = \text{rank } X$. To describe $\text{im } H_t$, in §2 we introduce the space $\mathcal{G}(\Xi)$ consisting of holomorphic functions F on Ξ satisfying the following two properties:

- $f := F|_X \in L^2(X)$
- $\int_{\mathcal{X}} |\hat{f}(b, \lambda)|^2 \psi_{\lambda}(iY) d\mu(b, \lambda) < \infty$ for all $Y \in \mathfrak{a}$.

Then we show that $\text{im } H_t$ is the space of those functions $F \in \mathcal{G}(\Xi)$ with $\int_{\mathfrak{a}} (D\mathcal{O}_{|F|^2})(iY) w_t(Y) dY < \infty$.

2. RIEMANNIAN SYMMETRIC SPACES

We recall the necessary background on the geometry and analysis on Ξ , the *complex crown* of X .

2.1. The complex crown. Let G be a connected semisimple Lie group. We will assume that G is contained in its universal complexification $G_{\mathbb{C}}$ and that $G_{\mathbb{C}}$ is simply connected. With $K < G$ a maximal compact subgroup we form the homogeneous space $X = G/K$. The complexification $X_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}}$ contains X as a totally real submanifold.

Write \mathfrak{g} (resp. \mathfrak{k}) for the Lie algebras of G (resp. K). We denote the Cartan-Killing form on \mathfrak{g} by $\langle \cdot, \cdot \rangle$. Then $\mathfrak{p} := \mathfrak{k}^{\perp}$ is a complementary subspace on which $\langle \cdot, \cdot \rangle$ defines an inner product and hence a Riemannian symmetric structure on $X = G/K$. Fix a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ and set

$$\Omega = \left\{ Y \in \mathfrak{a} : \text{Spec}(\text{ad}Y) \subset \left] -\frac{\pi}{2}, \frac{\pi}{2} \right[\right\}.$$

Ω is open, convex, and invariant under the Weyl group $\mathcal{W} = N_K(\mathfrak{a})/M$, $M = Z_K(\mathfrak{a})$.

Let $x_o = eK_{\mathbb{C}}$ be the base point in $X_{\mathbb{C}}$. The *complex crown* of X is defined by

$$\Xi = G \exp(i\Omega) \cdot x_o.$$

From several sources one knows that Ξ is a G -invariant Stein domain in $X_{\mathbb{C}}$ which contains X in its “middle” as a totally real submanifold. The definition of Ξ is independent of the choice of \mathfrak{a} (all maximal abelian subspaces in \mathfrak{p} are conjugate under K), hence Ξ is canonically defined by X .

Denote by $\Sigma \subset \mathfrak{a}^*$ the set of restricted roots. Then

$$\Omega = \left\{ Y \in \mathfrak{a} : (\forall \alpha \in \Sigma) |\alpha(Y)| < \frac{\pi}{2} \right\}.$$

In particular $\overline{\Omega}$ is a compact polyhedron.

To each $\alpha \in \Sigma$ we associate the root space $\mathfrak{g}^\alpha = \{Y \in \mathfrak{g} : (\forall H \in \mathfrak{a}) [H, Y] = \alpha(H)Y\}$. If \mathfrak{m} denotes the Lie algebra of M , one has the root space decomposition

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}^\alpha.$$

Fix a positive system $\Sigma^+ \subset \Sigma$ and set $\mathfrak{n} := \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}^\alpha$. If $A := \exp(\mathfrak{a})$ and $N := \exp(\mathfrak{n})$ are the (closed) analytic subgroups corresponding to \mathfrak{a} and \mathfrak{n} , and $A_{\mathbb{C}} := \exp(\mathfrak{a}_{\mathbb{C}})$ and $N_{\mathbb{C}} := \exp(\mathfrak{n}_{\mathbb{C}})$ are the corresponding complex groups, then one has the inclusion

$$\Xi \subset N_{\mathbb{C}} A_{\mathbb{C}} \cdot x_o$$

(cf. [11], [8], [14]). As Ξ is contractible one can define an N -invariant holomorphic map

$$a : \Xi \rightarrow A_{\mathbb{C}}$$

with $a(x_o) = \mathbf{1}$ and having $z \in N_{\mathbb{C}} a(z) \cdot x_o$ for all $z \in \Xi$. The image of $\text{Im} \log \circ a$ is described by the complex convexity theorem (cf. [5], [10]):

$$(2.1) \quad \text{Im} \log a(G \exp(iY) \cdot x_o) = \text{conv}(\mathcal{W} \cdot Y).$$

2.2. Spherical functions. Spherical functions on X have the property that they have a holomorphic extension to Ξ (cf. [11]). Moreover, Ξ can be characterized as the G -domain of holomorphy of the holomorphically extended spherical functions (cf. [12]). Thus, it seems reasonable to define spherical functions ab initio as holomorphic functions on Ξ . Detailed proofs of the material below can be found in [11].

Set $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} (\dim \mathfrak{g}^\alpha) \alpha \in \mathfrak{a}^*$. For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ we set $a(\cdot)^\lambda := e^{\lambda \log a(\cdot)}$. Notice that a^λ is a holomorphic function on Ξ .

The *spherical function with parameter* $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ is the holomorphic function on Ξ defined by

$$\varphi_\lambda(z) = \int_K a(kz)^{\rho+\lambda} dk \quad (z \in \Xi),$$

dk , as usual, a normalized Haar measure on the compact group K .

Later we shall restrict our attention to spherical functions with imaginary parameters, i.e. $\lambda \in i\mathfrak{a}^*$, as only these appear in the Plancherel decomposition of $L^2(X)$.

While Ξ is canonical as a G -domain in $X_{\mathbb{C}}$ for spherical functions, the situation is slightly different if one considers instead the action of $K_{\mathbb{C}}$ on $X_{\mathbb{C}}$. Write $\hat{X}_{\mathbb{C}, 2\Omega}$ for the smallest $K_{\mathbb{C}}$ -invariant domain in $X_{\mathbb{C}}$ which contains $A \exp(i2\Omega) \cdot x_o$ (for existence see [9]). Then a spherical function φ_λ can equally be considered as a $K_{\mathbb{C}}$ -invariant holomorphic function on $\hat{X}_{\mathbb{C}, 2\Omega}$. To avoid unnecessary notation, we write φ_λ also when viewed as a function on $\hat{X}_{\mathbb{C}, 2\Omega}$. For $u \in A$ and $Y \in \Omega$ we then have

$$(2.2) \quad |\varphi_\lambda(u \exp(i2Y) \cdot x_o)| \leq \varphi_\lambda(\exp(i2Y) \cdot x_o) = \int_K |a(k \exp(iY) \cdot x_o)^{\rho+\lambda}|^2 dk.$$

For $\lambda \in i\mathfrak{a}^*$ we set

$$(2.3) \quad \psi_\lambda(Z) = \sum_{w \in \mathcal{W}} e^{\lambda(wZ)} \quad (Z \in \mathfrak{a}_{\mathbb{C}}).$$

Notice that ψ_λ is a \mathcal{W} -invariant function that is positive on $i\mathfrak{a}$ and of exponential growth there.

Lemma 2.1. *For each $Y \in \Omega$ there exists a constant $C_Y > 0$ such that*

$$(2.4) \quad \varphi_\lambda(\exp(i2Y) \cdot x_o) \leq C_Y \cdot \psi_\lambda(i2Y)$$

for all $\lambda \in i\mathfrak{a}^*$.

Proof. This follows from equations (2.1) and (2.2). \square

Remark 2.2. C_Y is locally bounded on Ω .

2.3. Harmonic analysis on Ξ . We conclude this section with some aspects of harmonic analysis on Ξ . We begin with the Fourier transform and Plancherel theorem for X following Helgason and Harish-Chandra.

Let $B = M \backslash K$ and db a normalized Haar measure on B . Set $\mathcal{X} = B \times i\mathfrak{a}^*$ and define a measure μ on \mathcal{X} by $d\mu(b, \lambda) = db \otimes \frac{d\lambda}{|\mathfrak{c}(\lambda)|^2}$ where $d\lambda$ is the measure on $i\mathfrak{a}^*$ as normalized by Harish-Chandra, $\mathfrak{c}(\lambda)$ is the Harish-Chandra \mathfrak{c} -function. For an integrable function f on X we define its Fourier transform to be the function \hat{f} on \mathcal{X} defined by

$$\hat{f}(b, \lambda) = \int_{\mathcal{X}} f(x) a(bx)^{\rho-\lambda} dx, \quad (b, \lambda) \in \mathcal{X},$$

where dx is a G -invariant measure on X normalized so that $f \mapsto \hat{f}$ extends to an isometry $L^2(X) \rightarrow L^2(\mathcal{X}, \mu)$. The Fourier inversion theorem in this formulation becomes

$$f(x) = \int_{\mathcal{X}} \hat{f}(b, \lambda) a(bx)^{\rho+\lambda} d\mu(b, \lambda).$$

Let $\mathcal{G}(\Xi)$ be the space of holomorphic functions F on Ξ such that the following two properties hold:

- $f := F|_X \in L^2(X)$
- $\int_{\mathcal{X}} |\hat{f}(b, \lambda)|^2 \psi_\lambda(iY) d\mu(b, \lambda) < \infty$ for all $Y \in \mathfrak{a}$.

Lemma 2.3 (Gutzmer's identity d'après Faraut). *Let $F \in \mathcal{G}(\Xi)$. Then*

$$(2.5) \quad \int_G |F(g \exp(iY) \cdot x_o)|^2 dg = \int_{\mathcal{X}} |\hat{f}(b, \lambda)|^2 \varphi_\lambda(\exp(i2Y) \cdot x_o) d\mu(b, \lambda)$$

for all $Y \in \Omega$.

Proof. This is the Gutzmer identity of [3], Th. 1. \square

For a sufficiently decreasing function h on Ξ we consider its G -orbital integral

$$\mathcal{O}_h(iY) = \int_G h(g \exp(\frac{i}{2}Y) \cdot x_o) dg \quad (Y \in 2\Omega).$$

In particular, if $F \in \mathcal{G}(\Xi)$ one has from (2.5) that $\mathcal{O}_{|F|^2}(iY)$ is finite for all $Y \in 2\Omega$. We consider in $\mathfrak{a}_{\mathbb{C}}$ the abelian tube domain $\mathcal{T}(2\Omega) = \mathfrak{a} + i2\Omega$. Then for $F \in \mathcal{G}(\Xi)$ it follows from Lemma 2.3 and the holomorphic extension of spherical functions [11] that $\mathcal{O}_{|F|^2}$ admits a natural extension to a holomorphic function on $\mathcal{T}(2\Omega)$, namely

$$(2.6) \quad \mathcal{O}_{|F|^2}(Z) = \int_{\mathcal{X}} |\hat{f}(b, \lambda)|^2 \varphi_\lambda(\exp(Z) \cdot x_o) d\mu(b, \lambda) \quad (Z \in \mathcal{T}(2\Omega)).$$

3. THE HEAT KERNEL TRANSFORM

In this section we shall determine the image of the heat kernel transform on $L^2(X)$ and its holomorphic extension to Ξ . We start with a brief review of the results from [12].

3.1. Definition and basic properties. In the following t denotes a positive number. According to Gangolli [4] the heat kernel k_t , suitably normalized, on X has the spectral resolution

$$(3.1) \quad k_t(x) = \int_{i\mathfrak{a}^*} e^{-t(|\lambda|^2 + |\rho|^2)} \varphi_\lambda(x) \frac{d\lambda}{|\mathbf{c}(\lambda)|^2} \quad (x \in X).$$

If f is an analytic function on X which admits a holomorphic extension to Ξ , then we write \widetilde{f} for the holomorphically extended function. It is proved in [12] (but also immediate from (3.1) and the results collected in Subsection 2.2) that k_t admits an analytic continuation \widetilde{k}_t given by

$$\widetilde{k}_t(z) = \int_{i\mathfrak{a}^*} e^{-t(|\lambda|^2 + |\rho|^2)} \varphi_\lambda(z) \frac{d\lambda}{|\mathbf{c}(\lambda)|^2} \quad (z \in \Xi).$$

Our concern here is the heat kernel transform, viz. the operator given by convolution of k_t on $L^2(X)$:

$$(k_t * f)(x) = \int_X k_t(g^{-1}x) f(gK) d(gK) \quad (x \in X).$$

It was proved in [12] that $k_t * f$ admits a holomorphic extension given by

$$\widetilde{k_t * f}(z) = \int_X \widetilde{k}_t(g^{-1}z) f(gK) d(gK) \quad (z \in \Xi).$$

Thus one gets a linear map

$$H_t : L^2(X) \rightarrow \mathcal{O}(\Xi), \quad f \mapsto \widetilde{k_t * f}$$

which is referred to as the *heat kernel transform with parameter $t > 0$* . In what follows we consider $\mathcal{O}(\Xi)$ as a topological vector space endowed with the usual Fréchet topology of compact convergence.

In [12] the following properties of the heat kernel transform are proved:

- H_t is continuous;
- H_t is G -equivariant;
- H_t is injective.

Thus $\text{im } H_t$, when endowed with the Hilbert topology from $L^2(X)$, becomes a G -invariant Hilbert space of holomorphic functions on Ξ . As such it admits a reproducing kernel which can be spectrally characterized (cf. [12]).

Remark 3.1. The image of H_t was not described in [12]. Contrary to expectations, $\text{im } H_t$ is not a weighted Bergman space, i.e. there does *not* exist a measure μ_t on Ξ , absolutely continuous with respect to the standard Lebesgue measure, such that the norm on $\text{im } H_t$ is given by

$$\int_{\Xi} |F(z)|^2 d\mu_t(z) \quad (F \in \text{im } H_t).$$

This can be seen can verify this in the following way. As remarked earlier, the crown Ξ can be realized as a G -subdomain of the tangent bundle TX . However, the fibers of the natural fibration $\Xi \rightarrow X$ are not complete. In this way Ξ naturally resembles a strip domain in the complex plane, i.e. a domain of the type $S = \{z \in \mathbb{C} : |\text{Im } z| < \frac{\pi}{2}\}$. For strip domains S the non-existence of a measure μ_t becomes a simple exercise using the growth of Gaussians and the classical Paley-Wiener theorem.

3.2. The image of the heat kernel transform. Let $f \in L^2(X)$ and set $F = H_t(f)$. In [12] it was shown that $F \in \mathcal{O}(\Xi)$. As ψ_λ is only of exponential growth it is easily seen that $F \in \mathcal{G}(\Xi)$. In particular, we can form the orbital integral $\mathcal{O}_{|F|^2}$ (cf. (2.6)) and for all $Z \in \mathcal{T}(2\Omega)$ one has from Lemma 2.3

$$(3.2) \quad \mathcal{O}_{|F|^2}(Z) = \int_{\mathcal{X}} |\hat{F}(b, \lambda)|^2 \varphi_\lambda(\exp(Z) \cdot x_o) d\mu(b, \lambda)$$

$$(3.3) \quad = \int_{\mathcal{X}} |\hat{f}(b, \lambda)|^2 e^{-2t(|\lambda|^2 + |\rho|^2)} \varphi_\lambda(\exp(Z) \cdot x_o) d\mu(b, \lambda)$$

$$(3.4) \quad = \int_{i\mathfrak{a}^*} g(\lambda) \varphi_\lambda(\exp(Z) \cdot x_o) \frac{d\lambda}{|\mathfrak{c}(\lambda)|^2},$$

where the function g is given by

$$(3.5) \quad g(\lambda) = e^{-2t(|\lambda|^2 + |\rho|^2)} \int_B |\hat{f}(b, \lambda)|^2 db.$$

Next we define $\mathcal{F}(\mathcal{T}(\Omega))$ as the space of holomorphic functions h on $\mathcal{T}(2\Omega)$ such that

$$h(Z) = \int_{i\mathfrak{a}^*} g(\lambda) \varphi_\lambda(\exp(Z) \cdot x_o) \frac{d\lambda}{|\mathfrak{c}(\lambda)|^2} \quad (Z \in \mathcal{T}(2\Omega))$$

for an integrable function g on $i\mathfrak{a}^*$ with $\int_{i\mathfrak{a}^*} |g(\lambda)| \psi_\lambda(iY) \frac{d\lambda}{|\mathfrak{c}(\lambda)|^2} < \infty$ for all $Y \in \mathfrak{a}$. Notice that g is uniquely determined by h . Thus we can define a linear operator

$$D : \mathcal{F}(\mathcal{T}(\Omega)) \rightarrow \mathcal{O}(\mathfrak{a}_{\mathbb{C}}), \quad (Dh)(Z) = \int_{i\mathfrak{a}^*} g(\lambda) \psi_\lambda(Z) \frac{d\lambda}{|\mathfrak{c}(\lambda)|^2},$$

where ψ_λ is defined in 2.3. Observe that for $f \in L^2(X)$ and $F = H_t(f)$ we have $\mathcal{O}_{|F|^2} \in \mathcal{F}(\mathcal{T}(2\Omega))$ by (3.2) and (3.5), and so $D\mathcal{O}_{|F|^2} \in \mathcal{O}(\mathfrak{a}_{\mathbb{C}})$.

Remark 3.2. The operator D is a pseudo-differential shift operator and a differential operator if the multiplicities of all the restricted roots are even. One can consider D as an appropriate extension of the Abel transform. In fact let $h \in \mathcal{F}(\mathcal{T}(2\Omega))$. Then $h \circ \exp|_{\mathfrak{a}}$ is a \mathcal{W} -invariant smooth function on A and as such has a natural extension to a K -invariant function on X , say $H \in C^\infty(X)^K$. Moreover, it follows from the growth restriction on g that H belongs to the Harish-Chandra Schwartz space on X . Hence the Abel transform

$$(3.6) \quad (\mathcal{A}H)(u) = u^\rho \int_N H(un) dn \quad (u \in A)$$

of H is well defined. Notice that $\mathcal{A}H$ is a \mathcal{W} -invariant smooth function on A . Now with $g(\lambda)$ as above one has

$$(\mathcal{A}H)(u) = \int_{i\mathfrak{a}^*} g(\lambda) \psi_\lambda(\log u) \frac{d\lambda}{|\mathfrak{c}(\lambda)|^2}$$

(cf. [15]) which explains that D is an appropriate extension of \mathcal{A} . We remark that equation (3.6) cannot be extended to all of $u \in \exp(\mathcal{T}(2\Omega))$.

Finally, with $n = \dim \mathfrak{a}$ we define a weight function on \mathfrak{a} by

$$w_t(Y) = \frac{1}{|\mathcal{W}|} \cdot \frac{e^{2t|\rho|^2}}{(2\pi t)^{\frac{n}{2}}} \cdot e^{-\frac{|Y|^2}{2t}} \quad (Y \in \mathfrak{a}).$$

The main result of the paper is now as follows.

Theorem 3.3. For $f \in L^2(X)$ and $F = H_t(f)$ the following norm identity holds

$$(3.7) \quad \|f\|^2 = \int_{\mathfrak{a}} (D\mathcal{O}_{|F|^2})(iY) w_t(Y) dY.$$

Moreover, one has

$$(3.8) \quad \text{im } H_t = \{F \in \mathcal{G}(\Xi) : \int_{\mathfrak{a}} (D\mathcal{O}_{|F|^2})(iY) w_t(Y) dY < \infty\}.$$

Proof. As observed earlier, $F = H_t(f)$ is in $\mathcal{G}(\Xi)$. So consider the norm identity (3.7). By the Plancherel theorem the left hand side of (3.7) is given by

$$(3.9) \quad \|f\|^2 = \int_{\mathcal{X}} |\hat{f}(b, \lambda)|^2 d\mu(b, \lambda)$$

while, according to (3.2), the right hand side of (3.7) is given by

$$(3.10) \quad \int_{\mathfrak{a}} (D\mathcal{O}_{|F|^2})(iY) w_t(Y) dY = \int_{\mathfrak{a}} \int_{\mathcal{X}} |\hat{f}(b, \lambda)|^2 e^{-2t(|\lambda|^2 + |\rho|^2)} \psi_{\lambda}(i2Y) w_t(Y) d\mu(b, \lambda) dY.$$

Comparing (3.9) and (3.10), we see that (3.7) will be established provided we can show for each $\lambda \in i\mathfrak{a}^*$ that

$$\int_{\mathfrak{a}} \psi_{\lambda}(i2Y) w_t(Y) dY = e^{2t(|\lambda|^2 + |\rho|^2)}.$$

But, using the definition of w_t , we have

$$\begin{aligned} \int_{\mathfrak{a}} \psi_{\lambda}(i2Y) w_t(Y) dY &= \frac{1}{|\mathcal{W}|} \cdot \frac{e^{2t|\rho|^2}}{(2\pi t)^{\frac{n}{2}}} \int_{\mathfrak{a}} \psi_{\lambda}(i2Y) e^{-\frac{|Y|^2}{2t}} dY \\ &= \frac{e^{2t|\rho|^2}}{(2\pi t)^{\frac{n}{2}}} \int_{\mathfrak{a}} e^{\lambda(i2Y)} e^{-\frac{|Y|^2}{2t}} dY \\ &= \frac{e^{2t|\rho|^2}}{(2\pi)^{\frac{n}{2}}} \int_{\mathfrak{a}} e^{\lambda(i2\sqrt{t}Y)} e^{-\frac{|Y|^2}{2}} dY \\ &= e^{2t|\rho|^2} e^{\frac{12\sqrt{t}\lambda|^2}{2}} = e^{2t(|\lambda|^2 + |\rho|^2)}, \end{aligned}$$

establishing (3.7) and also proving that H_t is an injective isometry into the space described in (3.8).

For surjectivity, take an $F \in \mathcal{G}(\Xi)$. Then $f = F|_X \in L^2(X)$. We shall use the Paley-Wiener space from [12]. The main properties of $PW(G/K)$ needed here are that it is dense in $L^2(X)$ and that it consists of functions with compactly supported Fourier transforms. We will assume that f is in $PW(G/K)$. Multiply its Fourier transform by $e^{t(|\lambda|^2 + |\rho|^2)}$, then invert the result. We obtain a function g still in $PW(G/K)$. To g we apply the heat transform H_t . Then $H_t(g)$ is in $\mathcal{G}(\Xi)$ and its restriction to X has the same Fourier transform as f . Since $H_t(g)$ and F are in $\mathcal{G}(\Xi)$ and their restrictions to X have the same Fourier transform, we have $H_t(g) = F$. To conclude the argument we use that $PW(G/K)$ is dense and the identity of norms in (3.7). Finally, the characterization (3.8) follows from the observation that only equalities have been used in the above proof. \square

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OREGON, EUGENE OR 97401-1222
E-mail address: kroetz@math.uoregon.edu

DEPARTMENT OF MATHEMATICS, LOUISIANA STATE UNIVERSITY, BATON ROUGE, LA 70803
E-mail address: olafsson@math.lsu.edu

DEPARTMENT OF MATHEMATICS, OHIO STATE UNIVERSITY, 231 WEST 18TH AVENUE, COLUMBUS OH 43210-1174
E-mail address: stanton@math.ohio-state.edu