GENERATING FUNCTIONS ASSOCIATED TO HIGHEST WEIGHT REPRESENTATIONS

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Abstract. A bounded linear map Θ defined on an L^2-space with values in a reproducing kernel Hilbert space is necessarily given as an integral operator with kernel K^Θ. We discuss in general how the adjoint of K^Θ is the generating function associated to a basis of the domain space. Our primary applications are the highest weight representations for a Hermitian symmetric group G modelled by the geometric realization. We obtain new formulas relating the generating function to the action of an Abelian subalgebra $\mathfrak{p}_- \subset \mathfrak{g}_\mathbb{C}$, where $\mathfrak{g}_\mathbb{C}$ is the complexification on the Lie algebra of G.

Introduction

A generating function associated to a sequence $\{a_n\}$ is an analytic function $a(z)$ such that

$$a(z) = \sum_{i=0}^{\infty} a_n z^n$$

has a positive radius of convergence. When each a_n is a function on some set X one considers generating functions of the form

$$a(z,x) = \sum_{i=0}^{\infty} a_n(x) z^n.$$

Typically, the main problem is finding a closed explicit expression for $a(z,x)$. A case in point is the generating function for the Laguerre functions. The n^{th} Laguerre polynomial L_n^α is defined by

$$L_n^\alpha(x) = \frac{e^x x^{-\alpha}}{n!}\frac{d^n}{dx^n}(e^{-x}x^{n+\alpha})$$

and the n^{th} Laguerre function ℓ_n^α is defined by $\ell_n^\alpha(x) = L_n^\alpha(2x)e^{-x}$. In [5] we verified, in the context of a highest weight representation

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of $SL(2,\mathbb{R})$, that the generating function associated to $\{\ell_n^\alpha, n\in\mathbb{N}\}$ is given by the formula

$$(1-z)^{-(\alpha+1)}\exp\left(-\frac{1+z}{1-z}x\right)=\sum_{i=0}^{\infty}\ell_n^\alpha(x)z^n,\quad |z|<1 \tag{0.1}$$

from which one easily deduces the equivalent formula for the generating function associated to the Laguerre polynomials:

$$(1-z)^{-(\alpha+1)}\exp\left(\frac{xz}{z-1}\right)=\sum_{n=0}^{\infty}L_n^\alpha(x)z^n,\quad |z|<1,\ \alpha>-1. \tag{0.2}$$

Equations (0.2) is, of course, classical and its proof is straightforward in that it is possible to compute that the n^{th} term in the Maclaurin series of the generating function is the n^{th} Laguerre polynomial. Such a proof is offered in [5], among others. A more elegant approach that lends itself to a great deal of generalization, in particular to higher dimensional Laguerre functions [6, 8], is to realize that the generating function is the kernel of a integral transform defined on an L^2-space with values in a reproducing kernel Hilbert space. More specifically, let $L^2(\mathbb{R}^+, d\mu_\alpha)$ be the space of square integrable functions on $\mathbb{R}^+$ with respect to the measure $d\mu_\alpha = x^\alpha dx$, where dx is Lebesgue measure. The Laplace transform, $\mathcal{L}$, of a function in $L^2(\mathbb{R}^+, d\mu_\alpha)$ produces a holomorphic function on the upper half plane and the inverse of the Cayley transform, $\mathcal{C}$, carries the holomorphic functions on the upper half plane to holomorphic functions on the unit disk. The composition $\Theta = \mathcal{C}^{-1}\mathcal{L}$, which we will call the Cayley-Laplace transform, is a unitary map of $L^2(\mathbb{R}^+, d\mu_\alpha)$ with values in a reproducing kernel Hilbert space, $\mathcal{H}_\alpha(\mathcal{D})$. Necessarily, such a transform is given by a kernel (c.f. Proposition 2.1) and that kernel is precisely the generating function given in the left side of Equation (0.1). (In the general vector valued case one needs to consider the adjoint of the kernel. In the scalar case, such as the present example, this reduces to simple complex conjugation.) One can easily check that $\{\ell_n^\alpha, n\in\mathbb{N}\}$ is an orthogonal basis of $L^2(\mathbb{R}^+, d\mu_\alpha)$. If we define $\left\{\check{\ell}_n^\alpha\right\}, n\in\mathbb{N}$, to be the dual basis, which, in this case, is obtained by renormalizing each ℓ_n^α, then $\Theta(\check{\ell}_n^\alpha)(z)=z^n$. Theorem 3.1 then gives Equation (0.1). Although Theorem 3.1 is general in scope it follows from rather simple properties of Hilbert space theory.

This paper is organized into two main parts. In sections one through three we discuss the general properties of a bounded linear map Θ defined on a closed subspace $\mathbb{H}$ of $L^2(X, W, d\mu_\nu)$ with values in a reproducing kernel Hilbert space, $\mathcal{H}(\mathcal{S}, V)$. Such operators are necessarily given by a kernel $K(z,x)\in B(W,V)$, where $B(W,V)$ is the space of

V-valued bounded operators on W. The adjoint of this kernel is the generating function associated with a basis of the domain space. (c.f. Theorem 3.1 for the precise statement.) In the second part of the paper we explore the situation where the reproducing kernel Hilbert space, $\mathcal{H}(\mathcal{S}, V)$, supports a unitary highest weight representation of a group G, which is semisimple and Hermitian, and Θ is a unitary map. In this case one can define an equivalent representation of G on $\mathbb{H}$. The complexification of the Lie algebra of G decomposes into three subalgebras $\mathfrak{k}$, $\mathfrak{p}_+$, and $\mathfrak{p}_-$, where the action of $\mathfrak{p}_-$ and $\mathfrak{p}_+$ are, respectively, raising and lower operators. By Theorem 6.1, for each $T \in \mathcal{D} \subset \mathfrak{p}_+$ we can show that $K(T,x)^*v = \sum_{i=0}^{\infty} \frac{\bar{T}^n}{n!} \cdot v(x)$, where $\cdot$ indicates the Lie algebraic action on $\mathbb{H}$. This formula thus relates the action of G to the generating function in a precise way. Furthermore, when a basis $\mathcal{B} = \bigcup \mathcal{B}_n$ is chosen compatible with the natural grading on $\mathbb{H} = \oplus \mathbb{H}_n$ we obtain the formula

$$\frac{\overline{T}^n}{n!} \cdot v = \sum_{e_\alpha \in \mathcal{B}_n} (v|\, E_\alpha(T))\ e_\alpha.$$

These relations can be easily seen in our introductory example of the Laguerre functions. A covering group G of $SL(2,\mathbb{R})$ acts by a unitary highest weight representation $\lambda = \lambda_\alpha$ on $L^2(\mathbb{R}^+, d\mu_\alpha)$ and π_α on $\mathcal{H}_\alpha(\mathcal{D})$. Let $\mathfrak{s}l(2,\mathbb{R})$ be the Lie algebra of G and let $\mathfrak{s}l(2,\mathbb{C})$ be its complexification. There is a decomposition

$$\mathfrak{s}l(2,\mathbb{C}) = \mathfrak{p}_+ \oplus \mathfrak{k} \oplus \mathfrak{p}_-,$$

where $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are one dimensional Abelian subalgebras conjugate to one another with respect to $\mathfrak{s}l(2,\mathbb{R})$. Proposition (2.7) and Theorem (3.4) of [5] imply that there is a distinguished element $E \in \mathfrak{p}_+$ with the property that $\overline{\lambda(E)}\ell_n^\alpha = (n+1)\ell_{n+1}^\alpha$. By induction and linearity, we have that

$$\frac{\overline{\lambda(zE)}^n}{n!} \cdot \ell_0^\alpha = \overline{z}^n \ell_n^\alpha.$$

This allows us to reexpress Equation (0.1) in the following way:

$$(1-\overline{z})^{-(\alpha+1)} \exp\left(-\frac{1+\overline{z}}{1-\overline{z}}x\right) = \sum_{i=0}^{\infty} \frac{\overline{\lambda(zE)}^n}{n!} \cdot \ell_0^\alpha, \quad |z| < 1. \tag{0.3}$$

Such a formulation, in general, underscores the need to have explicit formulas for the Lie algebraic action, in particular, $\mathfrak{p}_-$. For $SL(2,\mathbb{R})$ the operator $\overline{\lambda(E)} = \lambda(\overline{E})$ is given by the second order differential operator $i(t\,D^2 - (2t - (\alpha+1))\,D + (t - (\alpha+1)))$, (c.f. [5]). We refer the reader to [3] where such formulas are worked out for $G = U(n,n)$.

The paper concludes with the example of the generalized Laguerre functions. In a future paper we will extend these ideas to evaluate in a natural way the generating functions for Hermite and generalized Hermite functions and other systems of special functions that arise in representation theory.

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1. Reproducing Kernel Hilbert Spaces

We begin by reviewing the essential properties of reproducing kernel Hilbert Spaces. A more detailed account can be found in (c.f. [10]). Let $\mathcal{S}$ be a locally compact Hausdorff space and V a complex Hilbert space with inner product given by $(\cdot|\cdot)_V$. Let $\mathcal{H}(\mathcal{S}, V)$ be a Hilbert space of continuous V-valued functions on $\mathcal{S}$. We say $\mathcal{H}(\mathcal{S}, V)$ is a *reproducing kernel Hilbert Space* if for each $T \in \mathcal{S}$ the linear map $E_T : \mathcal{H}(\mathcal{S}, V) \to V$ given by $E_T(f) = f(T)$ is continuous and has dense range. This assumption implies that the adjoint $Q(\cdot, T) := E_T^*$ is continuous, injective, and has the reproducing property: for each $f \in \mathcal{H}(\mathcal{S}, V)$ and $v \in V$ we have $(f(T)|\, v)_V = (f|\, Q(\cdot, T)v)$. A function $f \in \mathcal{H}(\mathcal{S}, V)$ orthogonal to $Q(\cdot, T)(V)$ satisfies $f(T) = 0$ and this implies that the set of all finite sums

$$f = \sum_i Q(\cdot, T_i)v_i,$$

with $v_i \in V$ and $T_i \in \mathcal{S}$, forms a dense subspace, $\mathcal{H}_\circ(\mathcal{S}, V)$, of $\mathcal{H}(\mathcal{S}, V)$. For such an f we have

$$\|f\|^2 = \sum_{i,j} (Q(T_i, T_j)v_j|\, v_i) \geq 0.$$

We thus have that the function $(T, S) \mapsto Q(T, S) = E_T E_S^*$ is positive definite; i.e.

$$\sum_{i,j} (Q(T_i, T_j)v_j|\, v_i) \geq 0,$$

for each $v_1, \ldots, v_n \in V$ and $T_1, \ldots, T_n \in \mathcal{S}$. Furthermore, the map $(T, S) \mapsto Q(T, S) : \mathcal{D} \times \mathcal{D} \to B(V)$ is continuous in the strong operator topology.

The operator valued kernel, Q, is called *the reproducing kernel* for $\mathcal{H}(\mathcal{S}, V)$. The continuity of E_T implies there is a bound B_T such that $\|f(T)\| \leq B_T \|f\|$, for all $f \in \mathcal{H}(\mathcal{D}, V)$. Thus convergence in $\mathcal{H}(\mathcal{D}, V)$ implies pointwise convergence. Moreover, since Q is continuous it is bounded on compact sets. The uniform boundedness principle then gives that convergence of $\{f_n\}$ in $\mathcal{H}(\mathcal{S}, V)$ implies uniform convergence on compact sets.

2. Bounded Operators on L^2 with values in a Reproducing Kernel Hilbert Space

Let W be a Hilbert Space and $\mathrm{Herm}^+(W)$ the convex cone of nonnegative definite operators on W; i.e. each $E \in \mathrm{Herm}^+(W)$ is a bounded operator and satisfies

$$(Ew|\, w)_W \geq 0,$$

for each $w \in W$. Let X be a measure space and $d\mu_\nu(x)$ a $\mathrm{Herm}^+(W)$-valued measure on X. If W is finite dimensional then we can write $d\mu_\nu(x) = \nu(x)d\mu$, where ν is a measurable $\mathrm{Herm}^+(W)$-valued function and $d\mu$ is a positive measure on X. We will henceforth assume that W is finite dimensional and the measure $d\mu_\nu(x)$ is written in this way (c.f. [11] for more details). We let $L^2(X, W, d\mu_\nu)$ denote the space of measurable W-valued functions f such that

$$\|f\|^2 = \int_X (\nu(x)f(x)|\, f(x))\, d\mu(x) < \infty.$$

A linear map Θ on $L^2(X, W, d\mu_\nu)$ with values in $\mathcal{H}(\mathcal{S}, V)$ is said to be an integral transform given by a kernel $K = K^\Theta$ if Θ can be written in the form

$$\Theta(f)(T) = \int_X K(T, x)\nu(x)f(x)\, d\mu(x),$$

for all $T \in \mathcal{S}$. We observe that $K(T, x)$ is a map from W to V.

Proposition 2.1. *Suppose $L^2(X, W, d\mu_\nu)$ is given as above and $\mathcal{H}(\mathcal{S}, V)$ is a reproducing kernel Hilbert space with V finite dimensional. Suppose $\Theta : L^2(X, W, d\mu_\nu) \to \mathcal{H}(\mathcal{S}, V)$ is a bounded linear map. Then Θ is an integral transform given by a kernel K. For each $T \in \mathcal{S}$ and $v \in V$ the map $x \mapsto K(T, x)^* v$ is in $L^2(X, W, d\mu_\nu)$.*

Proof. Let $T \in \mathcal{S}$ and $v \in V$. Then

$$\begin{aligned} (\Theta f(T)|\, v) &= (\Theta f|\, Q(\cdot, T)v) \\ &= (f|\, \Theta^*(Q(\cdot, T)v)). \end{aligned}$$

Let $k_{T,v} = \Theta^*(Q(\cdot, T)v)$. Then $k_{T,v} \in L^2(X, W, d\mu_\nu)$, for all $T \in \mathcal{S}$ and $v \in V$. For each $x \in X$ the map $v \mapsto k_{T,v}(x) : V \to W$ is linear and hence continuous as V is finite dimensional. Let $B(V, W)$ be the space of bounded W-valued linear maps on V and let $s_T(x) \in B(V, W)$ be given by $s_T(x)v = k_{T,v}(x)$. Let $K(T,x) \in B(W,V)$ be the adjoint of $s_T(x)$. We then have

$$\begin{aligned}(\Theta f(T)|\, v) &= (\, f|\, k_{T,v}) \\ &= \int_X (\nu(x)f(x)|\, s_T(x)v)\ d\mu(x) \\ &= \int_X (\, K(T,x)\nu(x)f(x)|\, v)\ d\mu(x)\end{aligned}$$

and hence

$$\Theta f(T) = \int_X K(T,x)\nu(x)f(x)\, d\mu(x),$$

for all $f \in L^2(X,W,d\mu_\nu)$ and $T \in \mathcal{S}$. Since $K(T,\cdot)^*v = k_{T,v}$ for all $T \in \mathcal{S}$ and $v \in V$ we have $K(T,\cdot)^*v \in L^2(X,W,d\mu_\nu)$ □

We will call K, given above, the kernel associated with Θ. When necessary we will denote it by K^Θ.

Corollary 2.2. *Suppose E is a bounded linear operator on $L^2(X,W,d\mu_\nu)$ and $\Theta : L^2(X,W,d\mu_\nu) \to \mathcal{H}(\mathcal{D},V)$ a bounded linear map. Then the kernel associated with ΘE is related to the kernel associated with Θ by the following formula:*

$$K^{\Theta E}(T,\cdot)^*v = E^*K^\Theta(T,\cdot)^*v,$$

for all $T \in \mathcal{S}$ and $v \in V$.

Proof. Let $k^\Theta_{T,v}$ be as in the proof of proposition 2.1. Then

$$\begin{aligned}k^{\Theta E}_{T,v} &= (\Theta E)^*(Q(\cdot,T)v) \\ &= E^*\Theta^*(Q(\cdot,T)v) \\ &= E^*k^\Theta_{T,v}.\end{aligned}$$

The corollary now follows from the fact that $K^\Theta(T,x)^*v = k^\Theta_{T,v}(x)$. □

Corollary 2.3. *Suppose $\mathbb{H}$ is a closed subspace of $L^2(X,W,d\mu_\nu)$ and $\Theta : \mathbb{H} \to \mathcal{H}(\mathcal{D},V)$ is a bounded operator. Then Θ is given by a kernel K^Θ, where $K^\Theta(T,\cdot)^*v \in \mathbb{H}$.*

Proof. Let P be orthogonal projection of $L^2(X,W,d\mu_\nu)$ onto $\mathbb{H}$. Then

$$\Theta P : L^2(X,W,d\mu_\nu) \to \mathcal{H}(\mathcal{D},V)$$

is a bounded operator. By proposition 2.1 ΘP is given by a kernel $K^{\Theta P}$. Since $P = P^2 = P^*$ we have by corollary 2.2

$$K^{\Theta P}(T,\cdot)^*v = PK^{\Theta P}(T,\cdot)^*v \in \mathbb{H},$$

for all $T \in \mathcal{S}$ and $v \in V$. Since Θ is the restriction of ΘP to $\mathbb{H}$ we have

$$\Theta(f)(T) = \int_X K^{\Theta P}(T,x)\nu(x)f(x)\,d\mu(x),$$

for all $f \in \mathbb{H}$. In this context we let $K^{\Theta} = K^{\Theta P}$. □

3. The Operator Valued Generating Function

Henceforth we assume V and W are finite dimensional complex inner product spaces.

Theorem 3.1. *Suppose $\mathbb{H} \subset L^2(X, W, d\mu_\nu)$ is a separable Hilbert subspace and $\Theta : \mathbb{H} \to \mathcal{H}(\mathcal{S}, V)$ is a bounded linear map into a reproducing kernel Hilbert space with kernel K^{Θ}. Suppose $\{e_i : i \in I\}$ is a basis of $\mathbb{H}$ that has a dual basis $\{\check{e}_i : i \in I\}$. Set $E_i = \Theta\check{e}_i$. Then*

$$K^{\Theta}(T,\cdot)^* v = \sum_{i\in I} (v|E_i(T))\, e_i,$$

where convergence is with respect to the L^2-norm as given in the previous section.

Proof. Let $v \in V$. For each $T \in \mathcal{S}$ we have, by corollary 2.3, $K^{\Theta}(T,\cdot)^* v \in \mathbb{H}$ and

$$\begin{aligned} K^{\Theta}(T,\cdot)^* v &= \sum_{i\in I} (K^{\Theta}(T,\cdot)^* v|\check{e}_i)\, e_i \\ &= \sum_{i\in I} (v|\Theta\check{e}_i(T))\, e_i \\ &= \sum_{i\in I} (v|E_i(T))\, e_i. \end{aligned}$$

□

We will call $K^{\Theta}(T,\cdot)^* v$ the *generating function* for Θ.

4. Highest Weight Representations

We now specialize to the situation where $\mathcal{H}(\mathcal{S}, V)$ is the geometric realization of a highest weight representation and Θ is a unitary map (c.f. [2]). By Proposition 2.1, Θ must be given by an integral transform and by Theorem (7.2) of [2] Θ is given in terms of the exponential of a certain Lie algebraic action. It is in the interplay between these two realizations of Θ that we obtain a more detailed formulation of the generating function that incorporates the action of the Lie algebra. In this and the next section we set down the preliminaries. See ([2, 6]) for more details.

Let G be a connected noncompact simple Lie group with finite center. Let K be a maximal compact subgroup. We will assume K has a one-dimensional center. This assumption implies that G/K is a Hermitian symmetric space. Let $T \subset K$ be a compact Cartan subgroup, i.e. a maximal Abelian subgroup in K. Let $\mathfrak{g}_\circ$, $\mathfrak{k}_\circ$, and $\mathfrak{t}_\circ$ be the Lie algebras of G, K, and T, respectively. We will adopt the convention that the removal of the subscript $_\circ$ will denote complexification. Our assumptions imply that there is a decomposition $\mathfrak{g} = \mathfrak{p}_+ \oplus \mathfrak{k} \oplus \mathfrak{p}_-$, where $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are Abelian subalgebras of $\mathfrak{p}$ satisfying

$$\begin{aligned} &[\mathfrak{k}, \mathfrak{p}_\pm] \subset \mathfrak{p}_\pm, \text{ and} \\ &[\mathfrak{p}_+, \mathfrak{p}_-] \subset \mathfrak{k}. \end{aligned} \tag{4.1}$$

Furthermore, if a bar $^-$ denotes conjugation on $\mathfrak{g}$ with respect to $\mathfrak{g}_\circ$, then we have

$$\overline{\mathfrak{p}_\pm} = \mathfrak{p}_\mp. \tag{4.2}$$

The subalgebra $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$. We let Φ be the roots corresponding to the pair $(\mathfrak{g}, \mathfrak{t})$. Let Φ_c and Φ_n denote the set of compact and noncompact roots, respectively. We denote the root space corresponding to $\alpha \in \Phi$ by $\mathfrak{g}_\alpha$. A positive system of roots Φ^+ may be chosen so that if $\Phi_n^+ = \Phi^+ \cap \Phi_n$ and $\Phi_n^- = (-\Phi^+) \cap \Phi_n$ then

$$\begin{aligned} \mathfrak{p}_+ &= \oplus_{\alpha \in \Phi_n^+} \mathfrak{g}_\alpha \text{ and} \\ \mathfrak{p}_- &= \oplus_{\alpha \in \Phi_n^-} \mathfrak{g}_\alpha \end{aligned} \tag{4.3}$$

To realize G/K as a bounded symmetric domain in $\mathfrak{p}_+$ it is convenient to assume that G has a faithful representation: thus $G \subset G_{\mathbb{C}}$. Let $P_\pm = \exp(\mathfrak{p}_\pm)$. Then the mapping $\mathfrak{p}_+ \times K_{\mathbb{C}} \times \mathfrak{p}_- \to G_{\mathbb{C}}$ defined by $(a, k, b) \mapsto \exp(a) k \exp(b)$ is a holomorphic diffeomorphism onto a dense open set $\Omega = P_+ K_{\mathbb{C}} P_-$ of $G_{\mathbb{C}}$. We uniquely write each $x \in \Omega$ as a product

$$x = \pi_+(x) \pi_\circ(x) \pi_-(x), \tag{4.4}$$

where $\pi_\pm(x) \in P_\pm$ and $\pi_\circ(x) \in K_{\mathbb{C}}$. It is well known that $G \subset \Omega$ and that the map $\zeta : \Omega \to \mathfrak{p}_+$ defined by $\zeta(x) = \log(\pi_+(x))$ induces a holomorphic diffeomorphism of G/K onto $\zeta(G)$. The set $\mathcal{D} = \zeta(G)$ is an open bounded symmetric domain in $\mathfrak{p}_+$ which we identify with G/K.

For each $(g, T) \in G_{\mathbb{C}} \times \mathfrak{p}_+$ for which $g \exp(t) \in \Omega$ we set

$$\begin{aligned} g \cdot T &= \log \pi_+(g \exp(T)) \text{ and} \\ j(g, T) &= \pi_\circ(g \exp(T)). \end{aligned} \tag{4.5}$$

The first formula in Equation (4.5) defines, by restriction, the action of G on $\mathcal{D}$ equivalent to the action of G on G/K. The map

$j : G_{\mathbb{C}} \times \mathfrak{p}_+ \to K_{\mathbb{C}}$ is a C^∞ map and is holomorphic in the $\mathfrak{p}_+$ variable. Furthermore, it satisfies the following properties:

$$\begin{aligned} j(k, T) &= k, \\ j(p, T) &= I, \text{ and} \\ j(g_1 g_2, T) &= j(g_1, g_2 \cdot T) j(g_2, T), \end{aligned} \tag{4.6}$$

where $k \in K_{\mathbb{C}}$, $T \in \mathfrak{p}_+$, $p \in P_+$, $g_1, g_2 \in G_{\mathbb{C}}$, and for which $g_1 g_2 \exp(T)$ and $g_2 \exp(T)$ are in Ω.

Let ω be a nontrivial irreducible unitary highest weight representation of G on a Hilbert space $\mathbb{H}$. We will let $d\omega$ denote both the derived representation of ω on $\mathfrak{g}_\circ$ and its complex linear extension to $\mathfrak{g}$. However, when there is no confusion we will usually write $X\phi$ or $X \cdot \phi$ instead of $d\omega(X)\phi$, where ϕ is a C^∞-vector in $\mathbb{H}$ and $X \in \mathcal{U}(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$. Our assumption on ω implies that there is a nonzero vector $v_\circ \in \mathbb{H}$ such that $Xv_\circ = 0$, for all $X \in \oplus_{\alpha \in \Phi^+} g_\alpha$ and $Xv_\circ = \lambda(X) v_\circ$, for all $X \in \mathfrak{t}$. The linear functional λ is called the highest weight and is uniquely determined by ω. The K-span of $v_\circ$ is denoted by V or $\mathbb{H}_0$. Let τ denote the restriction of $\omega|_K$ to V. Then τ is an irreducible representation of K on V with highest weight λ. We let Λ denote the set of highest weights that correspond to irreducible unitary highest weight representation. In the early 1980's Λ was computed by Enright Howe, and Wallach, [7] and Jakobsen [9]. We will sometimes write $\omega = \omega_\lambda$, $\mathbb{H} = \mathbb{H}_\lambda$, etc, if we want to emphasize the dependence on the weight λ.

In $\mathfrak{k}$ there is a central element ξ in which $\text{ad}(\xi)$ acts by $-2I$ on $\mathfrak{p}^+$ and $2I$ on $\mathfrak{p}^-$, where I is the identity operator. We let

$$\mathbb{H}_n = \{F \in \mathbb{H} : \pi_\nu(\xi) F = (\lambda(\xi) - 2n) F\}.$$

Equivalently $\mathbb{H}_n$ is the subspace spanned by all terms of the form $E^r v_\circ$, where $r = \{r_\alpha\}$, $\alpha \in \Phi^+$, is a multi-index and $E^r = \Pi_{\alpha \in \Phi^+} E_\alpha^{r_\alpha}$. By Equations (4.1) and (4.2) it is easy to verify

$$\begin{aligned} X &: \mathbb{H}_n \to \mathbb{H}_{n+1} \text{ for } X \in \mathfrak{p}_-, \\ X &: \mathbb{H}_n \to \mathbb{H}_{n-1} \text{ for } X \in \mathfrak{p}_+, \\ X &: \mathbb{H}_n \to \mathbb{H}_n \text{ for } X \in \mathfrak{k}, \end{aligned} \tag{4.7}$$

for each $n \in \mathbb{N}$. The space $\mathbb{H}_{-1}$ is understood to be the trivial space. From this it follows that $\oplus_{n \geq 0} \mathbb{H}_n$ is a $\mathfrak{g}$-invariant subspace of $\mathbb{H}$ which is dense by irreducibility.

5. The Geometric Realization

For each $\lambda \in \Lambda$ we shall define a concrete realization of $\mathbb{H}_\lambda$, which we will refer to as the *geometric realization.* Let $\tau = \tau_\lambda$ be an irreducible representation of K with highest weight $\lambda \in \Lambda$ on a finite dimensional space $V = V_\lambda$. Define $J = J_\lambda$ by $J(g, T) = \tau(j(g, T)$. Then J has the following properties:

$$\begin{aligned} &J(k, T) = k, \\ &J(p, T) = I, \text{ and} \\ &J(g_1 g_2, T) = J(g_1, g_2 \cdot T) J(g_2, T), \end{aligned} \tag{5.1}$$

where $k \in K_{\mathbb{C}}$, $T \in \mathfrak{p}_+$, $p \in P_+$, $g_1, g_2 \in G$, and for which $g_1 g_2 \exp(T)$ and $g_2 \exp(T)$ are in Ω. Let $\mathcal{H}ol(\mathcal{D}, V)$ be the set of V-valued holomorphic functions on $\mathcal{D}$. The formula

$$\pi_\lambda(g) F(T) = J(g^{-1}, T)^{-1} F(g^{-1} T),$$

where $g \in G$ and $F \in \mathcal{H}ol(\mathcal{D}, V)$, defines a representation of G on $\mathcal{H}ol(\mathcal{D}, V)$. The space of K-finite vectors in $\mathcal{H}ol(\mathcal{D}, V)$ consists of all V-valued polynomials on $\mathcal{D}$, which can be identified with $\mathcal{P}(\mathfrak{p}_+, V)$, the space of all V-valued polynomials on $\mathfrak{p}_+$, since $\mathcal{D}$ is open in $\mathfrak{p}_+$. For each $v \in V$ let $1_v \in \mathcal{P}(\mathfrak{p}_+, V)$ denote the constant function $1_v(T) = v$. We let $L(\lambda)$ be the unique g-irreducible subspace of $\mathcal{P}(\mathfrak{p}_+, V)$ which contains the constant functions. (For some $\lambda \in \Lambda$, the irreducible module, $L(\lambda)$, is just $\mathcal{P}(\mathfrak{p}_+, V)$ and for others is a proper subspace. For the latter case, c.f. [1] for a complete description of $L(\lambda)$ in terms of solutions to partial differential equations.) Our assumption on λ implies that $L(\lambda)$ is unitarizable. There is therefore an inner product, $(\cdot|\cdot)$ on $L(\lambda)$ which makes each $\pi_\lambda(X)$, $X \in \mathfrak{g}_\circ$, skew-Hermitian. Let $\mathcal{H}(\mathcal{D}, \tau)$ be the completion of $L(\lambda)$ with respect to this inner product. It can be identified with a subset of $\mathcal{H}ol(\mathcal{D}, V)$ and is well known to be a reproducing kernel Hilbert space with reproducing kernel $Q(S, T) = J(\exp(-S), T)^{*^{-1}}$. The restriction of π_λ on $\mathcal{H}(\mathcal{D}, \tau_\lambda)$, which we denote in the same way, is an irreducible unitary highest weight representation of G. We will refer to this representation as the *geometric realization.* If $v_\circ \in V$ is the highest weight vector under the action of K then $1_{v_\circ}$ is the highest weight vector under the action of G. We can identify $\mathcal{H}_n(\mathcal{D}, \tau) = (\mathcal{H}(\mathcal{D}, \tau))_n$, as defined in section 4, with the homogeneous polynomials of degree n in $L(\lambda)$.

Let ω be an irreducible unitary highest weight representation of G on a Hilbert space $\mathbb{H}_\lambda$. Let $V = V_\lambda$ be the K-span of the highest weight vector $v_\circ$. For each $T \in \mathfrak{p}_+$ and $v \in V$ we formally define

$$q_T v = \sum_{n=0}^{\infty} \frac{\bar{T}^n}{n!} \cdot v.$$

Theorem 5.1 ([2]). *With notation established above we have:*

(1) *The map $q_T : V \to \mathbb{H}_\lambda$ converges strongly in $\mathbb{H}_\lambda$ if and only if $T \in \mathcal{D}$.*

(2) *The map $\Xi : \mathbb{H}_\lambda \to \mathcal{H}(\mathcal{D}, \tau)$ given by*

$$\Xi(f)(T) = (q_T)^* f,$$

for each $T \in \mathcal{D}$, defines a unitary G intertwining map onto the geometric realization.

6. Generating functions and Highest Weight Representations

We now specialize section 3 to the setting of highest weight representations.

Theorem 6.1. *Suppose $\mathbb{H}_\lambda$ is a closed subspace of $L^2(X, W, d\mu_\nu)$ and supports a unitary highest weight representation of G. Suppose $\Theta : \mathbb{H}_\lambda \to \mathcal{H}(\mathcal{D}, V)$ is a unitary intertwining operator which maps each $v \in (\mathbb{H}_\lambda)_\circ$ to the constant function $1_v \in \mathcal{H}(\mathcal{D}, V)$. Let K^Θ be the kernel of Θ. Suppose $\mathcal{B}_n$ is a basis of $(\mathbb{H}_\lambda)_n$, $n \in \mathbb{N}$ and $\mathcal{B} = \bigcup \mathcal{B}_n$. Then for each $T \in \mathcal{D}$ and $v \in V$, we have*

(1) $q_T v = K^\Theta(T, \cdot)^* v = \sum_{\alpha \in \mathcal{B}} e_\alpha (v | E_\alpha(T))$

(2) $\frac{\bar{T}^n}{n!} \cdot v = \sum_{\alpha \in \mathcal{B}_n} e_\alpha (v | E_\alpha(T))$.

Proof. Unitary intertwining maps between irreducible representations are unique up to unitary scalar. Since $\Theta v = 1_v = \Xi v$, for each $v \in V$, we have $\Theta = \Xi$, where Ξ is the intertwining map given in Theorem 5.1. Set $k_{T,v} = K^\Theta(T, \cdot)^* v$ and let $f \in \mathbb{H}_\lambda$, $T \in \mathcal{D}$, and $v \in V$. Then

$$\begin{aligned} (\Theta f(T) | \, v) &= \int_X (K(T,x)\nu(x) f(x) | \, v) \; d\mu(x) \\ &= \int_X (\nu(x) f(x) | \, k_{T,x}) \; d\mu(x) \\ &= (f | \, k_{T,v}) \, . \end{aligned}$$

On the other hand,

$$(\Xi f(T | \, v) \;=\; (f | \, q_T v) \, ,$$

by Theorem 5.1. It follows then that

$$k_{T,v} = q_T v.$$

Since each space $\mathbb{H}_n$ is finite dimensional and the collection $\{\mathbb{H}_n\}_{n=0}^{\infty}$ are mutually orthogonal the basis $\mathcal{B}_n$ has a dual basis $\mathcal{B}'_n$ whose union over n form the dual basis for $\mathcal{B}$. By theorem 3.1 we have that

$$k_{T,v} = \sum_{\alpha\in\mathcal{B}} e_\alpha \left(v\right| E_\alpha(T)) = \sum_{i=0}^{\infty} \sum_{\alpha\in\mathcal{B}_n} e_\alpha \left(v\right| E_\alpha(T)) \,.$$

This gives (1) and projection of this formula onto $(\mathbb{H}_\lambda)_n$ gives (2). □

7. The generating function for the generalized Laguerre functions

The group representation theoretic aspects for $SL(2,\mathbb{R})$ and the Laguerre functions discussed in the introduction extend to a wider class of groups and a wider class of Laguerre functions. In [3, 4, 5, 6] we studied these extensions. The generalized Laguerre functions were a focus of some aspects of this work. In this section we derive the generating function associated to the generalized Laguerre functions.

The notation and results presented here are from [6]. We refer the reader to this article for more details. Let $\mathcal{D}$ be a Hermitian symmetric space isomorphic to a tube type domain $T(\Omega) = \Omega + iJ$, where Ω is an open symmetric cone in a real Jordan algebra J of rank r and dimension d. We let $L^2_\nu(\Omega) = L^2(\Omega, d\mu_\nu)$ be the space of square integrable functions on Ω with respect to the measure $d\mu_\nu(x) = \Delta^{\nu-\frac{d}{r}}(x)dx$, where Δ is the maximal principal minor on $V = J + iJ$. Define $\mathcal{L}_\nu : L^2_\nu(\Omega) \to \mathcal{O}(T(\Omega))$ by

$$\mathcal{L}_\nu(f)(z) \int_\Omega e^{-(z|x)} f(x)\, d\mu_\nu(x), \quad z \in T(\Omega).$$

Here $\mathcal{O}(T(\Omega)$ denotes the space of holomorphic functions on $T(\Omega)$.

We define the Hilbert space $\mathcal{H}_\nu(\mathcal{D})$ to be the space of holomorphic complex valued functions on $\mathcal{D}$ with norm given by

$$\|F\|^2_\nu = \|F\|^2_{\mathcal{H}_\nu(\mathcal{D})} := d_\nu \int_{\mathcal{D}} |F(z)|^2\, dm_\nu(z) \tag{7.1}$$

where

$$d_\nu = \frac{1}{\pi^d}\frac{\Gamma_\Omega(\nu)}{\Gamma_\Omega(\nu - d/r)},$$

$dm_\nu(z) = \Delta(e - z\bar{z})^{\nu-p}dz$, and $p = \frac{2d}{r}$. If $\nu > 1 + a(r-1)$ then $\mathcal{H}_\nu(\mathcal{D}) \neq 0$. The constant d_ν is chosen so that the constant function $z \mapsto 1$ has norm one. The Cayley transform $\gamma : \mathcal{D} \to T(\Omega)$, given by $\gamma \cdot z = \frac{e+z}{e-z}$ induces an isomorphism $\pi(\gamma^{-1})$ defined on the space of holomorphic function on $T(\Omega)$ to the space of holomorphic functions on $\mathcal{D}$ given by

$$\pi(\gamma^{-1})F(z) = 2^{\frac{r\nu}{2}}\Delta(e-z)^{-\nu}F(\gamma z).$$

Let $C = \frac{1}{\sqrt{\Gamma_\Omega(\nu)}}\pi(\gamma^{-1})$ and let $\Theta = C \circ \mathcal{L}_\nu$ be the Cayley-Laplace transform.

Theorem 7.1 ([6, 8])**.** *The map* $\Theta : L^2_\nu(\Omega) \to \mathcal{H}_\nu(\mathcal{D})$ *is a unitary map given by*

$$\Theta f(T) = \frac{2^{\frac{r\nu}{2}}}{\Gamma_\Omega(\nu)^{\frac{1}{2}}}\Delta(e-T)^{-\nu}\int_\Omega e^{-\frac{e+T}{e-T}\cdot x}f(x)d\mu_\nu(x).$$

Let G be the connected component of the identity of the group of biholomorphic functions on D and K the subgroup that fixes $0 \in \mathcal{D}$. We can define a unitary highest weight representation of G on $\mathcal{H}_\nu(\mathcal{D})$ by the formula

$$\pi_\nu(g)F(z) = J(g^{-1},z)^{\frac{\nu}{p}}F(g^{-1}z),$$

where $z \mapsto J(g,z)$ is the Jacobian of the action of G on $\mathcal{D}$. Let λ denote the unitarily equivalent representation of G on $L^2_\nu(\Omega)$ via the Cayley-Laplace transform. The Cayley transform γ can be identified with an element of $G_{\mathbb{C}}$. The group $G^c = \gamma G\gamma^{-1}$ acts on $T(\Omega)$. Let H be the subgroup of G^c that leaves Ω invariant and $L = H \cap K$. We let $L^2_\nu(\Omega)^L$ (resp. $\mathcal{H}_\nu(\mathcal{D})^L$) be the L-fixed vectors in $L^2_\nu(\Omega)$ (resp. $\mathcal{H}_\nu(\mathcal{D})$). If P_L is orthogonal projection of $L^2_\nu(\Omega)$ onto $L^2_\nu(\Omega)^L$ then we have $P_Lf(x) = \int_L f(lx)\,dl$, where dl is normalized Haar measure.

Let M be the collection of indices $\mathbf{m} = (m_1,\ldots,m_r)$ with $m_1 \geq m_2 \geq \cdots \geq m_r \geq 0$. For each $\mathbf{m} \in M$ we associate an L-invariant generalized power function $\Psi_{\mathbf{m}}$ with the property that $\Psi_{\mathbf{m}} \in (\mathcal{H}_\nu(\mathcal{D}))_n$ if and only if $|\mathbf{m}| = m_1 + \cdots + m_r = n$. The collection $\{\Psi_{\mathbf{m}} : \mathbf{m} \in M\}$ forms an orthogonal basis of $\mathcal{H}_\nu(\mathcal{D})^L$ and the collection $\{\Psi_{\mathbf{m}} : \mathbf{m} \in M \text{ and } |m| = n\}$ is an orthogonal basis of $(L^2_\nu(\Omega))_n$. For each $\mathbf{m} \in M$ we follow [8] and define the generalized Laguerre polynomials by the formula

$$L^\nu_{\mathbf{m}} = (\nu)_{\mathbf{m}}\binom{\mathbf{m}}{\mathbf{n}}\frac{1}{(\nu)_{\mathbf{n}}}\Psi_{\mathbf{n}}(-x),$$

and the generalized Laguerre functions by

$$\ell^\nu_{\mathbf{m}}(x) = e^{-Tr(x)}L^\nu_{\mathbf{m}}(2x).$$

By the unitarity of Θ and Propositions XIII.2.2 and XV.4.2 of [8] we have

Theorem 7.2. *The Laguerre functions form an orthogonal basis of* $L^2_\nu(\Omega)^L$,

$$\|\ell^\nu_{\mathbf{m}}\|^2 = \frac{\Gamma_\Omega(\mathbf{m}+\nu)}{2^{r\nu}} \frac{\left(\frac{d}{r}\right)_{\mathbf{m}}}{d_{\mathbf{m}}},$$

and

$$\Theta(\ell^\nu_{\mathbf{m}}) = \frac{\Gamma_\Omega(\mathbf{m}+\nu)}{\Gamma_\Omega(\nu) 2^{\frac{r\nu}{2}}} \Psi_{\mathbf{m}}.$$

We conclude with the following theorem.

Theorem 7.3. *Let* $T \in \mathcal{D}$ *and* $x \in \Omega$. *Then*

(1) $\Delta(e-T)^{-\nu} \int_L e^{-((1+T)(1-T)^{-1}|lx)}\, dl = \sum_{\mathbf{m}\geq 0} d_{\mathbf{m}} \frac{1}{(\frac{n}{r})_{\mathbf{m}}} \Phi_{\mathbf{m}}(T) \ell^\nu_{\mathbf{m}}(x)$.

(2) $\Delta(e-T)^{-\nu} \int_L e^{-(T(1-T)^{-1}|lx)}\, dl = \sum_{\mathbf{m}\geq 0} d_{\mathbf{m}} \frac{1}{(\frac{n}{r})_{\mathbf{m}}} \Phi_{\mathbf{m}}(T) L^\nu_{\mathbf{m}}(x)$.

(3) $\frac{\lambda(\overline{T})^n}{n!} \ell^\nu_{\mathbf{0}} = P_n \left(\Delta(e-\overline{T})^{-\nu} e^{-(\frac{1+\overline{T}}{1-\overline{T}})\cdot x} \right)$

(4) $P_L \left(\frac{\lambda(\overline{T})^n}{n!} \ell^\nu_{\mathbf{0}} \right) = \sum_{|\mathbf{m}|=n} d_{\mathbf{m}} \frac{1}{(\frac{n}{r})_{\mathbf{m}}} \overline{\Phi_{\mathbf{m}}(T)} \ell^\nu_{\mathbf{m}}(x)$,

where P_n *is orthogonal projection onto* $(L^2_\nu(\Omega))_n$.

Proof. Let $\check{\ell}^\nu_{\mathbf{m}} = \frac{2^{r\nu}}{\Gamma_\Omega(\mathbf{m}+\nu)} \frac{d_{\mathbf{m}}}{\left(\frac{d}{r}\right)_{\mathbf{m}}} \ell^\nu_{\mathbf{m}}$. Then $\left\{\check{\ell}^\nu_{\mathbf{m}}\right\}_{\mathbf{m}\in M}$ is the basis dual to $\{\ell^\nu_{\mathbf{m}}\}_{\mathbf{m}\in M}$ and, by Theorem 7.2, $\Theta\check{\ell}^\nu_{\mathbf{m}} = \frac{2^{\frac{r\nu}{2}}}{\sqrt{\Gamma_\Omega(\nu)}} \frac{d_{\mathbf{m}}}{\left(\frac{d}{r}\right)_{\mathbf{m}}} \Psi_{\mathbf{m}}$. Application of Theorem 3.1 gives

$$\frac{2^{\frac{r\nu}{2}}}{\sqrt{\Gamma_\Omega(\nu)}} \overline{\Delta(e-T)^{-\nu}} \int_L e^{-\frac{e+\overline{T}}{e-\overline{t}}\cdot x} f(x)\, d\mu_\nu(x) = \frac{2^{\frac{r\nu}{2}}}{\sqrt{\Gamma_\Omega(\nu)}} \sum_{\mathbf{m}\in M} \ell^\nu_{\mathbf{m}}(x)\, \overline{\Psi_{\mathbf{m}}(T)}.$$

Conjugation and multiplication by the reciprocal of $\frac{2^{\frac{r\nu}{2}}}{\sqrt{\Gamma_\Omega(\nu)}}$ gives formula (1). Formula (2) follows immediately from (1). (We observe that this formula is not new. See [8], page 347, and the references therein.) Formula (3) follows from Theorem 6.1. Formula (4) follows from (1) and (3), the fact that P_L and P_n commute, and $\ell^\nu_{\mathbf{m}} \in (L^2_\nu(\Omega))_n$ if and only if $|m| = n$. ☐

For a description of the action $\lambda(\overline{T})$ in the case $G = U(n,n)$ c.f. [3].

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