Calculus and Analytic Geometry, 4th Ed, by George B. Thomas, MIT, 1968

10. When $B^2 - 4AC$ is negative, the equation

$$Ax^2 + Bxy + Cy^2 = 1$$

represents an ellipse. If the semiaxes have lengths a and b, the area of the ellipse is πab. Show that the area of the ellipse given above is $2\pi/\sqrt{4AC - B^2}$.

11. Show, by reference to Eq. (6), Article 10.9 that

$$D'^2 + E'^2 = D^2 + E^2$$

for every angle of rotation α.

12. If $C = -A$ in Eq. (1), show that there is a rotation of axes for which $A' = C' = 0$ in the resulting Eq. (3). Find the angle α that makes $A' = C' = 0$ in this case.
Hint. Since $A' + C' = 0$, one need only make the further requirement that $A' = 0$ in Eq. (3).

10.11 SECTIONS OF A CONE

The circle, parabola, ellipse, and hyperbola are known as *conic* sections because each may be obtained by cutting a cone by a plane. If the cutting plane is perpendicular to the axis of the cone, the section is a circle.

In general, suppose the cutting plane makes an angle α with the axis of the cone and let the generating angle of the cone be β (see Fig. 10.51). Then the section is

(i) a circle, if $\alpha = 90°$,
(ii) an ellipse, if $\beta < \alpha < 90°$,
(iii) a parabola, if $\alpha = \beta$,
(iv) a hyperbola, if $0 \le \alpha < \beta$.

The connection between these curves as we have defined them and the sections from a cone is readily made by reference to Fig. 10.52. The figure is drawn to illustrate the case of an ellipse, but the argument holds for the other cases as well.

A sphere is inscribed tangent to the cone along a circle C, and tangent to the cutting plane at a point F. Point P is any point on the conic section. We shall see that F is a focus, and that the line L, in which the cutting plane and the plane of the circle C intersect, is a directrix of the curve. To this end let Q be the point where the line through P parallel to the axis of the cone intersects the plane of C, let A be the point where the line joining P to the vertex of the cone touches C, and let PD be perpendicular to line L at D. Then PA and PF are two lines tangent to the same sphere from a common point P and hence have the same length:

$$PA = PF.$$

Also, from the right triangle PQA, we have

$$PQ = PA \cos\beta;$$

and from the right triangle PQD, we find that

$$PQ = PD \cos\alpha.$$

Hence

$$PA \cos\beta = PD \cos\alpha,$$

or

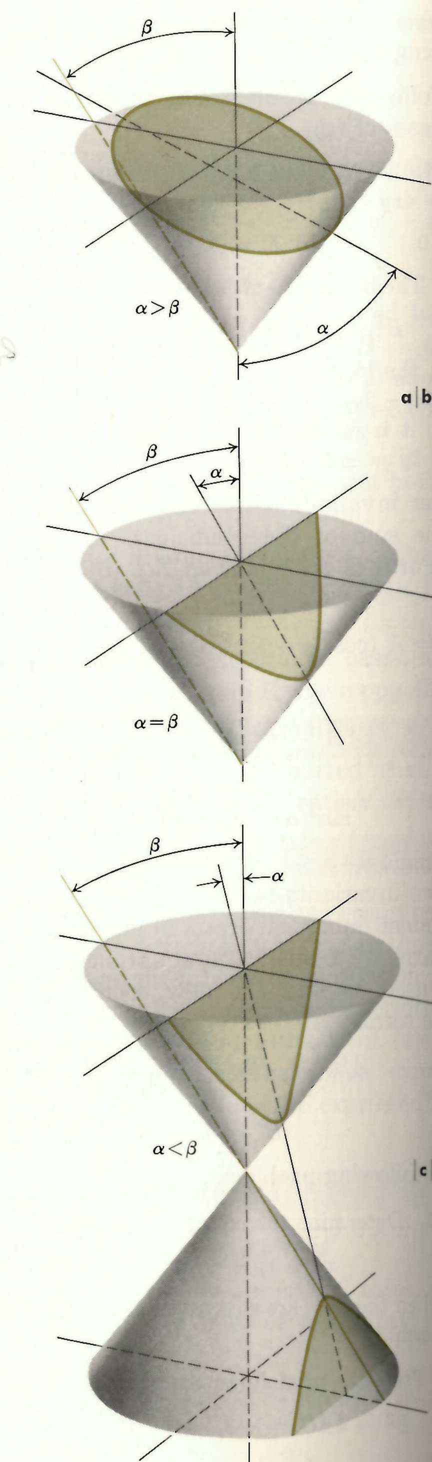
$$\frac{PA}{PD} = \frac{\cos\alpha}{\cos\beta}.$$

But since $PA = PF$, this means that

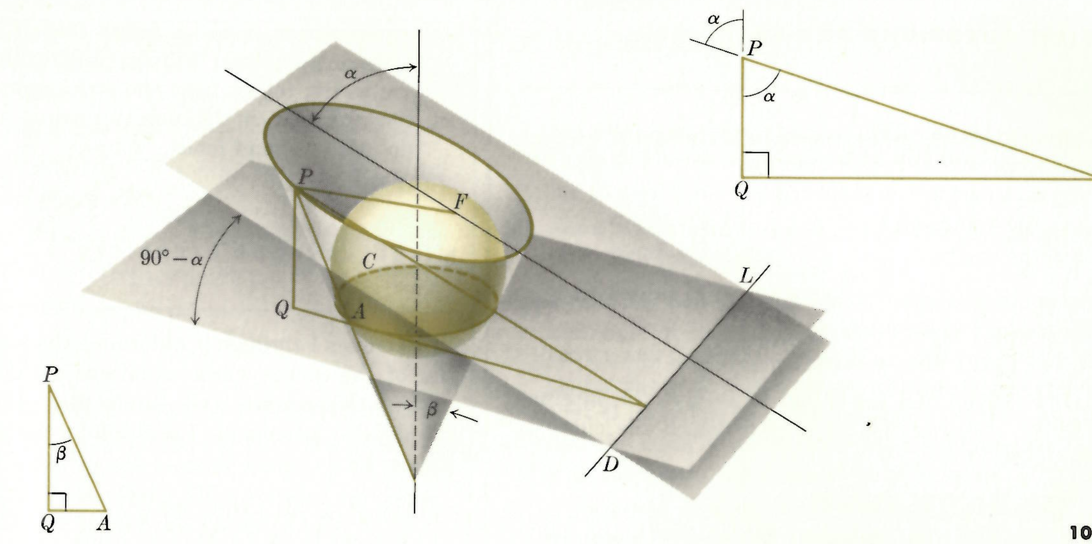
$$\frac{PF}{PD} = \frac{\cos\alpha}{\cos\beta}. \tag{1}$$

Since α and β are constant for a given cone and a given cutting plane, Eq. (1) has the form

$$PF = e \cdot PD.$$



10.51



10.52

This characterizes P as belonging to a parabola, an ellipse, or a hyperbola, with focus at F and directrix L, accordingly as $e = 1$, $e < 1$, or $e > 1$ respectively, where

$$e = \frac{\cos\alpha}{\cos\beta}$$

is thus identified with the eccentricity.

EXERCISES

1. Sketch a figure similar to Fig. 10.52 where the conic section is a parabola, and carry through the argument of Article 10.11 on the basis of such a figure.
2. Sketch a figure similar to Fig. 10.52 where the conic section is a hyperbola, and carry through the argument of Article 10.11 on the basis of such a figure.
3. Which parts of the construction described in Article 10.11 become impossible when the conic section is a circle?
4. Let one directrix be the line $x = -p$ and take the corresponding focus at the origin. Using Eq. (15) of Article 10.7, derive the equation of the general conic section of eccentricity e. If e is neither 0 nor 1, show that the center of the conic section has coordinates

$$\left(\frac{pe^2}{1 - e^2}, 0\right).$$