Print Your Name Here: ____________________

- **Show all work** in the space provided. We can give credit *only* for what you write! *Indicate clearly if you continue on the back side*, and write your name at the top of the scratch sheet if you will turn it in for grading.
- ***Books, notes (electronic or paper), cell phones, smart phones, and internet-connected devices are prohibited!*** A scientific calculator is allowed—but it is not needed. If you use a calculator, you *must still write out all operations performed* on the calculator. Please do ***not*** replace precise answers with decimal approximations.
- There are **five (5)** problems: 20 points each. The *Maximum total score* = 100.

1. (20) Consider the system of linear equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + 4x_2 + 8x_3 &= 2 \\ 3x_1 + 6x_2 + 13x_3 &= 3 \end{aligned}$$

a. (10) Write the *augmented* coefficient matrix A of this system and find the *reduced row echelon form* $\text{rref}(A)$. What is the *rank* of A?

b. (10) Use $\text{rref}(A)$ to find *all* solutions of the given system of equations. Describe fully the *kind of geometrical object* that is the set of all solutions.

2. (20)

a. (10) Evaluate the matrix product $\begin{bmatrix} 3 & 1 \\ -4 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & -2 \end{bmatrix}$.

b. (10) Find the 2×2 matrix of the orthogonal *projection* onto the line L containing the unit vector $\mathbf{u}$ where $\mathbf{u} = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. (Suggestion: What are $\text{proj}_{\mathbf{u}}\mathbf{e}_1$ and $\text{proj}_{\mathbf{u}}\mathbf{e}_2$?)

3. (20) The matrix of the rotation R_θ is $[R_\theta] = \begin{bmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{bmatrix}$

a. (10) Find $\sin\theta$. (Suggestion: What does the first column vector represent?)

b. (10) Find the matrix of R_θ^{-1}. (Suggestion: Use the geometry and check your work by multiplying.)

4. (20) Use the method of row reduction to find the *inverse* of the matrix $A = \begin{bmatrix} 2 & 5 & 2 \\ 0 & 0 & 1 \\ -1 & -2 & 0 \end{bmatrix}$. Be sure to show how you check your result by multiplying.

5. (20) Suppose A and B are $n \times n$ matrices and suppose $E_1, \ldots, E_k$ are $n \times n$ *elementary matrices* such that $E_1 \cdots E_k A = I_n$.

a. What is the product $E_1 \cdots E_k AB$?

b. What is the product $E_1 \cdots E_k I_n$?

Solutions

1.

a. $A = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 8 & 2 \\ 3 & 6 & 13 & 3 \end{array}\right]$ and $\text{rref}(A) = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$. The rank of A is 2.

b. The *free variable* is $x_2 = t$, where t can be any real number. The solution set is given by $x_1 = 1 - 2t;\ x_2 = t;\ x_3 = 0$ for all real numbers t. The solution set is a *straight line* in the plane $x_3 = 0$ in $\mathbb{R}^3$.

2.

a. $\begin{bmatrix} 2 & 7 & -5 \\ -4 & -8 & 4 \\ 3 & 3 & 0 \end{bmatrix}$. Since the row length of the first matrix matches the column length of the second matrix, the product is defined. Since the first matrix has 3 rows and the second has 3 columns, the product must be a 3×3 matrix.

b. $[\text{proj}_{\mathbf{u}}] = \frac{1}{4}\begin{bmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$.

3.

a. $\sin\theta = 0.8$ since the first column vector is $R_\theta(\mathbf{e}_1)$

b. Do it the easy way: $[R_\theta^{-1}] = [R_{-\theta}] = \begin{bmatrix} 0.6 & 0.8 \\ -0.8 & 0.6 \end{bmatrix}$.

4.

a. $A^{-1} = \begin{bmatrix} -2 & 4 & -5 \\ 1 & -2 & 2 \\ 0 & 1 & 0 \end{bmatrix}$.

5.

a. $E_1 \cdots E_k AB = B$

b. $E_1 \cdots E_k I_n = A^{-1}$.

Class Statistics

% Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	8				
80-89 (B)	6				
70-79 (C)	3				
60-69 (D)	4				
0-59 (F)	8				
Test Avg	74.2%	%	%	%	%
Cumulative HW Avg	84%	%	%	%	%