Print Your Name Here: ____________________

- **Show all work** in the space provided. We can give credit *only* for what you write! *Indicate clearly if you continue on the back side*, and write your name at the top of the scratch sheet if you will turn it in for grading.
- ***Books, notes (electronic or paper), cell phones, smart phones, and internet-connected devices are prohibited!*** A scientific calculator is allowed—but it is not needed. If you use a calculator, you *must still write out all operations performed* on the calculator. Please do ***not*** replace precise answers with decimal approximations.
- There are **five (5)** problems: *Maximum total score* = 100.

1. (25) Find vector(s) that *span* the kernel of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$. Find the *nullity* $N(A)$ and the *rank* $R(A)$.

2. (20) Find a *basis* for the image of the matrix $A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 1 & -4 \\ 1 & 4 & 5 \end{bmatrix}$.

3. (25) Find *all* values of k for which the set $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ k \end{bmatrix} \right\}$ is a *basis* for $\mathbb{R}^3$.

4. (20) Let $T(\vec{x}) = A\vec{x}$, where $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. So $A = [T]_{\mathfrak{S}}$, the matrix of T in the standard basis $\mathfrak{S}$. Let $B = \{\vec{v}_1, \vec{v}_2\}$ where $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, a new basis for $\mathbb{R}^2$. Find the matrix $[T]_B$, the matrix of T in the new basis B, using any method of your choice. But do show your work.

5. (10) Suppose $T : \mathbb{R}^3 \to \mathbb{R}^4$ is linear.

a. Find the value of $R(T) + N(T)$.

b. Is it possible that $R(T) = N(T)$? Explain.

Solutions

1. It is necessary to know the difference between the kernel and the image of $A : \mathbb{R}^3 \to \mathbb{R}^2$. To solve the equation $A\vec{x} = \vec{0}$ we reduce A to its $\text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$. Since there is no leading 1 of any row in column 3, we see that x_3 is a free variable, so $x_3 = t$, $x_1 = t$, and $x_2 = -2t$, where t can be any real number. Thus the vector $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ spans the kernel of A, as would any non-zero multiple of this vector. Also, $N(A) = 1$ and $R(A) = 2$ since $R(A) + N(A) = 3$.

2. The columns of A span the image of A. Since $\text{rref}(A) = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ we see that the third column of A is a linear combination of the first 2 columns, and the set $B = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \right\}$ is linearly independent and spans the image of A, making B a basis. Do not confuse the columns of A with the columns of $\text{rref}(A)$.

3. Letting A be the matrix having the given vectors for its 3 columns in the same order as given, we row-reduce A to $\begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & k - \frac{5}{2} \end{bmatrix}$. So the answer is all $k \neq \frac{5}{2}$ since then $\text{rref}(A) = I_3$ making the three columns linearly independent.

4. $[T]_B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. You may find this either by calculating $T(\vec{v}_1) = \vec{v}_2$ and $T(\vec{v}_2) = -\vec{v}_1$, or by finding $S^{-1}AS$, where S is the change of basis matrix. Note that the mapping T is actually a rotation $R_{-\pi/2}$, which makes the new matrix geometrically obvious. Compare with 3.4/19.

5.

- **a**. $R(T) + N(T) = 3$, the dimension of the ***domain***.
- **b**. This is not possible, since $R(T) + N(T) = 3$, which is an odd number. If $R(T) = N(T)$ then $R(T) + N(T) = 2R(T)$ would be even, which is impossible since the domain has dimension 3.

Class Statistics

% Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	8	14			
80-89 (B)	6	4			
70-79 (C)	3	3			
60-69 (D)	5	0			
0-59 (F)	8	5			
Test Avg	74%	84%	%	%	%
Cumulative HW Avg	79.5%	80.1%	%	%	%
HW/Test Correl	–	0.84			

The Correlation Coefficient is the cosine of the angle between two data vectors in $\mathbb{R}^{28}$–one dimension for each student enrolled. Thus this coefficient is between 1 and -1, with coefficients above 0.6 being considered strongly positive. The correlation coefficient shown indicates that the test grades in the course have a strongly positive correlation with performance on the homework.