

Print Your Name Here: \_\_\_\_\_

- **Show all work** in the space provided. We can give credit *only* for what you write! *Indicate clearly if you continue on the back side*, and write your name at the top of the scratch sheet if you will turn it in for grading.
- **Books, notes (electronic or paper), cell phones, smart phones, and internet-connected devices are prohibited!** A scientific calculator is allowed—but it is not needed. If you use a calculator, you *must still write out all operations performed* on the calculator. Please do **not** replace precise answers with decimal approximations.
- There are **four (4)** problems: *Maximum total score = 100.*

1. (20) Define  $T : P_2 \rightarrow P_3$  by  $T(p)(t) = tp(t)$ . (Recall  $P_2 = \{p(t) = at^2 + bt + c \mid a, b, c \in \mathbb{R}\}$ .)

a. (5) Is  $T$  a *linear* transformation? Please give a reason.

b. (10) Find  $\text{image}(T)$ ,  $R(T)$ ,  $N(T)$ , and  $\ker(T)$ .

c. (5) Is  $T$  an *isomorphism* from  $P_2$  to  $P_3$ ? Why or why not?

2. (30) Use the standard basis  $\mathfrak{B} = \left\{ e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$  for  $U^{2 \times 2}$ .

a. (20) Find the matrix  $[T]_{\mathfrak{B}}$  for the linear transformation  $T : U^{2 \times 2} \rightarrow U^{2 \times 2}$  defined by

$$T(M) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} M - M \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}.$$

b. (10) Find a basis  $\mathfrak{A}$  for  $\text{image}(T)$ , and find both  $R(T)$  and  $N(T)$ .

3. (20) Consider the orthonormal set  $\{\mathbf{u}_1 = \frac{1}{\sqrt{3}}(1, -1, 1), \mathbf{u}_2 = \frac{1}{\sqrt{2}}(1, 1, 0)\} \subset \mathbb{R}^3$ . Find *all* unit vectors  $\mathbf{u}_3$  such that  $\mathbf{u}_3$  is orthogonal to *both*  $\mathbf{u}_1$  and  $\mathbf{u}_2$ .

4. (30) Use the Gram-Schmidt Process to *construct* an *orthonormal basis*  $\mathfrak{B}$  for the *image* of the matrix  $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ . (Hint: The *image* is the *span* of the 3 column vectors,  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ .)

## Solutions

1. See 4.2/45, 61.

- a. Yes:  $T(cf + g)(t) = t(cf + g)(t) = tcf(t) + tg(t) = cT(f)(t) + (Tg)(t) = (cT(f) + T(g))(t)$  at each  $t \in \mathbb{R}$ . Note that  $T$  is linear if and only if it *preserves* vector addition and scalar multiplication. *Closure* is a property of a vector space, not a linear transformation.
- b.  $\text{image}(T) = \{at^3 + bt^2 + ct \mid a, b, c \in \mathbb{R}\}$ , so  $R(T) = 3$ ,  $N(T) = 0$ , and  $\ker(T) = \{0\}$ . Try not to confuse  $\{0\}$  with  $\mathbf{0}$ . One is a *set* and the other is a *vector*. Do not confuse  $\{1, t, t^2\}$  with  $\text{Span}\{1, t, t^2\}$ . The former is a set with 3 elements, whereas the latter is the 3-dimensional vector space  $P_2$ . Note that  $3+0=3$  as required by the rank + nullity theorem.
- c.  $T$  is *not* an isomorphism because it is *not* onto  $P_3$ , which is 4-dimensional. Also, the image of  $T$  lacks the non-zero *constant* polynomials and  $R(T) = 3$ , not 4, which is the dimension of  $P_3$ . Note that  $T$  is one-to-one since its kernel is  $\{0\}$ .

2. See 4.3/7. We calculate the coordinate vectors  $[T(e_j)]_{\mathfrak{B}}$ ,  $j = 1, 2, 3$ , to find the 3 columns of the  $3 \times 3$  matrix  $[T]_{\mathfrak{B}} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ . Thus  $\mathfrak{A} = \{e_2\}$ ,  $R(T) = 1$ ,  $N(t) = 3 - R(t) = 2$ . For the basis  $\mathfrak{A}$  one could use any nonzero multiple of  $e_2$ .

3. See 5.1/16. This is *not* a Gram-Schmidt problem. Solve the linear system obtained from setting  $\mathbf{u}_3 \cdot \mathbf{u}_1 = 0$  and  $\mathbf{u}_3 \cdot \mathbf{u}_2 = 0$ . Then use the condition that  $\|\mathbf{u}_3\| = 1$  to find the two choices for  $\mathbf{u}_3$ :  $\mathbf{u}_3 = \pm \frac{1}{\sqrt{6}}(-1, 1, 2)$ . In many cases, it would have been helpful to check that  $\|u_3\| = 1$  and that  $u_3 \cdot u_1 = 0 = u_3 \cdot u_2$  in order to correct errors. Note that  $u_1$  and  $u_2$  determine a plane, and there are two unit vectors perpendicular to a plane in  $\mathbb{R}^3$ .

4. See 5.2/35.  $\mathfrak{B} = \left\{ \mathbf{u}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \mathbf{u}_2 = \frac{1}{\sqrt{30}} \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix} \right\}$ . (Remark: Notice that  $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$ , so that  $\mathbf{v}_3 - \text{proj}_{\text{span}\{\mathbf{u}_1, \mathbf{u}_2\}} \mathbf{v}_3 = \mathbf{0}$  which cannot be a basis vector. That is how Gram-Schmidt eliminates any redundant vectors. All 3 column vectors,  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ , lie in the 2-dimensional subspace that is the plane  $z = 2x + y$ . If you found a different orthonormal basis from the one given above, both basis vectors must lie in that same plane.)

## Class Statistics

% Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	8	14	6		
80-89 (B)	6	4	4		
70-79 (C)	3	3	4		
60-69 (D)	5	0	6		
0-59 (F)	8	5	6		
Test Avg	74%	84%	72.7%	%	%
Cumulative HW Avg	79.5%	80.1%	80.6%	%	%
HW/Test Correl	-	0.84	0.81		

The Correlation Coefficient is the cosine of the angle between two data vectors in  $\mathbb{R}^{27}$ —one dimension for each student enrolled. Thus this coefficient is between 1 and -1, with coefficients above 0.6 being considered strongly positive. The correlation coefficient shown indicates that the test grades in the course have a strongly positive correlation with performance on the homework.