Print Your Name Here: ________________________

Show all work in the space provided and *keep your eyes on your own paper. Indicate clearly* if you continue on the back. Write your **name** at the **top** of the *scratch* sheet *if you will hand it in to be graded.* **No** books, notes, smart/cell phones, I-watches, communication devices, internet devices, or electronic devices are allowed except for a scientific calculator—which is not needed. The maximum total score is 100.

Part I: Short Questions. Answer **8** of the 12 short questions: 6 points each. **Circle** the **numbers** of the 8 questions that you want counted—*no more than 8*! Detailed explanations are not required, but they may help with partial credit and are *risk-free!* Maximum score: 48 points.

1. True or False: The empty set is a closed subset of the real line.

2. Describe an *open cover* $\mathcal{O} = \{O_n \mid n \in \mathbb{N}\}$ of the interval $(-1, 1)$ that has *no finite subcover.*

3. True or False: An open dense subset of $\mathbb{R}$ must be all of $\mathbb{R}$.

4. Give an example of a countable subset $S \subset \mathbb{R} \setminus \mathbb{Q}$ such that S is dense in $\mathbb{R}$.

5. True or False: The set $S = \{(x, y) \mid x, y \in \mathbb{Q}\}$ is uncountable.

6. True or False: The set S of all *infinite sequences* of the integers 0 through 9 is countable.

7. Find the set of all cluster points of the set $\mathbb{Z}$ of all integers.

8. True or False: The function $f(x) = \sin \dfrac{1}{x}$ is uniformly continuous on the interval $[.001, 1]$.

9. Let $f(x) = \begin{cases} 1+x & \text{if } x \in \mathbb{Q}, \\ x^2 & \text{if } x \notin \mathbb{Q}. \end{cases}$ Find all values of x at which f is continuous.

10. Suppose $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$ and suppose f is continuous. If $f(\pi) = 3$, find $f(2)$.

11. Let $f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0. \end{cases}$ True or False: f has the Intermediate Value Property on $[0,1]$.

12. Give an example of an interval I on which $f(x) = x^2$ is *not* uniformly continuous.

Part II: Proofs. *Prove carefully* **2** of the following 3 theorems for 26 points each. **Circle** the *letters* of the 2 proofs to be counted in the list below—*no more than 2!* You may write the proofs below, on the back, or on scratch paper. Maximum total credit: 52 points.

A. Let $E = \left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{N} \right\}$. *Find* an open cover $\mathcal{O} = \{O_n \mid n \in \mathbb{N}\}$ of E that has no finite subcover. *Prove* that $\mathcal{O}$ is an open cover *and* that $\mathcal{O}$ has no finite subcover.

B. Let f be monotone increasing on $\mathbb{R}$, meaning that whenever $x_1 < x_2$, $x_1, x_2 \in \mathbb{R}$, we have $f(x_1) \leq f(x_2)$. Prove: For all $a \in \mathbb{R}$, $\lim_{x \to a+} f(x)$ exists and is a real number. (Hint: Let $S = \{f(x) \mid x > a\}$ and *show* that S is bounded below so that $L = \inf(S) \in \mathbb{R}$. Then *show* that for *every* sequence $x_n \to a+$ we must have $f(x_n) \to L$.)

C. Let $f(x) = \dfrac{1}{x}$, for all $x \in (0, \infty)$. Prove *your choice* of *one* of the following statements.

(**i**) Prove that f is *uniformly* continuous on $(1, \infty)$.

(**ii**) Prove that f is *not* uniformly continuous on $(0, 1)$.

Solutions and Class Statistics

1. True: its complement is $\mathbb{R}$ which is open.

2. For example, let $O_n = \left(-1, 1 - \frac{1}{n}\right)$ for each $n \in \mathbb{N}$.

3. This is false. See Example 1.15 in the text.

4. For example, let $S = \left\{ \frac{p}{q}\sqrt{2} \mid p, q \in \mathbb{Z}, q \neq 0 \right\}$.

5. False: this can be expressed as a countable union of countable sets.

6. False: S can be identified with the set of all decimal expansions of real numbers in $[0, 1]$ for example.

7. $\emptyset$, since every point of $\mathbb{Z}$ is an isolated point and every point of $\mathbb{R} \setminus \mathbb{Z}$ has an open interval around it that excludes $\mathbb{Z}$ entirely.

8. True, because the interval is closed and finite and f is continuous on it.

9. $x = \dfrac{1 \pm \sqrt{5}}{2}$.

10. $f(2) = \frac{6}{\pi}$. (Use the fact that $f(x) = f(1)x$.)

11. True. This was a homework exercise.

12. For example, let $I = [0, \infty)$. Any infinite interval would suffice.

Remarks about proofs

For problem A it is essential to know the definition of an open cover and to distinguish between the cover and the union of the open sets belonging to it. If the underlying concepts are confused, it becomes impossible to write coherently about them. Proofs are graded for logical coherence. If you have questions about the grading of the proofs on this test, or if you are having difficulty writing satisfactory proofs, please bring me your test and also the graded homework from which the questions in Part II came. This will help us to see how you use the corrections to your homework in order to learn to write better proofs.

Class Statistics

Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	8	6			
80-89 (B)	3	2			
70-79 (C)	5	9			
60-69 (D)	5	5			
0-59 (F)	6	3			
Test Avg	74.1%	75.8%	%	%	%
HW Avg	5.7	5.2			
HW/Test Correl	0.90	0.89			