I have read, understood, and complied with the instructions in the box below. Legible

Signature **and LSU ID #:** ______________________

- Download and print a copy of this test on 8.5 by 11 inch letter size paper. If no printer is available, make a hand-written facsimile. Be sure to sign the statement above.
- **Show all work** in the space provided. We can give credit *only* for what you write! *Indicate clearly if you continue a problem on a second page.*
- ***Books/notes (electronic/paper), cell/smart phones, computers and all internet-connected devices are prohibited!*** A scientific calculator (*not capable* of graphing or symbolic calculations) is allowed—but it is not needed. If you use a calculator, you *must still write out all operations performed* on the calculator. *Do not replace* precise answers, such as $\sqrt{2}$, π, or $\cos\frac{\pi}{7}$ with decimal approximations. *Make all obvious simplifications.* Submit only your own work!
- Because this is a take-home test on an *honor system*, you have **90 minutes** to complete this hour-test, instead of the usual 50 minutes. Start counting time *after the test is printed* and you are ready to begin the mathematical work. *At the end of the 90 minutes, take extra time* to make a scan file or camera copy of your work (pdf file containing all the pages preferred) and email that file to me **rich@math.lsu.edu** as soon as possible but no later than 6 PM Monday April 13, 2020. *These instructions express my trust and confidence in your integrity and good character.*

Part I: Short Questions. Answer **8** of the 12 short questions: 6 points each. **Circle** the **numbers** of the 8 questions that you want counted—*no more than 8*! Detailed explanations are not required, but they may help with partial credit and are *risk-free!* Maximum score: 48 points.

1. True or False: If A_n is a countable set for each $n \in \mathbb{N}$, then $\mathcal{A} = \bigcup_{n=1}^{\infty} A_n$ is also a countable set.

2. True or False: An *open, dense* subset of $\mathbb{R}$ must be all of $\mathbb{R}$.

3. Find all cluster points of the set $I = \mathbb{R} \setminus \mathbb{Q}$ of irrational numbers.

4. Let $f : \mathbb{Z} \to \mathbb{R}$[1]. Find all points of *continuity* of f.

5. True or False: If $f(x) \to 0$ as $x \to a$ then $\dfrac{1}{f(x)} \to \infty$ as $x \to a$.

6. True or False: If $f(x) \to \infty$ as $x \to a$ then $\dfrac{1}{f(x)} \to 0$ as $x \to a$.

7. Let $f(x) = \begin{cases} 1 - x & \text{if } x \in \mathbb{Q}, \\ 1 - x^2 & \text{if } x \notin \mathbb{Q}. \end{cases}$ Find all points of continuity of f.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be any *continuous* function such that $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If $f(2) = 1$, find $f(-3)$.

9. Give and example of a function f and a domain D such that $f \in \mathcal{C}(D)$ but f is *not* uniformly continuous on D.

10. Let $f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}, \\ -x^2 & \text{if } x \notin \mathbb{Q}. \end{cases}$ Find $\|f\|_{\text{sup}}$.

[1] $\mathbb{Z}$ is the set of all integers.

11. Suppose $f_n \in \mathcal{C}[0,1]$ for all $n \in \mathbb{N}$ and suppose for each fixed $x \in [0,1]$, $f_n(x)$ is a sequence decreasing monotonically to the limit $f(x) = x$. True or False: f_n converges uniformly on $[0,1]$.

12. Let $f_n(x) = xe^{-nx}$ for all $n \in \mathbb{N}$ and all $x \in [0,\infty)$. Find $\|f_n\|_{\sup}$ for all $n \in \mathbb{N}$.

Part II: Proofs. *Prove carefully* **2** of the following 3 theorems for 26 points each. **Circle** the *letters* of the 2 proofs to be counted in the list below—*no more than 2!* You may write the proofs below, on the back, or on scratch paper. Maximum total credit: 52 points.

A. Prove: If f is *monotone increasing*[2] on $\mathbb{R}$, then for all $a \in \mathbb{R}$, $\lim_{x\to a+} f(x)$ exists and is a real number. The latter limit is called the *limit from the right*, and it is denoted by $f(a+)$. (Hint: Let $S = \{f(x) \mid x > a\}$ and show that $\inf(S)$ is a real number L. Then show that for every sequence $x_n \to a+$ we must have $f(x_n) \to L$.)

B. Prove the following *fixed point theorem*: Suppose $f \in \mathcal{C}[0,1]$ and suppose $0 \le f(x) \le 1$ for all $x \in [0,1]$. Then there exists $c \in [0,1]$ such that $f(c) = c$. (The point c is then called a *fixed point* for the function f. Hint: Consider $g(x) = f(x) - x$ and use the intermediate value theorem.)

C. Let $0 \le b < 1$ and let $f_n(x) = 1 - x^n$ for all $x \in [0,1)$.

- **(i)** Find the pointwise limit $f(x)$ of the sequence $f_n(x)$ for all $x \in [0,1)$.
- **(ii)** Let b be an arbitrary number in $[0,1)$. Prove that f_n converges *uniformly on* $[0,b]$.
- **(iii)** Determine whether or not f_n converges uniformly on $[0,1)$ and prove your conclusion.

[2] That is, if $x_1 < x_2$ then $f(x_1) \le f(x_2)$.

Solutions and Class Statistics

1. True: this was homework exercise 1.88.

2. False: Example 1.15 provides an example of a very *small* open dense subset of $\mathbb{R}$.

3. $\mathbb{R}$ is the set of all cluster points of the set I since the latter set is dense in $\mathbb{R}$.

4. Every point of Z is an isolated point, so f is continuous at every point of $\mathbb{Z}$.

5. False: the statement would be true if $f(x) > 0$ for all x. A counterexample would be $f(x) = x \sin \dfrac{1}{x}$ for all $x \neq 0$.

6. True.

7. f is continuous at p if and only if $p \in \{0, 1\}$.

8. $f(-3) = -\frac{3}{2}$ since $f(x) = cx$ for all $x \in \mathbb{R}$.

9. For example $f(x) = x^2$ is continuous on $\mathbb{R}$ but it is not uniformly continuous on $\mathbb{R}$.

10. $\|f\|_{\sup} = \infty$.

11. True: by Dini's theorem.

12. $\|f_n\|_{\sup} = \dfrac{1}{ne}$.

Remarks about the proofs

Proofs are graded for logical coherence. If you have questions about the grading of the proofs on this test, or if you are having difficulty writing satisfactory proofs, *please bring me your test and also the graded homework from which the questions in Part II came.* This will help us to see how you use the corrections to your homework in order to learn to write better proofs.

Class Statistics

Grade	Test#1	Test#2	Final Exam	Final Grade
90-100 (A)	6	9		
80-89 (B)	2	5		
70-79 (C)	1	0		
60-69 (D)	3	0		
0-59 (F)	3	0		
Test Avg	78.5%	90.8%	%	%
HW Avg	7.2	7.1		
HW/Test Correl	0.89	0.76		

The Correlation Coefficient is the cosine of the angle between two data vectors in $\mathbb{R}^{15}$–one dimension for each student enrolled. Thus this coefficient is between 1 and -1, with coefficients above 0.6 being considered strongly positive. The correlation coefficient shown indicates that the test grades in the course have a strongly positive correlation with performance on the homework.