I have read, understood, and complied with the instructions in the box below. Legible

***Signature* and LSU ID #:** ______________________________

- Download and print a copy of this test on 8.5 by 11 inch letter size paper. If no printer is available, make a hand-written facsimile. Be sure to sign the statement above.
- **Show all work** in the space provided. We can give credit *only* for what you write! *Indicate clearly if you continue a problem on a second page.*
- ***You may use your text book, class notes, and graded assignments!*** *However, no other sources or communication devices may be used.* ***All work must be your own.*** If you use a calculator, you *must still write out all operations performed* on the calculator. *Do not replace* precise answers, such as $\sqrt{2}$, π, or $\cos\frac{\pi}{7}$ with decimal approximations. *Make all obvious simplifications.* Submit only your own work!
- This is a take-home test on an *honor system.* You may take as much time as you like, but **I must receive your completed test by email no later than 2:45 PM on Thursday, May 7.** If you have no device that scans your work directly to a single pdf file, then photograph your work with your phone and save as jpeg, then try this please: put the jpeg files into your computer, highlight the whole group of pictures, right click PRINT and then select PRINT TO PDF. That way I can receive a multipage PDF file which is possible to grade in a way you will be able to read later. Email that file to me **rich@math.lsu.edu** as soon as possible but no later than 2:45 PM Thursday, May 7, 2020. *These instructions express my trust and confidence in your integrity and good character.*

Part I: Short Questions. Answer **12** of the 18 short questions: 8 points each. **Circle** the **numbers** of the 12 questions that you want counted—*no more than 12*! Detailed explanations are not required, but they may help with partial credit and are *risk-free!* Maximum score: 96 points.

1. True or Give a Counterexample: If $a_n b_n$ converges and $\dfrac{a_n}{b_n}$ converges, then a_n converges and b_n converges.

2. True or Give a Counterexample: If $|a_n - a_{n+1}| \to 0$ then the sequence a_n is Cauchy.

3. Suppose for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $m, n \geq N$ implies that $|a_m - a_n| < \epsilon$. True or give a counterexample: The sequence a_n is bounded.

4. Give an example of a *bounded* subset $S \neq \emptyset$ of $\mathbb{R}$ for which neither $\sup(S)$ nor $\inf(S)$ is an element of S.

5. Give an example of a Cauchy sequence of rational numbers that has no limit in $\mathbb{Q}$.

6. Give an example of a *countable* set S of *irrational* numbers such that S is *dense* in the real line.

7. Find the set of all *cluster points* of the set $\left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}$.

8. Find the set of all points of continuity of the function defined by $f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q}, \\ x^3 & \text{if } x \notin \mathbb{Q}. \end{cases}$

9. The inequality $\big||x| - |a|\big| \leq |x - a|$ expresses the *uniform continuity* of what common function $f(x)$ on the real line?

10. Give an example of a *continuous bounded* function $f(x)$ defined on the interval $(0, 1)$ that is *not* uniformly continuous on $(0, 1)$.

11. Let $f_n(x) = \sin^n x$ on the domain $\left[0, \frac{\pi}{2}\right)$. Find $\|f_n - 0\|_{\sup}$ and determine whether the convergence of f_n is *uniform* or only *pointwise* on the given domain.

12. Let $f_n(x) = \begin{cases} n & \text{if } 0 < x < \frac{1}{n}, \\ 0 & \text{if } x \in [0,1] \setminus \left(0, \frac{1}{n}\right), \end{cases}$ for all $n \in \mathbb{N}$. For each $x \in [0,1]$, find $\lim_{n\to\infty} f_n(x)$. Is the convergence uniform on $[0,1]$?

13. Let $p \in [a,b]$. If $f(x) = \begin{cases} 3 & \text{if } x = p, \\ 0 & \text{if } x \neq p. \end{cases}$ and if $\epsilon > 0$, how small must we make δ so that $\|P\| < \delta$ will imply that $|P(f, \{\bar{x_i}\})| < \epsilon$ for every partition P of $[a,b]$ and for all possible selections of evaluation points.

14. Express $\displaystyle\lim_{n\to\infty} \frac{3}{n} \sum_{k=1}^{n} \cos\left(1 + \frac{3k}{n}\right)$ as an integral. (Be sure to include the lower and upper limits of integration.)

15. Let $f(x) = x^4$ on $[a,b]$ and let $\epsilon > 0$. How small must we make $\|P\|$ to ensure that $U(f,P) - L(f,P) < \epsilon$?

16. Let $f = 1_{\mathbb{Q}} - 1_{\mathbb{R}\setminus\mathbb{Q}}$ on $[a,b]$. Find the numerical value of $\overline{\int_a^b} f - \underline{\int_a^b} f$. Is $f \in R[a,b]$ if $a < b$?

17. For each of the following statements, mark the statement True or False.

a. The functions $f_n(t) = \dfrac{1}{t \ln n} \to 0$ uniformly on $[1, \infty)$.

b. $\displaystyle\int_1^n f_n(t)\,dt \to \int_0^\infty 0\,dt = 0.$

18. Let $T : R[a,b] \to \mathbb{R}$ by $T(f) = f(b) - f(a)$, where $a < b$. Find a bound K for T, meaning that $|T(f)| \le K\|f\|_{\sup}$ for all $f \in R[a,b]$. Is T a continuous linear functional on $R[a,b]$?

Part II: Proofs. *Prove carefully* **4** of the following 6 theorems for 26 points each. **Circle** the *letters* of the 4 proofs to be counted in the list below—*no more than 4!* You may write the proofs below, on the back, or on scratch paper. Maximum total credit: 104 points.

A. *Prove* or else give a *counterexample:* If $2x_n + y_n$ converges and if $x_n - 2y_n$ converges, then x_n converges and y_n converges.

B. Let $E = \left\{ \frac{2}{n^2} \,\middle|\, n \in \mathbb{N} \right\}$. *Find* an open cover $\mathcal{O} = \{O_n \mid n \in \mathbb{N}\}$ of E that has no finite subcover. *Prove* that $\mathcal{O}$ is an open cover **and** that $\mathcal{O}$ has no finite subcover.

C. Let $q(t) = c_{2n}t^{2n} + \cdots + c_1 t + c_0$ be any polynomial of *even degree.* *Prove:* If $c_{2n} > 0$, then q has a *minimum* value on $\mathbb{R}$.

D. Suppose $f_n(x) = x^{2n}$. Decide whether or not f_n converges uniformly on each interval, and *prove* your conclusion.

(**i**) $[0, b]$, where $0 \le b < 1$

(**ii**) $[0, 1)$

E. Let $g \in \mathcal{C}[a,b]$. *Prove* that there exists $\bar{t} \in [a,b]$ such that $\int_a^b g(t)\,dt = g(\bar{t})(b-a)$ and *give and justify* a *counterexample* if $g \in \mathcal{R}[a,b] \setminus \mathcal{C}[a,b]$.

F. Let $f \in \mathcal{R}[a,b]$ and recall that $f^+(x) = \begin{cases} f(x) & \text{if } f(x) \ge 0, \\ 0 & \text{if } f(x) < 0. \end{cases}$ and $f^-(x) = \begin{cases} -f(x) & \text{if } f(x) < 0, \\ 0 & \text{if } f(x) \ge 0. \end{cases}$

Prove: f^+ and f^- are in $\mathcal{R}[a,b]$. (Hints: *Use* the Darboux integrability criterion and the fact that $f(x) = f^+(x) - f^-(x)$, but *prove* that $U(f^+, P) - L(f^+, P) \le U(f, P) - L(f, P)$ for every partition P of $[a,b]$ as part of your proof.)