## Print Your Name Here:

Show all work in the space provided and keep your eyes on your own paper. Indicate clearly if you continue on the back. Write your name at the top of the scratch sheet if you will hand it in to be graded. No books, notes, smart/cell phones, I-watches, communication devices, internet devices, or electronic devices are allowed except for a scientific calculator-which is not needed. The maximum total score is 100 .

Part I: Short Questions. Answer $\mathbf{8}$ of the 12 short questions: 6 points each. Circle the numbers of the 8 questions that you want counted-no more than 8 ! Detailed explanations are not required, but they may help with partial credit and are risk-free! Maximum score: 48 points.

1. Find $\lim _{x \rightarrow 2} \frac{x^{2}-x-2}{x^{2}-2 x}$.
2. If $P$ is any polynomial, find $\lim _{h \rightarrow 0} \frac{P(x+2 h)+P(x-2 h)-2 P(x)}{h^{2}}$.
3. Let $p$ be a polynomial of degree 100 and let $E=\left\{x \mid e^{x}=p(x)\right\}$. Find the largest possible number of elements in the set $E$.
4. Let $p(x)=3 x^{3}+2 x^{2}-x+1=\sum_{k=0}^{3} c_{k}(x-1)^{k}$. Find the numbers $c_{0}, c_{1}, c_{2}$ and $c_{3}$.
5. Does the following infinite series converge or diverge? $\sum_{n=1}^{\infty} \frac{1}{2 n-1}$
6. Is the following series divergent, conditionally convergent, or absolutely convergent? $\sum_{k=1}^{\infty} k r^{k-1}$, where $|r|<1$.
7. Does the following series converge or diverge? $\sum_{k=2}^{\infty}\left(\frac{1}{k \log k}\right)$
8. Is the following infinite series conditionally convergent, absolutely convergent, or divergent? $\sum_{n=1}^{\infty} \frac{(-1)^{n}}{2 n}$
9. Is the following series convergent or divergent? $\sum_{k=2}^{\infty} \frac{1}{(\log k)^{k}}$
10. Is the following series convergent or divergent? $\sum_{k=0}^{\infty} \frac{k!}{k^{k}}$. (Hint: Use the ratio test.)
11. Let $x_{k}>0$ for all $k \in \mathbb{N}$. True or Give a Counterexample: If $L=\lim \sup \frac{x_{k+1}}{x_{k}}>1$ then $\sum_{k=1}^{\infty} x_{k}$ diverges.
12. Find a summable sequence $x_{n}>0$ and a non-summable sequence $y_{n}>0$ for all $n \in \mathbb{N}$ such that $\frac{x_{n+1}}{x_{n}} \rightarrow 1$ as $n \rightarrow \infty$ and $\frac{y_{n+1}}{y_{n}} \rightarrow 1$ as $n \rightarrow \infty$.

Part II: Proofs. Prove carefully 2 of the following 3 theorems for 26 points each. Circle the letters of the 2 proofs to be counted in the list below-no more than 2! You may write the proofs below, on the back, or on scratch paper. Maximum total credit: 52 points.
A. Prove that $e$ is irrational. (Hint: Suppose false, so that $e=\frac{p}{q}$, where $p, q \in \mathbb{N}$. Write $e=e^{1}=P_{n}(1)+R_{n}(1)$, where $P_{n}$ is the $n$th Taylor polynomial and $R_{n}$ is the $n$th Taylor Remainder. Multiply both sides by $n!$, and deduce a contradiction when $n \in \mathbb{N}$ is sufficiently large.)
B. If $x$ and $y$ are sequences and $c \in \mathbb{R}$, we define $(c x)_{k}=c x_{k}$ and $(x+y)_{k}=x_{k}+y_{k}$. Prove: If $x$ and $y$ are summable, then
(i) $x+y$ is summable and $\sum_{k=1}^{\infty}\left(x_{k}+y_{k}\right)=\sum_{k=1}^{\infty} x_{k}+\sum_{k=1}^{\infty} y_{k}$.
(ii) $c x$ is summable and $\sum_{k=1}^{\infty} c x_{k}=c \sum_{k=1}^{\infty} x_{k}$. (Remark: We note that this exercise shows that the family of summable sequences is a vector space and the mapping $T(x)=\sum_{k=1}^{\infty} x_{k}$ is linear.)
C. Prove the following parts of the $n$th Root Test: Suppose $x_{k} \geq 0$ for all $k \in \mathbb{N}$, and suppose $\sqrt[k]{x_{k}} \rightarrow L$ as $k \rightarrow \infty$. Then we have the following conclusions.
(i) If $L>1, \sum_{k=1}^{\infty} x_{k}$ diverges.
(ii) If $L<1, \sum_{k=1}^{\infty} x_{k}$ converges.

## Solutions and Class Statistics

1. $\frac{3}{2}$. Be careful using L'Hospital's Rule!
2. $4 P^{\prime \prime}(x)$.
3. 101
4. The Taylor coefficients are $c_{0}=5, c_{1}=12, c_{2}=11$ and $c_{3}=3$. Note that $R_{3}(x)=0$.
5. Diverges
6. absolutely convergent
7. Diverges
8. Conditionally convergent
9. convergent
10. Convergent. The limit of the ratio is $\frac{1}{e}$.
11. Counterexample: Let $x_{k}=\left\{\begin{array}{ll}\frac{1}{k^{2}} & \text { if } \mathrm{k} \text { is odd } \\ \frac{1}{k^{3}} & \text { if } \mathrm{k} \text { is even. }\end{array}\right.$ Here $L=\infty$ but the series converges.
12. For example, let $x_{n}=\frac{1}{n^{2}}$ and $y_{n}=\frac{1}{n}$ for all $n \in \mathbb{N}$.

## Remarks about the proofs

Proofs are graded for logical coherence. Be sure to state what is your hypothesis (the assumption) and what conclusion you are seeking to prove. Then include justifications for each step. Your job is to show me through your writing that you understand the reasoning. If you have questions about the grading of the proofs on this test, or if you are having difficulty writing satisfactory proofs, please bring me your test and also the graded homework from which the questions in Part II came. This will help us to see how you use the corrections to your homework in order to learn to write better proofs. Also please bring your notebook showing how we presented the same proof in class after the homework was graded. It is important to learn from both sources.

A: For this proof one needs to write $e=e^{1}=P_{n}(1)+R_{n}(1)$. Then one needs to know what is $P_{n}(1)$ and $R_{n}(1)$, including the fact that the location of $\bar{x}$ is somewhere between 0 and 1 . Pick $n \geq q$ and multiply by $n!$ to force $R_{n}(1)$ to be an integer that is strictly between 0 and 1 , which is impossible once $n \geq 2$.

B: The key here is to use the definition $\sum_{k=1}^{\infty} x_{k}=\lim _{n \rightarrow \infty} \sum_{k=1}^{n} x_{k}$ IF the latter limit exists. That is, sums of infinite series are defined as limits of sequences of partial sums. Apply this definition to each of the two sequences $x+y$ and $c x$.

C: For the first part, show divergence by means of the $n$th term test: $x_{k}$ fails to converge to zero as $k \rightarrow \infty$. For the second part, pick a value of $r$ strictly between $L$ and 1 , show that there exists $N \in \mathbb{N}$ such that $k \geq N \Longrightarrow x_{k}<r^{k}$ and compare $\sum_{k=N}^{\infty} x_{k}$ with $\sum_{k=N}^{\infty} r^{k}$. Then take account of the first $N-1$ terms.

## Class Statistics

| Grade | Test\#1 | Test\#2 | Final Exam | Final Grade |
| :---: | :---: | :---: | :---: | :---: |
| $90-100$ (A) | 0 | 4 |  |  |
| $80-89$ (B) | 0 | 1 |  |  |
| $70-79$ (C) | 3 | 0 |  |  |
| $60-69$ (D) | 2 | 0 |  |  |
| $0-59$ (F) | 0 | 0 |  | $\%$ |
| Test Avg | $72 \%$ | $92.8 \%$ | $\%$ |  |
| HW Avg | 7.4 | $72.4 \%$ |  |  |
| HW/Test Correl | - | 0.58 |  |  |

The Correlation Coefficient is the cosine of the angle between two data vectors in $\mathbb{R}^{5}$-one dimension for each student enrolled. Thus this coefficient is between 1 and -1 , with coefficients above 0.6 being considered strongly positive. The correlation coefficient shown indicates that the test grades in the course have a strong positive correlation with performance on the homework.

