Print Your Name Here: ______________________________

Show all work in the space provided and *keep your eyes on your own paper. Indicate clearly* if you continue on the back. Write your **name** at the **top** of the *scratch* sheet *if you will hand it in to be graded.* **No** books, notes, smart/cell phones, I-watches, communication devices, internet devices, or electronic devices are allowed except for a scientific calculator—which is not needed. The maximum total score is 100.

Part I: Short Questions. Answer **8** of the 12 short questions: 6 points each. **Circle** the **numbers** of the 8 questions that you want counted—*no more than 8*! Detailed explanations are not required, but they may help with partial credit and are *risk-free!* Maximum score: 48 points.

1. Give an example of a sequence f_n for which $f_n' \to 0$ uniformly on $\mathbb{R}$, yet $f_n(x)$ *diverges* for all $x \in \mathbb{R}$.

2. True or Give a Counterexample: If f_n converges uniformly on $\mathbb{R}$ to a differentiable function g, then $f_n'(x)$ converges to $g'(x)$.

3. If P is any polynomial, find $\displaystyle\lim_{h\to 0} \frac{P(x+3h)+P(x-3h)-2P(x)}{h^2}$.

4. Find $\displaystyle\lim_{x\to 0+} x^x$.

5. Expand the polynomial $p(x) = 3x^3 + 2x^2 - x$ as a polynomial in powers of $(x-1)$: That is, express $p(x) = \sum_{k=0}^{3} c_k (x-1)^k$ and find the values of the constants $c_0, \ldots, c_3$.

6. Find $\lim_{x \to 2} \frac{x^2 - x - 2}{x^2 - 2x}$

7. Let $V = \left\{ x \,\middle|\, \sum_{k=1}^{\infty} x_k \text{ converges} \right\}$ be the vector space of summable sequences. True or False: $T : V \to \mathbb{R}$ by $T(x) = \sum_{k=1}^{\infty} x_k$ is linear.

8. Let $s_n = \sum_{k=1}^{n} \frac{1}{k}$. Show how you would insert parentheses in the summation that is s_7 to illustrate that $s_{2^n - 1} \leq n$ for $n = 3$.

9. True or give a counterexample: If $f : \mathbb{R} \to \mathbb{R}$ is a *contraction*, then f has at most one fixed point.

10. Suppose f is a decreasing positive function on the domain $[1, \infty)$. List the three terms $\sum_{k=1}^{n} \frac{1}{k}$, $\int_1^n \frac{1}{x}\,dx$, and $1 + \int_1^n \frac{1}{x}\,dx$ correctly in the form $a \le b \le c$.

11. Does the series $\sum_{0}^{\infty} \left(\frac{-3}{\pi}\right)^k$ converge absolutely, converge conditionally, or diverge? If it converges, what is the numerical value of the sum?

12. Find the numerical value of $\sum_{1}^{\infty} \frac{2}{k(k+2)}$. (Hint: telescoping series.)

Part II: Proofs. *Prove carefully* **2** of the following 3 theorems for 26 points each. **Circle** the *letters* of the 2 proofs to be counted in the list below—*no more than 2!* You may write the proofs below, on the back, or on scratch paper. Maximum total credit: 52 points.

A. Let $f_n(x) = \sin^n x$ for all $x \in [0, \pi]$. Prove that the sequence of *derivatives* f_n' is *not* uniformly convergent on $[0, \pi]$. (Hint: Suppose false and apply a Theorem about uniform convergence and derivatives to deduce a contradiction. Be explicit about how the hypotheses are satisfied.)

B. Let $f(x) = \sin x$. Prove that the nth Taylor Remainder converges to 0 as $n \to \infty$. Prove also that no polynomial $P(x) \equiv \sin x$ on any interval of positive length. (Hint: For the first proof, you may use a direct proof as in the text, or you may use a combination of the ratio test and the nth term test as we showed in class.)

C. Prove the *Limit Comparison Test:* Suppose $x_k \ge 0$ and $y_k > 0$ for all $k \in \mathbb{N}$. Suppose $\frac{x_k}{y_k} \to L \in \mathbb{R}$. If $L > 0$ then x is summable *if and only if* y is summable. (Hint: Apply the ordinary comparison test.)

Solutions and Class Statistics

1. For example, let $f_n(x) = n$ for all $x \in \mathbb{R}$ and for all $n \in \mathbb{N}$. See problem 4.36.

2. Counterexample: Let $f_n(x) = \frac{1}{n}\sin\left(n^2 x\right)$. Then the sequence $f_n'(0)$ diverges. See problem 4.35.

3. $\lim_{h\to 0} \frac{P(x+3h)+P(x-3h)-2P(x)}{h^2} = 9P''(x)$. See problem 4.45.

4. $\lim_{x\to 0+} x^x = 1$

5. The Taylor coefficients are $c_0 = 4,\ c_1 = 12,\ c_2 = 11, c_3 = 3$. Note that $R_3(x) = 0$.

6. $\lim_{x\to 2} \frac{x^2-x-2}{x^2-2x} = \frac{3}{2}$.

7. True

8. $s_7 = (1) + \left(\frac{1}{2}+\frac{1}{3}\right) + \left(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right)$ in which each of the three displayed parenthetic sums is less than or equal to 1.

9. True

10. $\int_1^n \frac{1}{x}\,dx \le \sum_{k=1}^{n} \frac{1}{k} \le 1 + \int_1^n \frac{1}{x}\,dx$, from the proof of the integral test. The question should have stated either that f is a decreasing *positive* function and then f would appear in the integrands, or else it could have simply omitted all reference to f and asked for the correct inequalities for the 3 terms listed. My apologies for the careless statement of the question..

11. $\sum_0^\infty \left(\frac{-3}{\pi}\right)^k = \frac{\pi}{\pi+3}$ with the convergence being absolute, by the geometric series test.

12. $\sum_1^\infty \frac{2}{k(k+2)} = \frac{3}{2}$

Remarks about the proofs

Proofs are graded for logical coherence. Be sure to state what is your hypothesis (the assumption) and what conclusion you are seeking to prove. Then include justifications for each step. Your job is to show me through your writing that you understand the reasoning. If you have questions about the grading of the proofs on this test, or if you are having difficulty writing satisfactory proofs, *please bring me your test and also the graded homework from which the questions in Part II came.*

This will help us to see how you use the corrections to your homework in order to learn to write better proofs. Also *please bring your notebook showing how we presented the same proof in class* after the homework was graded. It is important to learn from both sources.

A: I was astonished by the following false proof on several papers: $f_n'(x) = n\sin^{n-1}x\cos x$ and it is claimed that the factor n makes f_n' fail to converge on $[0, \pi]$. This is nonsense since $\sin^{n-1}x\cos x \to 0$ fast, even at $x = \pi/2$, where $\cos x = 0$. That is why I proved in class that f_n' is not uniformly convergent on $[0, \pi]$ correctly, using Theorem 4.4.1. If you gave this false proof and *if* it was (inappropriately) accepted by the grader, *please show me your graded assignment so I can take that into consideration and regrade your proof for A.*

B: The main things I looked for were the correct form of the nth Taylor Remainder, and then a proof that this remainder converges to 0 as $n \to \infty$

C: Remember that the Comparison Test is for *positive* term series. Then if $\frac{x_k}{y_k} \to L$ then there is a K such that $k \geq K$ implies that the ratio is sandwiched between $L/2$ and $3L/2$. We need to divide by L for one of the two inequalities in the ensuing proof, so that is why we assume $L > 0$. Don't omit the inequalities needed for each direction when we use the Comparison Test.

Class Statistics

Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	5	2			
80-89 (B)	3	3			
70-79 (C)	1	1			
60-69 (D)	0	3			
0-59 (F)	0	0			
Test Avg	89.6%	79.4%	%	%	%
HW Avg	86.8%	87.6%			
HW/Test Correl	0.45	0.63			

The Correlation Coefficient is the cosine of the angle between two data vectors in $\mathbb{R}^9$–one dimension for each student enrolled. Thus this coefficient is between 1 and -1, with coefficients above 0.6 being considered strongly positive. The correlation coefficient shown indicates that the test grades in the course have a strongly positive correlation with performance on the homework. (In my experience, this statistic is somewhat unstable in classes with low enrollment.)