Print Your Name Here: ______________________________

Show all work in the space provided and *keep your eyes on your own paper. Indicate clearly* if you continue on the back. Write your **name** at the **top** of the *scratch* sheet *if you will hand it in to be graded.* **No** books, notes, smart/cell phones, I-watches, communication devices, internet devices, or electronic devices are allowed except for a scientific calculator—which is not needed. The maximum total score is 100.

Part I: Short Questions. Answer **8** of the 12 short questions: 6 points each. **Circle** the **numbers** of the 8 questions that you want counted—*no more than 8*! Detailed explanations are not required, but they may help with partial credit and are *risk-free!* Maximum score: 48 points.

1. Find the sum of the infinite series that begins as follows: $\frac{1}{2}-\frac{1}{3}+\frac{1}{2^2}-\frac{1}{3^2}+-\cdots+\frac{1}{2^k}-\frac{1}{3^k}+-\cdots.$

2. Is the series in question 1 *conditionally* convergent or *absolutely* convergent?

3. True or False: There exists a rearrangement of $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}$ that converges to $-100\pi^2$.

4. True or False: If *every* rearrangement of $\sum_{k=1}^{\infty} a_k$ is convergent, then $\sum_{k=1}^{\infty} a_k$ is absolutely convergent.

5. Let $x, y \in \mathbb{R}$, and form the *sequences* e and f with $e_j = \dfrac{x^j}{j!}$ and $f_k = \dfrac{y^k}{k!}$, $j, k = 0, 1, 2, \ldots$. Write the *third* term, c_3, of the *Cauchy product* sequence c of e with f.

6. Give an example of a double series $\displaystyle\sum_{(j,k)\in\mathbb{N}\times\mathbb{N}} a_{j,k}$ in which each *column sum* $c_k = \displaystyle\sum_{j=1}^{\infty} a_{j,k} = 0$, making the sequence of column sums c_k absolutely summable, and yet $\displaystyle\sum_{(j,k)\in\mathbb{N}\times\mathbb{N}} |a_{j,k}|$ diverges.

7. For all $n \in \mathbb{N}$ define a sequence $x^{(n)} \in l_1$ by letting the kth term of $x^{(n)}$ be $x_k^{(n)} = \frac{n+1}{n2^k}$, for all $k \in \mathbb{N}$. Find $\|x^{(n)}\|_1$.

8. True or False: the infinite series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ has a rearrangement that converges to π.

9. True or False: If $0 \leq \alpha < 1$, then $\sum_{k=1}^{\infty} x^k$ converges *uniformly* on $[-\alpha, \alpha]$.

10. True or False: The sum $\sum_{k=1}^{\infty} x^k$ converges *uniformly* on $[0, 1)$.

11. Let x be sequence of real numbers for which $\displaystyle\sum_{k=1}^{\infty} x_k^+ = P \in \mathbb{R}$ and for which $\displaystyle\sum_{k=1}^{\infty} x_k^- = Q \in \mathbb{R}$. Find $\displaystyle\sum_{k=1}^{\infty} |x_k|$.

12. Using the sequence x from Question 11, *find* $\displaystyle\sum_{k=1}^{\infty} x_k$.

Part II: Proofs. *Prove carefully* **2** of the following 3 theorems for 26 points each. **Circle** the *letters* of the 2 proofs to be counted in the list below—*no more than 2!* You may write the proofs below, on the back, or on scratch paper. Maximum total credit: 52 points.

A. If c is a summable sequence and if $x \in [0,1]$, prove that $\sum_{k=0}^{\infty} c_k x^k$ converges.

B. For all $n \in \mathbb{N}$ define a sequence $x^{(n)} \in l_1$ by letting the kth term of $x^{(n)}$ be

$$x_k^{(n)} = \begin{cases} 1 & \text{if } k \le n, \\ \frac{1}{k^2} & \text{if } k > n. \end{cases}$$

- **(i)** Show that $x^{(n)} \in l_1$ by showing $\left\|x^{(n)}\right\|_1 < \infty$.
- **(ii)** Define a sequence x by letting $x_k = \lim_{n\to\infty} x_k^{(n)}$, for all $k \in \mathbb{N}$. Is $x \in l_1$? Prove your answer, yes or no.
- **(iii)** Is $x^{(n)}$ a Cauchy sequence in l_1? Prove your answer, yes or no.

C. If $\displaystyle\sum_{k=1}^{\infty} x_k$ is *conditionally* convergent, prove that $\displaystyle\sum_{k=1}^{\infty} x_k^+ = \infty = \sum_{k=1}^{\infty} x_k^-$. (Recall that $x_k = x_k^+ - x_k^-$ and $|x_k| = x_k^+ + x_k^-$.)

Solutions and Class Statistics

1. $\frac{1}{2}$

2. absolutely convergent

3. True, since the alternating harmonic series is conditionally convergent.

4. True: If the series were either divergent or only conditionally convergent, then it could not have the stated property.

5. $c_3 = \dfrac{x^3}{3!} + \dfrac{x^2y}{2} + \dfrac{xy^2}{2} + \dfrac{y^3}{3!}$.

6. For example, let each *column* sequence begin with 1 and then -1, and then all 0's.

7. $\|x^{(n)}\|_1 = 1 + \frac{1}{n}$

8. False: the series is absolutely convergent.

9. True, by the Weierstrass M-test.

10. False: we proved this in class!

11. $\displaystyle\sum_{k=1}^{\infty} |x_k| = P + Q$

12. $\displaystyle\sum_{k=1}^{\infty} x_k = P - Q$

Remarks about the proofs

Proofs are graded for logical coherence. Be sure to state what is your hypothesis (the assumption) and what conclusion you are seeking to prove. Then include justifications for each step. Your job is to show me through your writing that you understand the reasoning. If you have questions about the grading of the proofs on this test, or if you are having difficulty writing satisfactory proofs, *please bring me your test and also the graded homework from which the questions in Part II came.* This will help us to see how you use the corrections to your homework in order to learn to write better proofs. Also *please bring your notebook showing how we presented the same proof in class* after the homework was graded. It is important to learn from both sources.

A: If $x = 1$ then $\sum c_k x^k = \sum c_k$, which converges by *hypothesis.* If $0 \le x < 1$ then $\sum x^k$ is an *absolutely* convergent *geometric* series. Since $\sum c_k$ converges, $c_k \to 0$ and since the *sequence* c_k converges it is *bounded.* Therefore $\sum c_k x^k$ is absolutely convergent, and is thus convergent.

B: (i) $\|x^{(n)}\|_1 = n + \sum_{k=n+1}^{\infty} \frac{1}{k^2} < \infty$ by the *p-series* test or by the *integral* test. Thus $x^{(n)} \in l_1$ for each $n \in \mathbb{N}$. (ii) $x \notin l_1$ since $\|x\|_1 = \sum_{k=1}^{\infty} 1$ which diverges to infinity. (iii) $x^{(n)}$ is not Cauchy in the l_1-norm since if it were Cauchy then its limit x would have to be in l_1, which is a complete normed vector space. But $x \notin l_1$.

C: There are only 4 possibilities. If we denote $\sum_{k=1}^{\infty} x_k^+ = P$, and $\sum_{k=1}^{\infty} x_k^- = Q$ we have the following conclusions. If P and Q are both finite, we need to show that $\sum |x_k| = P + Q < \infty$, which is impossible because we are given that $\sum x_k$ is conditionally convergent. If $P = \infty$ and if Q is finite, prove that $\sum x_k$ diverges to infinity, which is impossible since the sum is given as (conditionally) convergent. If $Q = \infty$ and if P is finite, prove that $\sum x_k$ diverges to $-\infty$, which violates conditional summability. Thus we are left with the conclusion that $P = \infty = Q$.

Class Statistics

Grade	Test#1	Test#2	Final Exam	Final Grade
90-100 (A)	0	4	3	
80-89 (B)	0	1	1	
70-79 (C)	3	0	1	
60-69 (D)	2	0	0	
0-59 (F)	0	0	0	
Test Avg	72%	92.8%	90.6%	%
HW Avg	7.4	7.24	7.15	