## Print Your Name Here:

Show all work in the space provided and keep your eyes on your own paper. Indicate clearly if you continue on the back. Write your name at the top of the scratch sheet if you will hand it in to be graded. No books, notes, smart/cell phones, I-watches, communication devices, internet devices, or electronic devices are allowed except for a scientific calculator-which is not needed. The maximum total score is 200 .

Part I: Short Questions. Answer 12 of the 18 short questions: 8 points each. Circle the numbers of the 12 questions that you want counted-no more than 8 ! Detailed explanations are not required, but they may help with partial credit and are risk-free! Maximum score: 96 points.

1. If $f \in \mathcal{R}[0,1]$, use the Cauchy-Schwarz inequality to find a constant $K$ such that $\int_{0}^{1} x f(x) d x \leq K\left[\int_{0}^{1}[f(x)]^{2} d x\right]^{\frac{1}{2}}$.
2. Use the triangle inequality to find a value of $K$ such that for all $f \in \mathcal{R}[0,1]$, $\left[\int_{0}^{1}(\sqrt{\cos x}+f(x))^{2} d x\right]^{\frac{1}{2}} \leq K+\left[\int_{0}^{1}(f(x))^{2} d x\right]^{\frac{1}{2}}$.
3. The function $f(x)=\left\{\begin{array}{ll}0 & \text { if } x=0, \\ \frac{1}{n} & \text { if } x \in\left(\frac{1}{n+1}, \frac{1}{n}\right], \forall n \in \mathbb{N}\end{array}\right.$ belongs to a class of functions that is always Riemann integrable on $[0,1]$. To which class of integrable functions are we referring?
4. True or Give a Counterexample: Suppose $f \in \mathcal{C}(a, b), a, b \in \mathbb{R}$, and also that $f$ is bounded on $[a, b]$. Then $f \in \mathcal{R}[a, b]$.
5. Let $T: \mathcal{C}[0,1] \rightarrow \mathbb{R}$ be defined by $T(f)=\int_{0}^{1} f(x)\left(1+x^{2}\right) d x$. Find a constant $K$ for which $|T(f)| \leq K\|f\|_{\text {sup }}$ for all $f \in \mathcal{C}[0,1]$.
6. Let $f(x)=\left\{\begin{array}{ll}x \sin \left(\frac{\pi}{x}\right) & \text { if } x \in(0,1], \\ 0 & \text { if } x=0 .\end{array}\right.$ True or Give a Counterexample: $f^{\prime}(x)$ exists for all $x \in[0,1]$.
7. Does the following series converge or diverge? $\sum_{k=2}^{\infty}\left(\frac{1}{k \log k}\right)$
8. Is the following series divergent, conditionally convergent, or absolutely convergent? $\sum_{k=1}^{\infty} k r^{k-1}$, where $|r|<1$.
9. True or Give a Counterexample: Suppose $\left|f^{\prime}(x)\right| \leq M \in \mathbb{R}$ for all $x \in I$, an interval. Then $f$ is uniformly continuous on $I$.
10. True or False: Suppose $f \in \mathcal{R}[a, b]$ and let $F(x)=\int_{a}^{x} f(t) d t$ for all $x \in[a, b]$. Then $F \in \mathcal{C}[a, b]$.
11. Give an example of a sequence $f_{n}$ for which $f_{n}^{\prime} \rightarrow 0$ uniformly on $\mathbb{R}$, yet $f_{n}(x)$ diverges for all $x \in \mathbb{R}$.
12. If $P$ is any polynomial, find $\lim _{h \rightarrow 0} \frac{P(x+3 h)+P(x-3 h)-2 P(x)}{h^{2}}$.
13. Let $x$ be sequence of real numbers for which $\sum_{k=1}^{\infty} x_{k}^{+}=P \in \mathbb{R}$ and for which $\sum_{k=1}^{\infty} x_{k}^{-}=Q \in \mathbb{R}$. Find $\sum_{k=1}^{\infty}\left|x_{k}\right|$.
14. For all $n \in \mathbb{N}$ define a sequence $x^{(n)} \in l_{1}$ by letting the $k$ th term of $x^{(n)}$ be $x_{k}^{(n)}=\frac{n+1}{n 2^{k}}$, for all $k \in \mathbb{N}$. Find $\left\|x^{(n)}\right\|_{1}$.
15. For all $n \in \mathbb{N}$ define a sequence $x^{(n)} \in l_{1}$ by letting $x_{k}^{(n)}=\left\{\begin{array}{ll}\frac{1}{k} & \text { if } k \leq n, \\ 0 & \text { if } k>n .\end{array}\right.$ True or False: The sequence $x^{(n)}$ is a Cauchy sequence in $l_{1}$.
16. True or False: The series $\sum_{k=1}^{\infty} \frac{\sin k x}{k^{2}}$ converges uniformly on $\mathbb{R}$.
17. Find $\sum_{k=1}^{\infty} \frac{1}{k 2^{k}}$. (Hint: Begin with the series $\sum_{k=0}^{\infty} t^{k}$ and integrate between two appropriate limits.)
18. Find the interval $I$ of convergence for the power series $\sum_{k=1}^{\infty}(-1)^{k+1} \frac{x^{k}}{k}$. Be sure to check the endpoints.

Part II: Proofs. Prove carefully 4 of the following 6 theorems for 26 points each. Circle the letters of the 4 proofs to be counted in the list below-no more than 4! You may write the proofs below, on the back, or on scratch paper. Maximum total credit: 104 points.
A. Let $f_{n}(x)=\sin ^{n} x$ for all $x \in[0, \pi]$. Prove that $f_{n}^{\prime}$, the sequence of derivatives, is not uniformly convergent on $[0, \pi]$. (Hint: Suppose false and apply the theorem about uniform convergence and derivatives to deduce a contradiction. Be sure to show how you check that all the hypotheses of the theorem are satisfied.)
B. Let $F(x)=\left\{\begin{array}{ll}x^{2} \sin \frac{1}{x} & \text { if } 0<|x| \leq 1, \\ 0 & \text { if } x=0\end{array}\right.$ and denote $f(x)=F^{\prime}(x)$. (See Fig. 1.)


Figure 1: $y=f(x)$
(i) (10) Show that $F^{\prime}(x)$ exists for all $x \in[-1,1]$ and find it. Be sure to consider $x=0$.
(ii) (10)Prove $f \in \mathcal{R}[-1,1] \backslash \mathcal{C}[-1,1]$. (Hint: Use the Variant Form of the Darboux Integrability Criterion.)
(iii) (6) Find $\int_{-1}^{1} f(x) d x$.
C. Let $f(x)=\sin x$ and let $P_{n}(x)$ be the $n$th Taylor polynomial for $f(x)$ in powers of $x=x-0$. (That is, use $a=0$ ).
(i) (16) Prove that $P_{n}(x) \rightarrow \sin x$ as $n \rightarrow \infty$ for all $x \in \mathbb{R}$.
(ii) (10) Prove that no polynomial $P(x) \equiv \sin x$ on any interval of positive length.
D. Use the Comparison Test to prove the following parts of the $n$th Root Test: Suppose $x_{k} \geq 0$ for all $k \in \mathbb{N}$, and suppose $\sqrt[k]{x_{k}} \rightarrow L$ as $k \rightarrow \infty$. Then we have the following conclusions.
(i) (13) If $L>1, \sum_{k=1}^{\infty} x_{k}$ diverges.
(ii) (13) If $L<1, \sum_{k=1}^{\infty} x_{k}$ converges.
E. Prove parts of the Limit Comparison Test: Suppose $x_{k} \geq 0$ and $y_{k}>0$ for all $k \in \mathbb{N}$. Suppose $\frac{x_{k}}{y_{k}} \rightarrow L \in \mathbb{R}$. If $L>0$, prove that $x$ is summable if and only if $y$ is summable. (Hint: Apply the ordinary comparison test.)
F. Let $f_{k}(x)=\frac{(-1)^{k+1}}{k} x^{k}$.
(i) Prove: $\sum_{k=1}^{\infty} f_{k}(x)$ converges uniformly on $[0,1]$. (Hint: Use the error estimate from the alternating series test.)
(ii) Find $\left\|f_{k}\right\|_{\text {sup }}$ on $[0,1]$. Can the Weierstrass M-test be used to prove the uniform convergence of $\sum_{k=1}^{\infty} f_{k}(x)$ on $[0,1]$ ? Why or why not?

## Solutions and Class Statistics

1. $K=\frac{1}{\sqrt{3}}$.
2. $K=\sqrt{\sin 1}$.
3. $f$ is monotone on $[0,1]$
4. True
5. $K=\frac{4}{3}$ works.
6. Counterexample: $f^{\prime}(0)$ does not exist.
7. Diverges
8. absolutely convergent
9. True
10. True
11. For example, let $f_{n}(x)=n$ for all $x \in \mathbb{R}$ and for all $n \in \mathbb{N}$.
12. $\lim _{h \rightarrow 0} \frac{P(x+3 h)+P(x-3 h)-2 P(x)}{h^{2}}=9 P^{\prime \prime}(x)$.
13. $\sum_{k=1}^{\infty}\left|x_{k}\right|=P+Q$
14. $\left\|x^{(n)}\right\|_{1}=1+\frac{1}{n}$
15. False: the sequence is not even bounded in the $l_{1}$-norm.
16. True, by the Weierstrass M-test.
17. The sum is $\ln 2$.
18. $\quad I=(-1,1]$.

## Remarks about the proofs

Proofs are graded for logical coherence. Be sure to state what is your hypothesis (the assumption) and what conclusion you are seeking to prove. Then include justifications for each step. Your job is to show me through your writing that you understand the reasoning. If you have questions about the grading of the proofs on this test, or if you are having difficulty writing satisfactory proofs,
please bring me your test and also the graded homework from which the questions in Part II came. This will help us to see how you use the corrections to your homework in order to learn to write better proofs. Also please bring your notebook showing how we presented the same proof in class after the homework was graded. It is important to learn from both sources.

A: Since $f_{n}(x)$ converges at least at one point (for example, at $x=0$ ), if $f_{n}^{\prime}$ did converge uniformly it would follow that $f_{n}$ converges uniformly on $[0, \pi]$. But this is impossible since the pointwise limit of $f_{n}(x)$ is discontinuous at $x=\frac{\pi}{2}$. See the homework solutions for more details.

B: It is important to be able to calculate a derivative correctly using the chain rule. Also, at $x=0$ it is necessary to use the limit of a difference quotient. Phony reasons, such as thinking that $F^{\prime}(0)$ is the derivative of the constant 0 are unacceptable. Riemann integrability of $f$ comes from the fact that $f$ is continuous on $(0,1)$ and bounded on $[0,1]$ and also continuous on $(-1,0)$ and bounded on $[-1,0]$. See the homeowrk exercise 3.26 . The value of the integral is $2 \sin (1)$ by the Fundamental Theorem of the Calculus.

C: For part (i), one must show that the $n$th Taylor Remainder $R_{n}(x) \rightarrow 0$ as $n \rightarrow \infty$ for each fixed $x \in \mathbb{R}$. Of course one needs to know the formula for $R_{n}(x)$ and that $f(x)=P_{n}(x)+R_{n}(x)$, which is a generalization of the Mean Value Theorem for Derivatives. For part (ii), the $n$th derivative of any $n$th degree polynomial is a constant, but this is not true for $\sin x$. Part (ii) is not about Taylor polynomials: It is about all polynomials!

D: For part (i), show that for big enough $k$ we have $\sqrt[k]{x_{k}}>1$ and then apply the $n$th term test. For part (ii) pick $r$ between $L$ and 1 and show that for big enough $k$ we have $\sqrt[k]{x_{k}}<r$ and use the comparison test with the geometric series $\sum_{k} r^{k}$.
E: Show that for big enough $k$ we have $\frac{L}{2}<\frac{x_{k}}{y_{k}}<\frac{3}{2} L$. Multiply this double inequality by the positive number $y_{k}$ and use both inequalities. One can use other inequalities as above, but make sure the left-most term is greater than 0 . To use the comparison test, it is essential that the inequalities be in the appropriate direction!

F: The series converges at least pointwise by the alternating series test. Letting $S_{n}(x)$ be the $n$th partial sum and $S(x)$ be the final sum, we have $\left|S_{n}(x)-S(x)\right|<\left|f_{n+1}(x)\right| \leq \frac{1}{n+1}$ for all $x \in[0,1]$, so $\left\|S_{n}-S\right\|_{\text {sup }}=\frac{1}{n+1} \rightarrow 0$ as $n \rightarrow \infty$. Thus the series converges uniformly. The uniform convergence cannot be proved using the Weierstrass M-Test since $\sum\left\|f_{n}\right\|_{\text {sup }}=\sum \frac{1}{n}=\infty$.

## Class Statistics

| Grade | Test\#1 | Test\#2 | Test \#3 | Final Exam | Final Grade |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $90-100$ (A) | 0 | 4 | 3 | 2 | 4 |
| $80-89$ (B) | 0 | 1 | 1 | 3 | 1 |
| $70-79$ (C) | 3 | 0 | 1 | 0 | 0 |
| $60-69$ (D) | 2 | 0 | 0 | 0 | 0 |
| $0-59$ (F) | 0 | 0 | 0 | 0 | 0 |
| Test Avg | $72 \%$ | $92.8 \%$ | $90.6 \%$ | $88.4 \%$ | $93.9 \%$ |
| HW Avg | 7.4 | 7.24 | 7.15 | 7.15 | 7.15 |

