Name: ____

Instructions. *Show all work in the space provided. Indicate clearly if you continue on the back side, and write your name at the top of the scratch sheet if you will turn it in for grading*. No books or notes are allowed. A scientific calculator is ok - but not needed . The *maximum total score* is 100.

Part I - 48 points. Answer 8 of the following 12 questions. **Circle** the numbers of the 8 questions you want counted - *no more than 8*! Detailed explanations are not required, but they may help with partial credit and are *risk free!*

1. Express $\lim_{n\to\infty} \frac{2}{n}\sum_{k=1}^{n}\cos(1+\frac{2k}{n})$ as an integral. (Be sure to include the lower and upper limits of integration.)

2. Let $f(x) = \begin{cases} 1, \text{ if } x \in \{\frac{1}{n} | n \in \mathbb{N}\} \\ 0, \text{ if } x \in [0,1] \setminus \{\frac{1}{n} | n \in \mathbb{N}\} \end{cases}$. Either find $\int_0^1 f(x)dx$ if $f \in$ *R*$[0,1]$ or state that $f \notin$ *R*$[0,1]$.

3. Let $f(x) = \begin{cases} 0, \text{ if } x = 0 \\ \frac{1}{\sqrt{x}}, \text{ if } 0 < x \le 1 \end{cases}$ Either find $\int_0^1 f(x)dx$ if $f \in$ *R*$[0,1]$, or else state that $f \notin$ *R*$[0,1]$.

4. Give an example of f such that $|f| \in R[a,b]$ yet $f \notin R[a,b]$.

5. Let $f(x) = \begin{cases} \sin\frac{\pi}{x}, \text{ if } 0 < x \leq 1 \\ 0, \text{ if } x = 0 \end{cases}$. Find $\overline{\int_a^b} f - \underline{\int_a^b} f$.

6. Answer True or False for each separate part below.

   (a) If $f \in R[a,b]$ then $f \in C[a,b]$.

   (b) If f is monotone decreasing on $[a,b]$, then $f \in R[a,b]$.

7. Let $f_n(x) = \begin{cases} n, \text{ if } 0 < x \leq \frac{1}{n} \\ 0, \text{ if } \frac{1}{n} < x \leq 1 \\ 0, \text{ if } x = 0 \end{cases}$ for all $n \in \mathbb{N}$. For each $x \in [0,1]$, *find* $f(x) = \lim_{n\to\infty} f_n(x)$, the point-wise limit.

8. Let $T : C[0,3] \to \mathbb{R}$ be defined by $T(f) = f(1) - f(2)$. True or False: T is a bounded linear functional on $C[0,3]$, equipped with the sup-norm.

9. $T : C[0,1] \to \mathbb{R}$ by $T(f) = \int_0^1 (1+x^2) f(x) dx$, for all $f \in C[0,1]$. True or False: T is a continuous linear functional on $C[0,1]$, equipped with the sup-norm.

10. True or False: If f and g are in $R[0,1]$, then

$$\int_0^1 [f(x)+g(x)]^2 dx \leq \int_0^1 [f(x)]^2 dx + \int_0^1 [g(x)]^2 dx$$

11. True or False: If $f \in R[a,b]$, then $\int_a^b \frac{1}{1+[f(x)]^2} dx$ exists.

12. True or False: If $f \in \mathrm{R}[0,1]$, then $\int_0^1 x f(x) dx \leq \frac{1}{\sqrt{3}} \left(\int_0^1 [f(x)]^2 dx \right)^{\frac{1}{2}}$.

Part II - 52 points. Prove carefully *two* of the following three theorems. **Circle** the letters of the two proofs to be counted - *no more than two!* You may write the proofs below, on the back, or on scratch paper.

A. Prove the *Mean Value Theorem for Integrals:* Let $f \in C[a,b]$. Prove: There exists $\bar{x} \in [a,b]$ such that $\int_a^b f(x)dx = f(\bar{x})(b-a)$. (Hints: Let $m = \min\{f(x)|x \in [a,b]\}$ and $M = \max\{f(x)|x \in [a,b]\}$. You may assume that $C[a,b] \subseteq R[a,b]$.)

B. Suppose $f \in C(a,b)$ and also that f is bounded on $[a,b]$. Prove: $f \in R[a,b]$. (Hint: Use a version of the Darboux Criterion.)

C. Prove that $\|\cdot\|_2$ does satisfy the *triangle inequality:* $\|f+g\|_2 \leq \|f\|_2 + \|g\|_2$. (Hint: Write $\|f+g\|_2^2 = \langle f+g, f+g\rangle$, expand using linearity in each variable, and apply the Cauchy-Schwarz inequality.)