Print Your Name Here: ______________________________

Show all work in the space provided and *keep your eyes on your own paper. Indicate clearly* if you continue on the back. Write your **name** at the **top** of the *scratch* sheet *if you will hand it in to be graded.* **No** books, notes, smart/cell phones, I-watches, communication devices, internet devices, or electronic devices are allowed except for a scientific calculator—which is not needed. The maximum total score is 200.

Part I: Short Questions. Answer **12** of the 18 short questions: 8 points each. **Circle** the **numbers** of the 12 questions that you want counted—*no more than 12*! Detailed explanations are not required, but they may help with partial credit and are *risk-free!* Maximum score: 96 points.

1. The (*non-Euclidean*) *taxicab norm* is defined on the plane $\mathbb{R}^2$ by $\|\mathbf{x}\|_t = |x_1| + |x_2|$. True or Give a Counterexample: The taxicab norm satisfies the *parallelogram law:* The sum of the squares of the two diagonals of every parallelogram equals the sum of the squares of the four sides.

2. Let $f(x) = \begin{cases} \sin\frac{\pi}{x} & \text{if } x \in (0,1], \\ 0 & \text{if } x = 0. \end{cases}$ So the graph $G_f = \{(x, f(x)) \mid x \in [0,1]\}$ Is G_f:

a. closed?

b. connected?

c. compact?

3. Let $E \subseteq \mathbb{E}^n$ be any connected set and let $\mathbf{p}$ be a cluster point of E that is not in E itself. *True or False*: $E \cup \{\mathbf{p}\}$ is connected.

4. Let S^1 denote the unit circle centered at the origin in $\mathbb{E}^2$. Let $f : [0, 2\pi) \to S^1$ by $f(x) = (\cos x, \sin x)$. At which point of S^1 is the inverse map, f^{-1}, not continuous?

5. A region $S \subseteq \mathbb{E}^n$ is called *star-shaped* provided that there exists a point $P \in S$ such that for every $Q \in S$ the straight line segment $PQ \subseteq S$. True or False: Every star-shaped region is connected. Draw a sketch of any star-shaped region that is not convex.

6. Find the *interior*, $(\mathbb{Q}^n)^\circ$, and the *closure*, $\overline{\mathbb{Q}^n}$, of $\mathbb{Q}^n$.

7. As clearly as you can, write a valid definition of $\|T\|$ where $T \in \mathcal{L}(\mathbb{E}^n, \mathbb{E}^m)$.

8. $\mathcal{GL}(n, \mathbb{R}) = \{T \in \mathcal{L}(\mathbb{E}^n) \mid \det(T) \neq 0\}$. True or False: $\mathcal{GL}(n, \mathbb{R})$ is:

a. a vector space.

b. a closed subset of $\mathcal{L}(\mathbb{E}^n)$.

c. an open subset of $\mathcal{L}(\mathbb{E}^n)$.

9. True or False: Every *open* set $S \subseteq \mathbb{E}^n$ can be expressed as the union of a family of *mutually disjoint* nonempty open balls.

10. As clearly as you are able, write a valid definition of what it means for $f : \mathbb{E}^n \to \mathbb{E}^m$ to be differentiable at x with $f'(x) = A \in \mathcal{L}(\mathbb{E}^n, \mathbb{E}^m)$.

11. Suppose $\mathbf{f} : \mathbb{E}^2 \to \mathbb{E}^2$ and $\mathbf{g} : \mathbb{E}^3 \to \mathbb{E}^2$ are both differentiable. If $\mathbf{g}(\mathbf{0}) = \mathbf{x}_0$, $[\mathbf{g}'(\mathbf{0})] = \begin{pmatrix} 1 & -1 & 0 \\ -2 & 2 & 3 \end{pmatrix}$ and $[\mathbf{f}'(\mathbf{x}_0)] = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$ find the matrix $[(\mathbf{f} \circ \mathbf{g})'(\mathbf{0})]$ using the standard bases.

12. True or False: There exists $X \in \mathcal{L}\left(\mathbb{E}^2\right)$ for which $\|X + X\| < \|X\| + \|X\|$.

13. Give an example of $T \in \mathcal{L}(\mathbb{E}^2)$ for which $\|T^2\| = 0$ but $\|T\| > 0$.

14. Find the matrix $[\mathbf{f}'(\mathbf{x})]$ and calculate $\det \mathbf{f}'(\mathbf{x})$ if $\mathbf{f} : \mathbb{E}^3 \to \mathbb{E}^3$ by $\mathbf{f}(\mathbf{x}) = (x_1 \cos x_2, x_1 \sin x_2, x_3)$.

15. Suppose $D \subset \mathbb{E}^n$ is compact and $\mathbf{f} \in \mathcal{C}\left(D, \mathbb{E}^m\right)$ is one-to-one. True or Give a Counterexample: If $\mathbf{f}(D)$ is connected, then D is connected.

16. Suppose $\mathbf{f} : \mathbb{E}^n \to \mathbb{E}^m$ is such that every *directional derivative* $D_{\mathbf{v}}\mathbf{f}(\mathbf{x})$ exists for all $\mathbf{v}, \mathbf{x} \in \mathbb{E}^n$. True or False: f must be differentiable on $\mathbb{E}^n$.

17. Let $f \in \mathcal{C}^1\left(\mathbb{E}^1, \mathbb{E}^1\right)$ be defined by $f(x) = \sin x$ for all $x \in \mathbb{E}^1$. Is $f\left(\mathbb{E}^1\right)$ open? If yes, say so. If not, state which hypothesis of the Open Mapping Theorem fails to be true in this example.

18. Suppose $\mathbf{f} \in \mathcal{C}^1(\mathbb{E}^3, \mathbb{E}^2)$ and the matrix $[\mathbf{f}'(\mathbf{0})] = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$. If $(a,b) \in \mathbb{E}^2$, express the matrix $\left[((a,b)\cdot \mathbf{f})'(\mathbf{0})\right]$ in terms of a and b.

Part II: Proofs. *Prove carefully* **4** of the following 6 theorems for 26 points each. **Circle** the *letters* of the 4 proofs to be counted in the list below—*no more than 4!* You may write the proofs below, on the back, or on scratch paper. Maximum total credit: 104 points.

A. Suppose a vector space V is equipped with an inner product $\langle \cdot, \cdot \rangle$, and define the corresponding norm by $\|\mathbf{x}\|^2 = \langle \mathbf{x}, \mathbf{x} \rangle$.

- (**i**) Prove the *Parallelogram Law:* $\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2$. (This law says that the sum of the squares of the diagonals of a parallelogram must equal the sum of the squares of the four sides.)
- (**ii**) Prove the identity $\langle \mathbf{x}, \mathbf{y} \rangle = \dfrac{1}{4}\left(\|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}-\mathbf{y}\|^2\right)$.

B. Prove that part of the Heine-Borel theorem in $\mathbb{E}^n$ that claims that *if* $E \subseteq \mathbb{E}^n$ is *compact, then* E is *closed*. (Hint: Use the definition of compactness to show that E^c, the complement of E, is open.)

C. Let $f(x) = \begin{cases} \sin \frac{\pi}{x} & \text{if } x > 0, \\ 0 & \text{if } x = 0. \end{cases}$ Prove that the graph $G_f = \{(x, f(x)) \mid x \geq 0\}$ of f is a connected subset of $\mathbb{E}^2$. (Hint: Try writing $G_f \subseteq A \cup B$ with A and B open and disjoint. Prove that G_f must be contained entirely within one of the two open sets.)

D. Let $L \in \mathcal{L}(\mathbb{E}^n, \mathbb{E}^m)$ and $T \in \mathcal{L}(\mathbb{E}^m, \mathbb{E}^k)$, so that $T \circ L \in \mathcal{L}(\mathbb{E}^n, \mathbb{E}^k)$.

- (**i**) Prove: $\|T \circ L\| \leq \|T\|\|L\|$.
- (**ii**) Now let $k = m = n$. Denote $T^2 = T \circ T$ and $T^{j+1} = T^j \circ T$ for all $j \in \mathbb{N}$. Show that $\|T^j\| \leq \|T\|^j$.

E. Suppose $\mathbf{f}'(\mathbf{x})$ exists for all $\mathbf{x}$ in a nonempty *open* set $D \subseteq \mathbb{E}^n$, where $\mathbf{f} : D \to \mathbb{E}^m$.

- (**i**) Suppose D is *convex*. If $\mathbf{f}'(\mathbf{x}) \equiv 0 \in \mathcal{L}(\mathbb{E}^n, \mathbb{E}^m)$, prove that $\mathbf{f}$ is a constant function on D.
- (**ii**) Suppose next that D is a *connected* set in $\mathbb{E}^n$, but not necessarily convex. If $\mathbf{f}'(\mathbf{x}) \equiv 0 \in \mathcal{L}(\mathbb{E}^n, \mathbb{E}^m)$ on D, prove that $\mathbf{f}$ is a constant function on D. (Hint: Show that $\mathbf{f}^{-1}(\mathbf{c})$ is open for each $\mathbf{c} \in \mathbf{f}(D)$.)

F. Suppose $D \subset \mathbb{E}^n$ and $S \subset \mathbb{E}^m$ are both open sets. Suppose $\mathbf{f} : D \to S$ is differentiable, one-to-one and onto S, and suppose $\mathbf{f}^{-1}$ is differentiable also. *Prove* that $n = m$ **and** that if $m \neq n$, then $\mathbb{E}^n$ and $\mathbb{E}^m$ are *not diffeomorphic*. (Hints: Apply the *Chain Rule* to the composition of $\mathbf{f}$ and $\mathbf{f}^{-1}$ in both orders. Recall and use a theorem from linear algebra concerning the *rank* and the *nullity* of $T \in \mathcal{L}(\mathbb{E}^n, \mathbb{E}^m)$.)

Solutions and Class Statistics

1. Counterxample: For the unit square $[0,1]\times[0,1]$ each diagonal has length 2 in the taxicab norm and the length of each side is 1 in the taxicab norm. But $2^2+2^2 \neq 1^2+1^2+1^2+1^2$.

2.

a. closed? NO

b. connected? YES

c. compact? NO

3. True.

4. $(1,0)$

5. True, since each of the line segments PQ is connected and they all intersect at P. For a sketch you could pick your favorite Marshall or Sheriff's badge, or the star of David, for example.

6. $(\mathbb{Q}^n)^\circ = \emptyset$, and $\overline{\mathbb{Q}^n} = \mathbb{E}^n$.

7. See Definition 10.1.1 in the text!

8.

a. False: If $L \in \mathcal{GL}(n,\mathbb{R})$ then $L - L = 0 \in \mathcal{L}(\mathbb{E}^n)$ is not because its determinant is 0.

b. False

c. True, since det is a continuous function of T and this is the preimage under det of an open set.

9. False: This would force every open set to be disconnected.

10. See Definition 10.2.1 in the text.

11. $\begin{pmatrix} -3 & 3 & 6 \\ 4 & -4 & -3 \end{pmatrix}$

12. False.

13. For example, let T have the matrix $[T] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

14. $[f'(x)] = \begin{bmatrix} \cos x_2 & -x_1 \sin x_2 & 0 \\ \sin x_2 & x_1 \cos x_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\det f'(\mathbf{x}) = x_1$.

15. True, since f must be a homeomorphism.

16. False. See Exercise 10.39 in the text.

17. $f'(x)$ fails to be nonzero at the points $\left\{\frac{\pi}{2}+n\pi\right\}$.

18. $\left[((a,b)\cdot\mathbf{f})'(\mathbf{0})\right]=\left(\begin{array}{ccc} a+4b & 2a+5b & 3a+6b\end{array}\right)$.

Remarks about the proofs

Proofs are graded for logical coherence. If you have questions about the grading of the proofs on this test, or if you are having difficulty writing satisfactory proofs, *please bring me your test and also the graded homework from which the questions in Part II came.* This will help us to see how you use the corrections to your homework in order to learn to write better proofs. Here are some general remarks about the proofs I have just read.

A. Both identities follow from the *linearity* of the scalar product in both the first and second variables, and from the identity $\|\mathbf{x}\|^2=\langle\mathbf{x},\mathbf{x}\rangle$.

B. If $p\notin E$ we need an open ball around p that lies in E^c. Construct an open covering of E consisting of the open sets $\bar{B}_{\frac{1}{n}}(p)^c$ with $n\in\mathbb{N}$. Now use the existence of a finite subcover of E.

C. If we let g denote the restriction of f to $(0,\infty)$ then the continuity of g implies that G_g is connected. Now suppose $G_f\subset A\dot{\cup}B$, where A and B are both open. Without loss of generality, we suppose that $(0,0)\in A$. There is $r>0$ such that $B_r(0,0)\subseteq A$ and then show that G_g has nonempty intersection with $B_r(0,0)$. Since G_g is connected it follows that $B\cap G_f=\emptyset$ showing that G_f is connected.

D. The key is to use the definition of the norm of a linear function as the infimum of all bounds for that linear function, along with the fact that the norm is itself a bound.

E. For the first part use the Mean Value inequality. For the second part follow the hint.

F. Since Rank + Nullity = dimension of the domain, the dimension of the image of the derivative can be no larger than the dimension of the domain. Use the chain rule in both orders of composition and note that the derivative of a linear map such as the identity must be itself.

Class Statistics

Grade	Test#1	Test#2	Final Exam	Final Grade
90-100 (A)	3	6	3	5
80-89 (B)	2	1	3	1
70-79 (C)	2	0	1	2
60-69 (D)	2	1	1	0
<60 (F)	0	0	0	0
Test Avg	80.6%	89.1%	86.65%	92.4%
HW Avg	6.7	6.2	6.2	6.2
HW/Test Correl	0.91	0.94	0.86	0.86

The Correlation Coefficient is the cosine of the angle between two data vectors in $\mathbb{R}^8$–one dimension for each student enrolled. Thus this coefficient is between 1 and -1, with coefficients above 0.6 being considered strongly positive. The correlation coefficient shown indicates that the test grades in the course have a very strongly positive correlation with performance on the homework.