Print Your Name Here: ______________________________

- **Show all work:** *Answers without work are not sufficient.* We can give credit *only* for what you write! *Indicate clearly if you continue on the back side*, and *write your name at the top of the scratch sheet if you will turn it in for grading.*
- ***Books, notes (electronic or paper), cell phones, smart phones, and internet-connected devices are prohibited!*** A scientific calculator is allowed—but it is not needed. If you use a calculator, you must write out the operations performed on the calculator to show that you know how to solve the problem. *Please do not replace precise answers with decimal approximations.*
- There are **three (3)** problems: maximum total score = 100.

1. (30) All solutions to Equation (1) below are analytic, and have the form $y = \sum_{k=0}^{\infty} a_k x^k$.

$$y'' - xy' + y = 0 \tag{1}$$

a. (15) Find a *recursion formula* for all a_{k+2} in terms of a_k and/or a_{k+1}.

b. (10) Find all those *odd-indexed* coefficients a_{2k-1} that *must* be zero.

c. (5) Write the general solution for y in terms of a_0, a_1 and all those powers of x up to x^8.

2. (40) The differential equation $2xy'' + y' + y = 0$ has a regular singular point at $x = 0$. The Method of Frobenius assures the existence of at least one solution of the form $y = \sum_{m=0}^{\infty} a_m x^{m+r}$.

a. (15) Find the *indicial equation* and its two roots r_1 and r_2, $r_1 \geq r_2$, and note that they do not differ by an integer.

b. (15) Find the *recursion relation* that determines a_{m+1} in terms of a_m and the general value of r. (Suggestion: If you express a_{m+1} in terms of $\mathbf{2}a_m$ this will help you to find the factorial in the denominator of the general coefficient a_m in part (c) below.)

c. (10) *Using the **smaller** root* r_2, and choosing $a_0 = 1$ for convenience, find a_1, a_2, and a_3, and find the *general coefficient* a_m in terms of m. Express the solution y_2 corresponding to r_2 as the sum of a series using the preceding information. *(5 point **bonus** if you express y_2 as an elementary function.)*

3. (30) Use the steps in (a)-(c) to solve

$$xy'' - y' - y = 0. \tag{2}$$

a. (10) Use a new independent variable $t = 2\sqrt{x}$ to transform Eq. (2) into an equation in y, t and derivatives of y with respect to t.

b. (10) Use a new dependent variable $w = \dfrac{y}{x}$ to transform the result of (a) into a modified Bessel equation of the form $t^2 \dfrac{d^2w}{dt^2} + t\dfrac{dw}{dt} - (t^2 + \nu^2)w = 0$.

c. (10) Write the general solution to Equation (2) for y in terms of x.

Solutions

1.

a. $a_{k+2} = \dfrac{(k-1)}{(k+2)(k+1)} a_k$ for all $k \geq 0$, or any equivalent formulation.

b. $a_3 = 0 = a_5 = a_7 = \cdots = a_{2k-1}$ for all $k \geq 2$.

c. $y = a_1 x + a_0 \left[1 - \dfrac{x^2}{2} - \dfrac{x^4}{4!} - \dfrac{3x^6}{6!} - \dfrac{5 \cdot 3x^8}{8!} - \cdots\right]$. The 3 dots indicate that the series goes on ad infinitum. It is a good thing to write the solution in this form, as the general linear combination of two linearly independent solutions. The solution space for a second order linear homogeneous equation must be two-dimensional. Please do *not* multiply out factorials!

2.

a. The *indicial equation* is $r(2r-1) = 0$ so that $r_1 = \frac{1}{2}$ and $r_2 = 0$.

b. The *recursion relation* is $a_{m+1} = \dfrac{-a_m}{(2m+2r+1)(m+r+1)} = \dfrac{-2a_m}{(2m+2r+1)(2m+2r+2)}$. The right-most form above is most useful because the denominator leads readily to a factorial expression in the formula for a_m.

c. Using the *smaller* root $r_2 = 0$, and using $a_0 = 1$, $a_1 = -\frac{2}{2!} = -1$, $a_2 = \frac{2^2}{4!}$, $a_3 = -\frac{2^3}{6!} = -\frac{1}{90}$ and $a_m = (-1)^m \frac{2^m}{(2m)!}$. Thus $y_2 = \displaystyle\sum_{m=0}^{\infty} \frac{(-1)^m 2^m}{(2m)!} x^m = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} \left(\sqrt{2x}\right)^{2m} = \cos\sqrt{2x}$. We remark that the latter expression for y_2 as an elementary function is not required for full credit.

3.

a. $\dfrac{d^2y}{dt^2} - \dfrac{3}{t}\dfrac{dy}{dt} - y = 0$. Be careful with the Chain Rule.

b. $t^2 \dfrac{d^2w}{dt^2} + t\dfrac{dw}{dt} - (t^2+4)w = 0$. Note that the complicated formula in the text is useless for a *modified* Bessel equation. But this shouldn't matter since I have given you the correct changes of variable here.

c. $y = x\left[c_1 I_2(2\sqrt{x}) + c_2 K_2(2\sqrt{x})\right]$. Of course we need to use the *modified* Bessel functions here.

Class Statistics

% Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	4				
80-89 (B)	4				
70-79 (C)	12				
60-69 (D)	3				
0-59 (F)	2				
Test Avg	76.7%	%	%	%	%
HW Avg	97.6%	%	%		