a) 1) $y'' - xy' + y = 0$ $\quad y = \sum_{k=0}^{\infty} a_k x^k$; $y' = \sum_{k=1}^{\infty} k a_k x^{k-1}$; $y'' = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}$

$$\sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} - \sum_{k=1}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} a_k x^k$$

$m = k-2$

$$\sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m - \sum_{m=0}^{\infty} m a_m x^m + \sum_{m=0}^{\infty} a_m x^m = 0$$

$$(m+2)(m+1) a_{m+2} = (m-1) a_m$$

$$\boxed{a_{m+2} = \frac{m-1}{(m+2)(m+1)} a_m} \quad \forall m \geq 0$$

b) $a_3 = \frac{0}{3 \cdot 2} a_1 \Rightarrow a_3 = 0 \Rightarrow \boxed{a_{2k-1} = 0 \ \forall k \geq 2}$

c) a_1 arb. $\quad a_2 = \frac{-1}{2 \cdot 1} a_0 = -\frac{a_0}{2}$

$$a_4 = \frac{1}{4 \cdot 3} a_2 = -\frac{a_0}{4!}$$

$$a_6 = \frac{3}{6 \cdot 5} a_4 = \frac{-3 a_0}{6!}$$

$$a_8 = \frac{5 a_6}{8 \cdot 7} = \frac{-5 \cdot 3}{8!} a_0$$

$$\boxed{y = a_1 x + a_0 \left[1 - \frac{x^2}{2} - \frac{x^4}{4!} - \frac{3}{6!} x^6 - \frac{5 \cdot 3}{8!} x^8 - \cdots\right]}$$

2] $2xy'' + y' + y = 0 \qquad y = \sum_{m=0}^{\infty} a_m x^{m+r}; \; y' = \sum_{m=0}^{\infty} (m+r) a_m x^{m+r-1}$

$y'' = \sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r-2}$

a)

$$\sum_{m=0}^{\infty} 2(m+r)(m+r-1) a_m x^{m+r-1} + \sum_{m=0}^{\infty} (m+r) a_m x^{m+r-1} + \sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

$k = m-1$ reindex

rename

$$\sum_{k=-1}^{\infty} 2(k+1+r)(k+r) a_{k+1} x^{k+r} + \sum_{k=-1}^{\infty} (k+1+r) a_{k+1} x^{k+r} + \sum_{k=0}^{\infty} a_k x^{k+r} = 0$$

Let $k=-1$: $2r(r-1)a_0 + ra_0 = 0 \qquad a_0 \neq 0$

$2r^2 - 2r + r = 0$

$$\boxed{2r^2 - r = 0 \quad (2r-1)r = 0 \quad r_1 = \tfrac{1}{2};\; r_2 = 0}$$

b) $\forall k \geq 0$: $[2(k+1+r)(k+r) + (k+1+r)]\, a_{k+1} = -a_k$

$(k+1+r)[2(k+r)+1]\, a_{k+1} = -a_k$

$(2k+2r+1)(k+1+r)\, a_{k+1} = -a_k$

$$\boxed{a_{k+1} = \frac{-a_k}{(2k+2r+1)(k+1+r)} = \frac{-2a_k}{(2k+2r+2)(2k+2r+1)}}$$

c) use $r_2 = 0$, $a_0 = 1$

$a_{k+1} = \dfrac{-2a_k}{(2k+2)(2k+1)}$

$a_1 = \dfrac{-2}{2} = -1$

$a_2 = \dfrac{-2a_1}{4 \cdot 3} = \dfrac{2^2}{4!} =$

$a_3 = \dfrac{-2a_2}{6 \cdot 5} = \dfrac{-2^3 a_0}{6!} = \dfrac{-8}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} = \dfrac{-1}{90}$

$\vdots$

$$\boxed{a_m = \frac{(-1)^m 2^m}{(2m)!}}$$

$$y_2 = \sum_{m=0}^{\infty} \frac{(-1)^m 2^m x^m}{(2m)!} = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} (2x)^m = \sum_{m=0}^{\infty} \frac{(-1)^m}{2m!} \left[\sqrt{2x}\right]^{2m}$$

$$= \cos\sqrt{2x}$$

3] $xy'' - y' - y = 0$ $\qquad t = 2\sqrt{x} = 2x^{1/2}$ $\qquad x = \frac{t^2}{4}$

a) $y' = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$ $\qquad \frac{dt}{dx} = x^{-1/2} = \frac{1}{\sqrt{x}} = \frac{2}{t}$

$$y' = \frac{2}{t}\frac{dy}{dt}$$

$$y'' = \frac{2}{t}\frac{d}{dt}\left(\frac{2}{t}\frac{dy}{dt}\right) = \frac{2}{t}\left(\frac{2}{t}\frac{d^2y}{dt^2} - \frac{2}{t^2}\frac{dy}{dt}\right) = \frac{4}{t^2}\frac{d^2y}{dt^2} - \frac{4}{t^3}\frac{dy}{dt}$$

$$\frac{t^2}{4}\left(\frac{4}{t^2}\frac{d^2y}{dt^2} - \frac{4}{t^3}\frac{dy}{dt}\right) - \frac{2}{t}\frac{dy}{dt} - y = 0$$

$$\frac{d^2y}{dt^2} - \frac{1}{t}\frac{dy}{dt} - \frac{2}{t}\frac{dy}{dt} - y = 0$$

$$\boxed{\frac{d^2y}{dt^2} - \frac{3}{t}\frac{dy}{dt} - y = 0}$$

b) $w = \frac{y}{x} = \frac{4y}{t^2}$ $\qquad y = \frac{t^2}{4}w$

$$\frac{dy}{dt} = \frac{t^2}{4}\frac{dw}{dt} + \frac{t}{2}w$$

$$\frac{d^2y}{dt^2} = \frac{t^2}{4}\frac{d^2w}{dt^2} + t\frac{dw}{dt} + \frac{1}{2}w$$

$$\frac{t^2}{4}\frac{d^2w}{dt^2} + t\frac{dw}{dt} + \frac{t}{2}w - \frac{3}{t}\left(\frac{t^2}{4}\frac{dw}{dt} + \frac{1}{2}w\right) - \frac{t^2}{4}w = 0$$

4 $\quad \frac{t^2}{4}\frac{d^2w}{dt^2} + \frac{t}{4}\frac{dw}{dt} + \frac{1}{2}w - \frac{t^2}{4}w - \frac{3}{2}w = 0$

$$t^2\frac{d^2w}{dt^2} + t\frac{dw}{dt} + 2w - t^2w - 6w = 0$$

$$t^2\frac{d^2w}{dt^2} + t\frac{dw}{dt} - (t^2+4)w = 0$$

$$w = c_1 I_2(t) + c_2 K_2(t)$$

$$\boxed{y = x\left[c_1 I_2(2\sqrt{x}) + c_2 K_2(2\sqrt{x})\right]}$$