Print Your Name Here: _____

- **Show all work:** *Answers without work are not sufficient.* We can give credit *only* for what you write! *Indicate clearly if you continue on the back side*, and write your name at the top of the scratch sheet if you will turn it in for grading.
- ***Books, notes (electronic or paper), cell phones, smart phones, and internet-connected devices are prohibited!*** A scientific calculator is allowed—but it is not needed. If you use a calculator, you must write out the operations performed on the calculator to show that you know how to solve the problem. *Please do not replace precise answers with decimal approximations.*
- There are **three (3)** problems: maximum total score = 100.

1. (30) Let $f(x) = 1 - |x|$ on $[-1, 1]$. Find *all* the Fourier coefficients in the Fourier series $f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\pi x + b_n \sin n\pi x)$.

2. (30) Consider the *square wave* $f(x) = \begin{cases} 1 & \text{if } 0 \leq x < \pi \\ 0 & \text{if } -\pi \leq x < 0 \end{cases}$ with period 2π. Express $f(x)$ as the sum of a Fourier series.

3. (40) Consider the (Cauchy-Euler) differential equation

$$x^2y'' + xy' + \lambda y = 0,\ \lambda > 0. \tag{1}$$

a. (15) Find the general real-valued solution to (1). (Hint: Recall that $x^r = e^{r \ln x}$ and use Euler's formula.)

b. (15) Find all the *eigenvalues* λ_n and corresponding *eigenfunctions* y_n satisfying the boundary conditions $y'(1) = 0$ and $y'(e) = 0$ on the derivative of y.

c. (10) Put the equation (1) into *self-adjoint form* and identify the *weight-function* $p(x)$ for which $\int_1^e y_m(x)y_n(x)\,p(x)dx = 0$ whenever $m \neq n$. Now write out the latter integral using your solution functions and your $p(x)$ and use elementary calculus to reduce this integral to one that is already known to be zero.

Solutions

1. $a_0 = \text{avg}_{[-1,1]}(f) = \dfrac{1}{2}$ and $b_n = 0$ since f is even. If $n \geq 1$ we have $a_n = \int_{-1}^{1}(1-|x|)\cos n\pi x\,dx = 2\int_0^1 (1-x)\cos n\pi x\,dx = \frac{2}{n^2\pi^2}(1-(-1)^n) = \begin{cases} 0 & \text{if } n \text{ even} \\ \frac{4}{\pi^2 n^2} & \text{if } n \text{ odd} \end{cases}$ if $n \geq 1$. Here we have used the evenness of f to simplify $|x|$ by doubling the integral from 0 to 1. Remember that $|x| = -x$ if $x < 0$. One needs to use integration by parts. Many students introduced unnecessary work leading to errors because of failing to use the benefits of an even or an odd integrand.

2. Since $f(x) - \frac{1}{2}$ is an *odd* function it has a pure sine series: $f(x) - \dfrac{1}{2} = \displaystyle\sum_1^\infty b_n \sin nx$, where $b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}\left[f(x) - \frac{1}{2}\right]\sin nx\,dx = \frac{2}{\pi}\int_0^\pi \frac{1}{2}\sin nx\,dx$. Thus $f(x) = \dfrac{1}{2} + \displaystyle\sum_{n=1}^\infty \frac{1-(-1)^n}{n\pi}\sin nx$.

3. This is almost the same as Webassign 12.5 problem 2.

a. (15) Substituting $y = x^r$ into (1), we find $r = \pm i\sqrt{\lambda}$, and $x^{i\sqrt{\lambda}} = e^{i\sqrt{\lambda}\ln x} = \cos(\sqrt{\lambda}\ln x) + i\sin(\sqrt{\lambda}\ln x)$. This yields the general real solution $y = c_1\cos(\sqrt{\lambda}\ln x) + c_2\sin(\sqrt{\lambda}\ln x)$.

b. (15) $\lambda_n = (n\pi)^2$ and $y_n = \cos(n\pi\ln x)$, $n = 1, 2, 3, \ldots$.

c. (10) The self-adjoint form is $(xy')' + \dfrac{\lambda}{x}y = 0$ and $p(x) = \frac{1}{x}$. Finally, $\displaystyle\int_1^e \cos(m\pi\ln x)\cos(n\pi\ln x)\frac{1}{x}\,dx = \int_0^1 \cos m\pi u\cos n\pi u\,du = 0$ where we have substituted $u = \ln x$.

Class Statistics

% Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	14	10			
80-89 (B)	2	3			
70-79 (C)	2	1			
60-69 (D)	0	4			
0-59 (F)	1	0			
Test Avg	88.8%	86.8%	%	%	%
HW Avg	93.6%	92.27%	%		
HW/Test Correl.	–	0.71			

The Correlation Coefficient is the cosine of the angle between two data vectors in 18-dimensional space: one dimension for each student enrolled. Thus this coefficient is between 1 and -1, with coefficients above 0.6 being considered strongly positive. The correlation coefficient shown indicates that the test grades in the course have a strongly positive correlation with performance on the homework.