I have read, understood, and complied with the instructions in the box below. Legible

***Signature* and LSU ID #:** ______________________________

- Download and print a copy of this test on 8.5 by 11 inch letter size paper. If no printer is available, make a hand-written facsimile. Be sure to sign the statement above.
- **Show all work** in the space provided. Grading is based on the correctness of the work shown. We can give credit *only* for what you write! *Indicate clearly if you continue a problem on a second page.*
- ***You may use your text book, Zoom recordings of our class meetings, your class notes, and your WebAssign Homework!*** *However, no other sources or communication devices may be used.* ***All work must be your own.*** If you use a calculator, you *must still write out all operations performed* on the calculator. *Do not replace* precise answers, such as $\sqrt{2}$, π, or $\cos \frac{\pi}{7}$ with decimal approximations. *Make all obvious simplifications.* Submit only your own work!
- This is a take-home test on an *honor system.* You may take as much time as you like, but **I must receive your completed test by email no later than 2:30 PM on Saturday, November 21.** If you have no device that scans your work directly to a single pdf file, then photograph your pages in the correct order with your phone and save as jpeg, then try this please: put the jpeg files into your computer, highlight the whole group of pictures, right click PRINT and then select PRINT TO PDF. That way I can receive a multipage PDF file which is possible to grade in a way you will be able to read later. Email that file to me **rich@math.lsu.edu** as soon as you are ready but no later than Saturday, November 21, at 2:30 PM. *These instructions express my trust and confidence in your integrity and good character.*

Before you send me your pdf file containing all your pages as one single file, please make sure everything is legible. Use a sufficiently dark writing instrument for your test and make sharp, clear images, so I can read them. I simply cannot grade what I cannot read. Thank you for your consideration in this!

1. (30) The function $f(x) = 1 - x$ on the domain $(0, \pi)$ has a Fourier half-range expansion

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx. \tag{1}$$

a. (5) Evaluate $\displaystyle\int_0^{\pi} \sin^2 nx\, dx$, which is the square of the norm of the function $\sin nx$.

b. (20) Find the value of b_n for all positive integers n.

c. (5) The sum of the series in Equation (1) is equal for all x in $(-\pi, 0)$ to a linear function $f_o(x) = ax + b$. Find a and b.

2. (30) For the Sturn-Liouville problem $y'' + \lambda y = 0,\ y(0) = 0,\ y'(\pi) = 0$ on $[0, \pi]$ find all the eigenvalues λ_n and the corresponding eigenfunctions y_n. Remember to show how you determined the complete set of eigenvalues and eigenfunctions. by considering all the possible cases:

a. $\lambda = -\mu^2 < 0$

b. $\lambda = 0$

c. $\lambda = \mu^2 > 0$

3. (40) Consider the heat equation $u_t = \kappa^2 u_{xx}$ on the interval $[0, L]$ with the *initial condition* $u(x, 0) = f(x)$ and the *boundary conditions* $u(0, t) = 0,\ u_x(L, t) = 0$ for all $t \geq 0$. Thus the left end is held at temperature 0 and the right end is *insulated*.

a. (10) Seeking solutions that split as $u(x, t) = F(x)G(t)$, separate variables using the separation constant λ (***not*** with a minus sign) and show that $\lambda = 0$ is *not* an eigenvalue.

b. (10) You may assume there are no positive eigenvalues. Find all the negative eigenvalues $\lambda_n = -\mu_n^2$ and the corresponding eigenfunctions $F_n(x)$.

c. (10) Find the eigenfunctions $G_n(t)$ corresponding to each eigenvalue λ_n.

d. (10) Express the final solution $u(x, t) = \displaystyle\sum_{n=?}^{\infty} b_n G_n(t) F_n(x)$, filling in the results from the preceding parts. Use the initial condition to express the Fourier coefficients b_n in terms of an appropriate integral.

Solutions

1.

a. $\displaystyle\int_o^{\pi} \sin^2 nx\,dx = \frac{\pi}{2}.$

b. $b_n = \dfrac{2}{n\pi}[1 + (\pi - 1)(-1)^n].$

c. Since the sum of a convergent sine series is an odd function $f_o(x) = -f(-x) = -(1 - -x) = -1 - x$ for all x in $(-\pi, 0)$). That is, $a = -1 = b$. See Fig. (1).

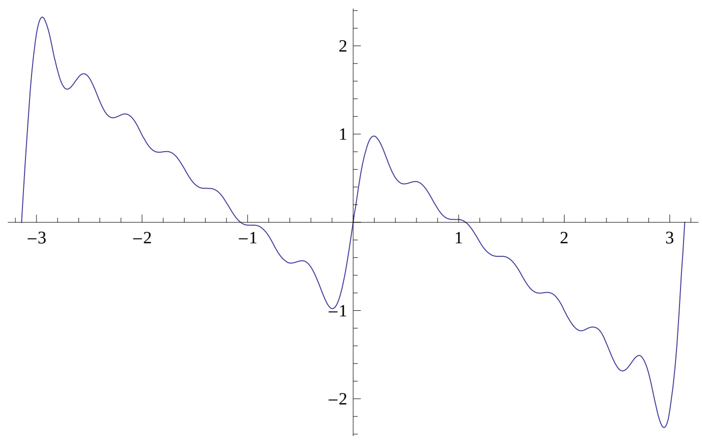

Figure 1: Fourier Series summed to N=15 in problem 1.

2. If $\lambda = -\mu^2$, a negative number, the general solution is $y = c_1 e^{\mu x} + c_2 e^{-\mu x}$ yielding two linear equations from the boundary condions. Elementary algebra shows that both coefficients must be zero. If $\lambda = 0$ then $y = mx + b$ and again the boundary conditions yield two linear equations resulting in $m = 0 = b$ so that there is only a trivial solution. The only case that yields eigenvalues for this Sturm-Liouville problem is $\lambda = \mu^2 > 0$ so that $y = c_1 \cos\mu x + c_2 \sin\mu x$. The boundary conditions then show that $\lambda_n = \left(n + \frac{1}{2}\right)^2$ for all $n = 0, 1, 2, \ldots$ and $y_n = \sin\left(n + \frac{1}{2}\right)x$. These formulas will be slightly different if you start counting from $n = 1$ instead of counting from $n = 0$.

3.

a. $u(x,t) = F(x)G(t)$, so $\dfrac{F''(x)}{F(x)} = \lambda = \dfrac{\dot{G}(t)}{G(t}$. If $\lambda = 0$ then $F(x) = ax + b$ and the two boundary conditions force $a = 0 = b$.

b. $\lambda_n = -\left(n + \frac{1}{2}\right)^2 \dfrac{\pi^2}{L^2}$ and the corresponding eigenfunctions $F_n(x) = \sin\left(n + \frac{1}{2}\right)\dfrac{\pi}{L}x$ for all $n = 0, 1, 2, \ldots$.

c. $G_n(t) = e^{-\left(n+\frac{1}{2}\right)^2 (\kappa\frac{\pi}{L})^2 t}$ corresponding to λ_n for all $n = 0, 1, 2, \ldots$.

d. $u(x,t) = \displaystyle\sum_{n=0}^{\infty} b_n e^{-\left(n+\frac{1}{2}\right)^2 (\kappa\frac{\pi}{L})^2 t} \sin\left(n + \frac{1}{2}\right)\frac{\pi}{L}x$. Finally, $b_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\left(n + \frac{1}{2}\right)\frac{\pi}{L}x\,dx.$

Class Statistics

% Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	15	10			
80-89 (B)	7	10			
70-79 (C)	2	4			
60-69 (D)	2	2			
0-59 (F)	0	0			
Test Avg	91.7%	86.1%	%	%	%
HW Avg	84.43%	85.13%	%		
HW/Test Correl	–	0.65			

The Correlation Coefficient is the cosine of the angle between two data vectors in $\mathbb{E}^{26}$–one dimension for each student enrolled in a 26-dimensional Euclidean space. Thus this coefficient is between 1 and -1, with coefficients above 0.6 being considered strongly positive. The correlation coefficient shown indicates that the test grades in the course have a strongly positive correlation with performance on the homework.