Print Your Name Here: ______________________________

- **Show all work:** *Answers without work are not sufficient.* We can give credit *only* for what you write! *Indicate clearly if you continue on the back side*, and *write your name at the top of the scratch sheet if you will turn it in for grading.*
- ***Books, notes (electronic or paper), cell phones, smart phones, and internet-connected devices are prohibited!*** A scientific calculator is allowed—but it is not needed. If you use a calculator, you must write out the operations performed on the calculator to show that you know how to solve the problem. *Please do not replace precise answers with decimal approximations.*
- There are **three (3)** problems: maximum total score = 100.

1. (30) If $f(x) = \begin{cases} \pi & \text{if } -\pi \le x \le 0 \\ \pi - x & \text{if } 0 < x \le \pi \end{cases}$ then $f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ for all x in $[-\pi, \pi]$.

a. (5) Find the value of a_0.

b. (15) Find the value of a_n for all $n = 1, 2, 3, \ldots$.

c. (10) Express b_n in terms of definite integral(s) for all $n = 1, 2, 3, \ldots$ *but do not evaluate.*

2. (35) Let $f(x) = x$ on $[0, 1]$. There is a Fourier *cosine half-range* expansion $f(x) = \sum_{n=0}^{\infty} a_n \cos n\pi x$.

a. (5) Find a_0.

b. (10) For $n \geq 1$, express a_n in terms of a definite integral, but do not evaluate this integral yet.

c. (10) Evaluate the integral to find a_n for all $n \geq 1$.

d. (10) $\sum_{n=0}^{\infty} a_n \cos n\pi x$ converges for all real numbers x. *Either* sketch the graph of the sum of this Fourier series on the interval [-1,1], *or simply give the familiar name* to this sum on [-1,1].

3. (35) Consider the Sturm-Liouville problem

$$y'' + \lambda y = 0, \quad y(0) = 0 = y'(1) \tag{1}$$

Note that the *right* boundary condition is on the *derivative* y'.

a. (25) Find *all* the *positive* eigenvalues $\lambda_n = \nu_n^2 > 0$ *and the corresponding eigenfunctions* y_n. Specify the range of values for n so that the eigenfunctions y_n are all distinct.

b. (5) The differential equation in (1) is already in the self-adjoint form $(ry')' + (q + \lambda p)y = 0$. In (1), what are the functions $r(x), q(x)$ and $p(x)$?

c. (5) If $m \neq n$ write out the weighted inner product $\langle y_m, y_n \rangle_p = 0$ as a definite integral, showing the correct upper and lower limits of integration and the correct integrand, *but do not evaluate the integral.*

Solutions

1.

a. $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{3}{4}\pi$. This is immediate from the graph which is clearly the upper boundary of a square region of side π plus an isosceles right triangular region with legs π. This gives us $\frac{3}{2}$ of a square, totaling $\frac{3}{2}\pi^2$ area. No calculus is needed for this. Then we divide by the length of the interval to get the average height of $f = \frac{3}{4}\pi$.

b. $a_n = \frac{1-(-1)^n}{\pi n^2}$ for all $n = 1, 2, 3, \ldots$.

c. $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \pi \sin nx\,dx - \frac{1}{\pi}\int_0^{\pi} x\sin nx\,dx \left(= \text{(not required) } \frac{(-1)^n}{n}\right)$ for all $n = 1, 2, 3, \ldots$. Remark: A missing comma crashed a Mars Rover into Mars. Just think what a missing or unclosed bracket would do. It will serve you well to write accurately.

2.

a. $a_0 = \frac{1}{2}$, since the constant term in every Fourier series is the *average* value of $f(x)$ on the given interval.

b. If $n \geq 1$, $a_n = \dfrac{2}{1}\displaystyle\int_0^1 x\cos n\pi x\,dx$.

c. $a_n = \dfrac{2}{\pi^2 n^2}\left((-1)^n - 1\right)$ for all $n \geq 1$.

d. A pure Fourier cosine series converges to an *even* function, which must be $|x|$ on [-1,1]. (Remark: This graph of $|x|$ on [-1,1] is repeated with period 2 on the whole real line, making a saw-tooth pattern.)

3.

a. $\lambda_n = (n+\frac{1}{2})^2\pi^2,\ n = 0, 1, 2, 3, \ldots$, and the corresponding eigenfunctions $y_n = \sin(n+\frac{1}{2})\pi x$.

b. (5) $r(x) = 1$, $q(x) = 0$ and $p(x) = 1$.

c. (5) If $m \neq n$, $\displaystyle\int_0^1 \sin\left(n+\frac{1}{2}\right)\pi x\,\sin\left(m+\frac{1}{2}\right)\pi x\,dx = 0$

Class Statistics

% Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	6	14			
80-89 (B)	3	6			
70-79 (C)	12	3			
60-69 (D)	3	1			
0-59 (F)	3	0			
Test Avg	77.1%	89.6%	%	%	%
Cumulative HW Avg	96.94%	94.87%	%	%	%
HW/Test Correl	–	0.91			

The Correlation Coefficient is the cosine of the angle between two data vectors in $\mathbb{R}^{28}$–one dimension for each student enrolled. Thus this coefficient is between 1 and -1, with coefficients above 0.6 being considered strongly positive. The correlation coefficient shown indicates that the test grades in the course have a very strongly positive correlation with performance on the homework.