Print Your Name Here: ______________________________

- **Show all work:** *Answers without work are not sufficient.* We can give credit *only* for what you write! *Indicate clearly if you continue on the back side*, and *write your name at the top of the scratch sheet if you will turn it in for grading.*
- ***Books, notes (electronic or paper), cell phones, smart phones, and internet-connected devices are prohibited!*** A scientific calculator is allowed—but it is not needed. If you use a calculator, you must write out the operations performed on the calculator to show that you know how to solve the problem. *Please do not replace precise answers with decimal approximations.*
- There are **three (3)** problems: maximum total score = 100.

1. (20) Use *separation of variables* with *separation constant* $\lambda \neq 0$ to find all split solutions $u(x,y) = F(x)G(y)$ to the equation $xu_x = yu_y$.

2. (20) Use *separation of variables* with *separation constant* $\lambda \neq 0$ to find all split solutions $u(x, y) = F(x)G(y)$ to the equation $u_x + u_y = u$.

3. (60) Consider the heat equation $u_t = \kappa^2 u_{xx}$ on the interval $[0, \pi]$ for all $t \geq 0$, with the *boundary* conditions $u(0,t) = 0 = u(\pi, t)$ for all $t \geq 0$ and the *initial* condition $u(x, 0) = f(x)$.

a. (20) First consider all *split* solutions of the form $u(x,t) = F(x)G(t)$. Separate variables with a *separation constant* λ. Write an ordinary differential equation for $F(x)$ with boundary conditions *and* an ordinary differential equation for $G(t)$.

b. (20) Find all *negative* eigenvalues $\lambda_n = -\mu^2 < 0$ and the corresponding eigenfunctions $F_n(x)$ along with the matching functions $G_n(t)$. Write $u_n(x,t)$ and give the correct range of values for n.

c. (10) Find the solution $u(x,t) = \sum_{n=1}^{\infty} b_n u_n(x,t)$ for the boundary value problem satisfying the *initial condition* $u(x, 0) = f(x)$. Express the coefficients b_n in terms of an integral involving $f(x)$.

d. (10) Find $u(x,t)$ if the initial condition is $u(x, 0) = f(x) = 3\sin(3x)$.

Solutions

1. $u(x,y) = c(xy)^\lambda$ where c is a constant. The equations for $F(x)$ and $G(y)$ are first order Cauchy-Euler equations that can be solved by substituting $F(x) = x^r$ and $G(y) = y^r$ There is no reason to consider different cases for $\lambda > 0$ or $\lambda < 0$, since for first order ODEs the equation for r is not quadratic so r can't be complex. (This was WebAssign #3 for Sec. 13.1.)

2. $u(x,y) = ce^{\lambda(x-y)+y}$ where c is a constant. (The roles of x and y can be reversed depending on how you write the separation.) Since the ODEs for F and G are constant coefficient linear ODEs they can be solved by substituting e^{rx} or e^{ry} respectively. Or, being first order, you can use an elementary version of separation of variables for ODEs of first order. (This was WebAssign #2 for Sec. 13.1.)

3. (This was WebAssign #5 for Sec. 13.3.)

a. $F''(x) - \lambda F(x) = 0$, with $F(0) = 0 = F(\pi)$ and $G'(t) - \lambda\kappa^2 G(t) = 0$.

b. $\lambda_n = -n^2 < 0$, $F_n(x) = \sin(nx)$, $G_n(t) = e^{-(n\kappa)^2 t}$, and $u_n(x,t) = e^{-(n\kappa)^2 t}\sin(nx)$, and $n = 1, 2, 3, \ldots$.

c. $u(x,t) = \sum_{n=1}^{\infty} b_n e^{-(n\kappa)^2 t}\sin(nx)$, where $b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\,dx$.

d. $u(x,t) = 3e^{-9\kappa^2 t}\sin(3x)$. Note that all but one of the integrals from part (c) vanish because of the orthogonality of the sine waves on $[0,\pi]$.

Class Statistics

% Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	6	14	15		
80-89 (B)	3	6	4		
70-79 (C)	12	3	3		
60-69 (D)	3	1	1		
0-59 (F)	3	0	0		
Test Avg	77.1%	89.6%	90.1%	%	%
Cumulative HW Avg	96.94%	94.87%	97.76%	%	%
HW/Test Correl	–	0.91	0.65		

The Correlation Coefficient is the cosine of the angle between two data vectors in $\mathbb{R}^{25}$–one dimension for each student enrolled. Thus this coefficient is between 1 and -1, with coefficients above 0.6 being considered strongly positive. The correlation coefficient shown indicates that the test grades in the course have a very strongly positive correlation with performance on the homework.