Print Your Name Here: ______________________________

- **Show all work:** *Answers without work are not sufficient.* We can give credit *only* for what you write! *Indicate clearly if you continue on the back side*, and write your name at the top of the scratch sheet if you will turn it in for grading.
- ***Books, notes (electronic or paper), cell phones, smart phones, and internet-connected devices are prohibited!*** A scientific calculator is allowed—but it is not needed. If you use a calculator, you must write out the operations performed on the calculator to show that you know how to solve the problem. *Please do not replace precise answers with decimal approximations.*
- There are **six (6)** problems: maximum total score = 200.

1. (30) The differential equation $y'' + 2xy' - 2y = 0$ has analytic solutions of the form $y = \sum_{n=0}^{\infty} a_n x^n$.

a. (15) Find a recursion formula for a_{n+2} in terms of a_n, $n \geq 0$.

b. (10) The general solution has the form $y = c_1 y_1(x) + c_2 y_2(x)$ where $y_1(x)$ is a polynomial of odd degree, and $y_2(x)$ is not a polynomial. Find a suitable polynomial solution $y_1(x)$, showing why all the higher odd-indexed coefficients must be zero.

c. (5) Choosing $a_0 = 1$ write out the first 3 nonzero terms of the non-polynomial power series solution $y_2(x)$.

2. (30) The equation $2xy'' - y' + 2y = 0$ has a regular singular point at $x = 0$. Using the method of Frobenius, there is a solution of the form $y = x^r \sum_{n=0}^{\infty} a_n x^n$.

a. (10) Write the indicial equation.

b. (10) Find the allowable values of r in the indicial equation.

c. (10) Find the recursion formula for a_{n+1} in terms of a_n, n and r. You are *not* asked to find the general solution for this problem.

3. (30) Let $f(x) = \begin{cases} 1, & 0 < x \le \pi \\ 0, & x = 0 \\ -1, & -\pi \le x < 0. \end{cases}$

a. (20) Find all the Fourier coefficients in the expansion

$$f(x) = a_0 + \sum_1^\infty (a_n \cos nx + b_n \sin nx) \tag{1}$$

.

b. (5) Calculate the (very easy) integral $\int_{-\pi}^{\pi} f(x)^2\, dx$.

c. (5) By Eq. (1),
$\int_{-\pi}^{\pi} f(x)^2\, dx \;=\; \langle f, f\rangle \;=\; \left\langle a_0 + \sum_1^\infty (a_n \cos nx + b_n \sin nx), a_0 + \sum_1^\infty (a_n \cos nx + b_n \sin nx)\right\rangle$. Use the orthogonality relations and the results of (a) and (b) to find the value of
$\sum_{n \text{ odd}} \frac{1}{n^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$.

4. (40) The differential equation with *boundary conditions*

$$y'' + 4y' + (\lambda + 4)y = 0, \quad y(0) = 0 = y(1) \tag{2}$$

becomes a Sturm-Liouville problem (in self-adjoint form) with equation $(r(x)y')' + (q(x) + \lambda p(x))y = 0$ if you multiply both sides of the equation by e^{kx} for some k.

a. (10) *Choose* the correct value of k and do the multiplication to *write* Eq. (2) in Sturm-Liouville (self-adjoint) format. *Find the weight function* $p(x)$.

b. (15) Set $y = e^{rx}$ to solve the *original* Eq. (2) to find a *complete set of eigenvalues* λ_n and corresponding (real-valued) eigenfunctions $y_n(x)$ for the interval $[0, 1]$. (You will use Euler's formula here to evaluate $e^{(a+ib)x} = e^{ax}e^{ibx}$.)

c. (10) Write the orthogonality relation $\langle y_n, y_m \rangle_p = 0$ if $m \neq n$ in the form of an integral vanishing if $m \neq n$. (You do not need to evaluate the resulting integral.)

d. (5) If f is integrable on $[0, 1]$, it can be expanded in the form $f(x) = \sum_{m=1}^{\infty} c_m y_m(x)$. Using what you know about orthogonal function expansions, express c_m in terms of $\|y_m\|$ and an integral involving both $f(x)$ and $y_m(x)$. (*Do not* evaluate the integral or the value of $\|y_m\|$ for this problem.)

5. (30) Consider the wave equation $u_{tt} = u_{xx}$ on $[0, 1]$ with the *boundary conditions* $u(0, t) = 0 = u(1, t),\ t \geq 0$ and the *initial conditions* $u(x, 0) = \sin 2\pi x,\ u_t(x, 0) = \sin 3\pi x,\ 0 \leq x \leq 1$.

a. (15) Use the separation constant λ and find all eigenvalues λ_n with all corresponding split solutions $u_n(x, t) = F_n(x)G_n(t)$ satisfying the *boundary conditons.*

b. (15) Write the general solution satisfying the *boundary conditions* as a sum of all the split solutions $u_n(x, t)$ (using the principle of superposition) and determine all the necessary coefficients so as to find the unique solution satisfying the *initial conditions* as well as the boundary conditions.

6. (40) Consider the wave equation $u_{tt} = u_{xx} + u_{yy}$ on the square $0 \leq x, y \leq \pi$ with the *boundary conditions* $u(0, y, t) = 0 = u(\pi, y, t)$ and $u(x, 0, t) = 0 = u(x, \pi, t)$ for all $t \geq 0$, and with the *initial conditions* $u(x, y, 0) = f(x, y)$ and $u_t(x, y, 0) = 0$. We will begin by finding all the split solutions $u(x, y, t) = F(x)G(y)H(t)$ as follows.

a. (10) Use the constant λ to separate the functions of x from the functions of y and t. (Time Saver: λ will be negative.) Determine all the eigenvalues λ_m and the corresponding eigenfunctions $F_m(x)$.

b. (10) Given λ_m as in part (a), use the constant μ to separate the functions of y from the functions of t. (Time Saver: μ will be negative.) Determine all the eigenvalues μ_n and the corresponding eigenfunctions $G_n(y)$.

c. (10) Given λ_m and μ_n from parts (a) and (b), Find the corresponding $H_{m,n}(t)$ and write out the general split solutions $u_{m,n}(x, y, t)$ satisfying the boundary conditions.

d. (10) Write out the general solution $u(x, y, t)$ as the sum of an infinite series satisfying all the boundary conditions. Now express all the coefficients as double integrals involving $f(x, y)$ to find the unique solution satisfying *both initial conditions* as well as the boundary conditions.

Solutions

1.

a. $a_{n+2} = -\dfrac{2(n-1)a_n}{(n+2)(n+1)}$

b. $y_1(x) = x$ or any nonzero constant multiple thereof. The recursion formula shows that $a_3 = 0$ and the same for all higher odd-indexed coefficients.

c. (5) $y_2(x) = 1 + x^2 - \dfrac{1}{6}x^4 - \cdots$.

2.

a. $2r^2 - 3r = 0$.

b. $r_1 = \frac{3}{2}$ and $r_2 = 0$.

c. $a_{n+1} = \dfrac{-2a_n}{(n+r+1)(2(n+r)-1)}$.

3.

a. $a_n = 0$ for all n since f is an odd function. $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} \sin nx\,dx = 0$ if n is even, but $b_n = \dfrac{4}{\pi n}$ for each odd value of n. It helps to avoid errors if one remembers that the product of two odd functions is even, simplifying the domain of integration as shown above.

b. $\displaystyle\int_{-\pi}^{\pi} f(x)^2\,dx = \int_{-\pi}^{\pi} 1\,dx = 2\pi$

c. $\displaystyle\int_{-\pi}^{\pi} f(x)^2\,dx = 2\pi = \left\langle \sum_{n\text{ odd}} \frac{4}{\pi n}\sin nx, \sum_{n\text{ odd}} \frac{4}{\pi n}\sin nx \right\rangle = \sum_{n\text{ odd}} \frac{16}{\pi^2 n^2}\int_{-\pi}^{\pi} \sin^2 nx\,dx$ and $\displaystyle\sum_{n\text{ odd}} \frac{1}{(n)^2} = \frac{\pi^2}{8}$.

4.

a. $k = 4$: The differential equation is rewritten as $\left(e^{4x}y'\right)' + \left(e^{4x} + \lambda e^{4x}\right)y = 0$. This has the Sturm-Liouville (self-adjoint) format, with the weight function $p(x) = e^{4x}$.

b. $\lambda_n = n^2\pi^2$ and $y_n(x) = e^{-2x}\sin n\pi x$ for all $n = 1, 2, 3, \ldots$.

c. $\displaystyle\langle y_n, y_m\rangle_p = 0 = \int_0^1 e^{-2x}\sin m\pi x\, e^{-2x}\sin n\pi x\, e^{4x}dx = \int_0^1 \sin m\pi x\,\sin n\pi x\,dx = 0$ if $m \neq n$. Remember that the p-scalar product includes the weight function $p(x)$!

d. $\displaystyle c_m = \frac{1}{\|y_m\|^2}\int_0^1 f(x)\,e^{-2x}\sin m\pi x\, e^{4x}dx = \frac{1}{\|y_m\|^2}\int_0^1 f(x)\,e^{2x}\sin m\pi x\,dx$. Remember that the p-scalar product includes the weight function $p(x)$!

5. It is important to know that $\sin n\pi x$ is orthogonal to $\sin m\pi x$ on $[0,1]$ when the positive integers m an n are different. Thus the Fourier coefficients are unique which makes $a_m = 0$ unless $m = 2$ and $b_n = 0$ unless $n = 3$. There was a WebAssign problem similar to this, so I did not anticipate that this question would cause so much difficulty! Several students erred with the sign of the eigenvalues thereby producing the wrong equation for G_n resulting in G_n being incorrectly identified as a hyperbolic function. This should be recognized as impossible because we are dealing with the wave equation for an oscillation!

a. $\lambda_n = -n^2\pi^2$ and $u_n(x,t) = (a_n\cos n\pi t + b_n \sin n\pi t)\sin n\pi x,\ n \geq 1$.

b. The general solution satisfying the *boundary conditions* is $\displaystyle u(x,t) = \sum_{n=1}^{\infty}(a_n\cos n\pi t + b_n\sin n\pi t)\sin n\pi x$ and the unique solution satisfying the *initial conditions* as well as the boundary conditions is $u(x,t) = \cos 2\pi t \sin 2\pi x + \frac{1}{3\pi}\sin 3\pi t \sin 3\pi x$.

6. Several students erred in this problem as well with the sign of the eigenvalues thereby producing the wrong equation for G_n resulting in G_n being incorrectly identified as a hyperbolic function. This should be recognized as impossible because we are dealing with the wave equation for an oscillation!

a. $\lambda_m = -m^2$ and the corresponding eigenfunctions $F_m(x) = \sin mx$, $m \geq 1$.

b. $\mu_n = -n^2$ and the corresponding eigenfunctions $G_n(y) = \sin ny$, $n \geq 1$.

c. $H_{m,n}(t) = a_{m,n} \cos \sqrt{m^2+n^2}t + b_{m,n} \sin \sqrt{m^2+n^2}t$ and
$u_{m,n}(x,y,t) = (a_{m,n} \cos \sqrt{m^2+n^2}t + b_{m,n} \sin \sqrt{m^2+n^2}t) \sin mx \sin ny$

d. $u(x,y,t) = \sum_{m,n\geq 1} (a_{m,n} \cos \sqrt{m^2+n^2}t + b_{m,n} \sin \sqrt{m^2+n^2}t) \sin mx \sin ny$ where all $b_{m,n} = 0$ and
$a_{m,n} = \frac{4}{\pi^2} \int_0^\pi \int_0^\pi f(x,y) \sin mx \sin ny \, dxdy$.

Class Statistics

% Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	14	10	9	6	13
80-89 (B)	2	3	7	8	2
70-79 (C)	2	1	1	2	3
60-69 (D)	0	4	0	1	0
0-59 (F)	1	0	1	1	0
Test Avg	88.8%	86.8%	87.6%	85.7%	90.5%
HW Avg	93.6%	92.27%	96.27%	96.27%	96.27%
HW/Test Correl.	–	0.71	0.65	0.45	0.45
Absences/Test Correl.	–	–	–	–	-0.44

The Correlation Coefficient is the cosine of the angle between two data vectors in 18-dimensional space: one dimension for each student enrolled. Thus this coefficient is between 1 and -1, with coefficients above 0.6 being considered strongly positive. The correlation coefficient shown indicates that the test grades in the course have a positive correlation with performance on the homework and a negative correlation with the number of absences from class.