Print Your Name Here: ____________________

- **Show all work:** *Answers without work are not sufficient.* We can give credit *only* for what you write! *Indicate clearly if you continue on the back side*, and *write your name at the top of the scratch sheet if you will turn it in for grading.*
- ***Books, notes (electronic or paper), cell phones, smart phones, and internet-connected devices are prohibited!*** A scientific calculator is allowed—but it is not needed. If you use a calculator, you must write out the operations performed on the calculator to show that you know how to solve the problem. *Please do not replace precise answers with decimal approximations.*
- There are **six (6)** problems: maximum total score = 200.

1. (30) Use the steps (a)-(c) below to solve

$$xy'' + 5y' + xy = 0. \tag{1}$$

a. (10) Define a new *dependent* variable $w = x^2y$ and express y, y' and y'' in terms of w and its derivatives with respect to x.

b. (10) Rewrite Equation (1) showing that it is a Bessel equation of order ν in terms of w and x.

c. (10) Write the general solution to Equation (1) for y in terms of x.

2. (40) The differential equation $2xy'' + y' - 3y = 0$ has a regular singular point at $x = 0$. The Method of Frobenius assures the existence of at least one solution of the form $y = \sum_{m=0}^{\infty} a_m x^{m+r}$.

a. (15) Find the *indicial equation* and its two roots r_1 and r_2, $r_1 \geq r_2$.

b. (15) Find the *recursion relation* that determines the later coefficients a_m in terms of the earlier coefficients and the general value of r.

c. (10) Using the larger root r_1, and choosing $a_0 = 1$ for convenience, find a_1, a_2, and a_3, and find the *general coefficient* a_m in terms of m. Express the solution y_1 corresponding to r_1 as the sum of a series using the preceding information.

3. (30) Let $f(x) = \begin{cases} 1, & 0 < x \leq \pi \\ 0, & x = 0 \\ -1, & -\pi \leq x < 0. \end{cases}$

a. (5) Fill in the two blanks in the Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos \_x + b_n \sin \_x) \tag{2}$$

.

b. (5) Calculate the coefficients a_0 and also a_n for all $n \geq 1$.

c. (20) Calculate the coefficients b_n for all $n \geq 1$.

d. (*Optional* 10 point BONUS) By Eq. (2),
$\displaystyle\int_{-\pi}^{\pi} f(x)^2\,dx \;=\; \langle f, f\rangle \;=\; \left\langle a_0 + \sum_1^{\infty}(a_n \cos \_x + b_n \sin \_x),\, a_0 + \sum_1^{\infty}(a_n \cos \_x + b_n \sin \_x)\right\rangle$. Use the *orthogonality* relations and the results of (a) – (c) to find the value of
$\displaystyle\sum_{n \text{ odd}} \frac{1}{n^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots.$

4. (40) The differential equation with *boundary conditions*

$$y'' + 4y' + (\lambda + 4)y = 0, \quad y(0) = 0 = y(1) \tag{3}$$

becomes a Sturm-Liouville problem (in self-adjoint form) with equation $(r(x)y')' + (q(x) + \lambda p(x))y = 0$ if you multiply both sides of the equation by e^{kx} for some k.

a. (10) *Choose* the correct value of k and do the multiplication to *write* Eq. (3) in Sturm-Liouville (self-adjoint) format. *Find the weight function* $p(x)$.

b. (15) Set $y = e^{rx}$ to solve the *original* Eq. (3) to find a *complete set of eigenvalues* λ_n and corresponding (real-valued) eigenfunctions $y_n(x)$ for the interval $[0, 1]$. (You will use Euler's formula here to evaluate $e^{(a+ib)x} = e^{ax}e^{ibx}$.)

c. (10) Write the orthogonality relation $\langle y_n, y_m \rangle_p = 0$ if $m \neq n$ in the form of an integral vanishing if $m \neq n$. (You do not need to evaluate the resulting integral.)

d. (5) If f is integrable on $[0, 1]$, it can be expanded in the form $f(x) = \sum_{m=1}^{\infty} c_m y_m(x)$. Using what you know about orthogonal function expansions, express c_m in terms of $\|y_m\|$ and an integral involving both $f(x)$ and $y_m(x)$. (*Do not* evaluate the integral or the value of $\|y_m\|$ for this problem.)

5. (30) Consider the wave equation $u_{tt} = u_{xx}$ on $[0, 1]$ with the *boundary conditions* $u(0, t) = 0 = u(1, t),\ t \geq 0$, and the *initial conditions* $u(x, 0) = 0,\ u_t(x, 0) = \sin 3\pi x$, $0 \leq x \leq 1$.

a. (15) Use the *separation constant* λ and find all *negative* eigenvalues $\lambda_n = -\mu^2 < 0$ with all corresponding split solutions $u_n(x, t) = F_n(x)G_n(t)$ satisfying the *boundary conditons*.

b. (15) Write the general solution satisfying the *boundary conditions* as $u(x, t) = \sum_{n=1}^{\infty} u_n(x, t)$ and determine all the necessary coefficients so as to find the unique solution satisfying the *initial conditions* as well as the boundary conditions.

6. (30) Consider Laplace's equation $u_{xx} + u_{yy} = 0$ on the square region $0 \le x \le \pi,\ 0 \le y \le \pi$ with the boundary conditions $u(x, 0) = 0 = u(x, \pi)$ and $u(0, y) = 1 = u(\pi, y)$.

a. (10) Using a *separation constant* $-\lambda$ write an ordinary differential equation for $G(y)$ *with boundary conditions.*

b. (10) Solve the Sturm-Liouville problem in part (a) to find all the *negative eigenvalues* $\lambda_n = -\mu_n^2$ and eigenfunctions $G_n(y)$.

c. (5) For each eigenvalue λ_n, show that the corresponding factor $F_n(x) = a_n \cosh \_x + b_n \sinh \_x$, filling in the two blanks.

d. (5) Write $u(x, y) = \sum_{n=1}^{\infty} (a_n \cosh \_x + b_n \sinh \_x) G_n(y)$ filling in for the blanks and the function $G_n(y)$. Set $x = 0$ to find a_n and then set $x = \pi$ to find b_n.

Solutions

1. The most common errors were not knowing how to differentiate either a quotient or a product.

a. $y = wx^{-2}$, $y' = w'x^{-2} - 2x^{-3}w$, and $y'' = w''x^{-2} - 4x^{-3}w' + 6x^{-4}w$.

b. $x^2w'' + xw' + (x^2 - 4)w = 0$

c. $y = \dfrac{c_1 J_2(x) + c_2 Y_2(x)}{x^2}$

2.

a. The *indicial equation* is $r(2r-1) = 0$ so that $r_1 = \frac{1}{2}$ and $r_2 = 0$.

b. The *recursion relation* is $a_{m+1} = \dfrac{3a_m}{(2m+2r+1)(m+r+1)}$.

c. Using the larger root $r_1 = \frac{1}{2}$, and using $a_0 = 1$, $a_1 = \frac{6}{3!}$ $(= 1)$, $a_2 = \frac{6^2}{5!}$ $(= \frac{3}{10})$, $a_3 = \frac{6^3}{7!}$ $(= \frac{3}{70})$ and $a_m = \frac{6^m}{(2m+1)!}$. Thus $y_1 = \sqrt{x}\displaystyle\sum_{m=0}^{\infty} \frac{6^m}{(2m+1)!}x^m$. We remark that $y_1 = \dfrac{1}{\sqrt{6}}\sinh(\sqrt{6x})$, though the latter information is not required for full credit. Also, you should *not* simplify the powers and factorials in finding the first few coefficients, because that will make it *much harder* to find the general coefficient a_m. I put the simplified fractions in parentheses above only to check the work of students who have the unfortunate habit of throwing away the information that leads to the general formula for a_m. Also, many students forgot the $\sqrt{x}$.

3.

a. Both blanks should be filled in with n.

b. $a_n = 0$ for all n since f is an odd function. Note: a_0 is obviously zero since that is the average value of the function.

c. $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} \sin nx\,dx = 0$ if n is even, but $b_n = \dfrac{4}{\pi n}$ for each odd value of n. It helps to avoid errors if one remembers that the product of two odd functions is even, simplifying the domain of integration as shown above.

d. $\displaystyle\int_{-\pi}^{\pi} f(x)^2\,dx = 2\pi = \left\langle \sum_{n \text{ odd}} \frac{4}{\pi n}\sin nx, \sum_{n \text{ odd}} \frac{4}{\pi n}\sin nx \right\rangle = \sum_{n \text{ odd}} \frac{16}{\pi^2 n^2}\int_{-\pi}^{\pi} \sin^2 nx\,dx$ and $\displaystyle\sum_{n \text{ odd}} \frac{1}{(n)^2} = \frac{\pi^2}{8}$. *Congratulations to the several students who solved the Bonus Problem!*

4.

a. $k = 4$: The differential equation is rewritten as $\left(e^{4x}y'\right)' + \left(e^{4x} + \lambda e^{4x}\right)y = 0$. This has the Sturm-Liouville (self-adjoint) format, with the weight function $p(x) = e^{4x}$.

b. $\lambda_n = n^2\pi^2$ and $y_n(x) = e^{-2x}\sin n\pi x$ for all $n = 1, 2, 3, \ldots$. It is important to know how to solve a quadratic equation (from 9th grade algebra) and one must apply the boundary conditions to find the eigenvalues and eigenfunctions.

c. $\langle y_n, y_m\rangle_p = 0 = \displaystyle\int_0^1 e^{-2x}\sin m\pi x\, e^{-2x}\sin n\pi x\, e^{4x}\,dx = \int_0^1 \sin m\pi x\,\sin n\pi x\,dx = 0$ if $m \neq n$. Remember that the p-scalar product includes the weight function $p(x)$!

d. $c_m = \dfrac{1}{\|y_m\|^2}\displaystyle\int_0^1 f(x)\,e^{-2x}\sin m\pi x\, e^{4x}dx = \dfrac{1}{\|y_m\|^2}\int_0^1 f(x)\,e^{2x}\sin m\pi x\,dx$. Remember that the p-scalar product includes the weight function $p(x)$!

5. It is important to know that $\sin n\pi x$ is orthogonal to $\sin m\pi x$ on $[0,1]$ when the positive integers m an n are different.

a. $\lambda_n = -n^2\pi^2$ and $u_n(x,t) = (a_n \cos n\pi t + b_n \sin n\pi t)\sin n\pi x,\ n \geq 1$.

b. The general solution satisfying the *boundary conditions* is $u(x,t) = \displaystyle\sum_{n=1}^{\infty}(a_n \cos n\pi t + b_n \sin n\pi t)\sin n\pi x$ and the unique solution satisfying the *initial conditions* as well as the boundary conditions is $u(x,t) = \frac{1}{3\pi}\sin 3\pi t \sin 3\pi x$.

6. Note that the boundary conditions for $x = 0$ and $x = \pi$ are not homogeneous. These will be satisfied using the infinite series at the end.

a. $G''(y) - \lambda G(y) = 0$ with boundary conditions $G(0) = 0 = G(\pi)$.

b. $\lambda_n = -n^2,\ n \geq 1$, and $G_n(y) = \sin ny$.

c. $F_n''(x) - n^2 F_n(x) = 0$, so that $F_n(x) = a_n \cosh nx + b_n \sinh nx$.

d. Setting $x = 0$ we find $\displaystyle\sum_{n=1}^{\infty} a_n \sin ny = 1$, so that $a_n = \dfrac{2(1-(-1)^n)}{n\pi}$ and setting $x = \pi$ we find $b_n = \dfrac{2(1-(-1)^n)(1-\cosh n\pi)}{n\pi \sinh n\pi}$.

Class Statistics

% Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	6	14	15	9	13
80-89 (B)	3	6	4	6	8
70-79 (C)	12	3	3	5	4
60-69 (D)	3	1	1	4	0
0-59 (F)	3	0	0	1	0
Test Avg	77.1%	89.6%	90.1%	89.18%	89.18%
Cumulative HW Avg	96.94%	94.87%	97.76%	98.22%	98.22%
HW/Test Correl	–	0.91	0.65	0.37	0.37

The Correlation Coefficient is the cosine of the angle between two data vectors in $\mathbb{R}^{25}$–one dimension for each student enrolled. Thus this coefficient is between 1 and -1, with coefficients above 0.6 being considered strongly positive. The correlation coefficient shown indicates that the test grades in the course have a positive correlation with performance on the homework. My interpretation is that the correlation coefficient is unstable because in this class almost all the homework scores are very high so that the number of points above or below the homework average of the class is too small to be a reliable indicator of test performance.