**Instructions**: Do any ***two*** (2) problem from the list below. *Start each chosen problem on a fresh sheet of paper.* **Write your name on each sheet at the top**. Be sure to cite all theorems that you apply, checking that all hypotheses are satisfied. You have 50 minutes for this test. If you have time, **please check your work *carefully***. Clarity is important. If you find an error but are unsure how to fix it in the time allotted, say so, because it is better if you recognize it!

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1. Let $\mathfrak{A}$ be a field of sets in $\mathfrak{P}(X)$, where $X$ is an abstract set, and let $\mathfrak{B} = \mathbb{B}(\mathfrak{A})$, the $\sigma$-field generated by $\mathfrak{A}$. If $\mu_1$ and $\mu_2$ are two countably additive measures for which $\mu_1(A) = \mu_2(A) < \infty$ for all $A \in \mathfrak{A}$, prove that
$$\mu_1(E) = \mu_2(E), \tag{1}$$
for all $E \in \mathfrak{B}$. (Hint: Show that the family $\mathcal{S}$ of all sets $E \in \mathfrak{P}(X)$ satisfying Equation (1) is a monotone class.)

2. Let $E \subset [0,1]$ be measurable, having positive Lebesgue measure:
$$0 < l(E) \leq 1. \text{ Let } \ f(x) = l\left(E \cap [0,x]\right), \forall x \in [0,1].$$
   (a) Prove that $f$ is a continuous function on $[0,1]$.
   (b) Let $\alpha \in (0,1)$. Prove that there exists a subset $E_\alpha \subset E$ such that $l(E_\alpha) = \alpha l(E)$.

3. Use the Baire Category Theorem to prove that:
   (a) The set $S = (\mathbb{R} \setminus \mathbb{Q}) \cap [0,1]$ of irrational numbers in $[0,1]$ is not an $F_\sigma$–set.
   (b) The set $T = \mathbb{Q} \cap [0,1]$ is not a $G_\delta$-set.

4. Suppose that $f_n : X \to \mathbb{R}$ is a measurable function on a measure space $(X, \mathfrak{A}, \mu)$, for each $n \in \mathbb{N}$. Prove that the set
$$S = \left\{ x \;\middle|\; \lim_{n\to\infty} f_n(x) \text{ exists} \right\}$$
is measurable.

5. Let $(X, \mathfrak{A}, \mu)$ be a *finite* measure space, and let $f : X \to \mathbb{R}$ be measurable. Prove that
$$\mu\left(f^{-1}(n,\infty)\right) \to 0,$$
as $n \to \infty$.

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