Representing Finite Groups: A Semisimple Introduction

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Sample pages

Preface

Geometry is nothing but an expression of a symmetry group. Fortunately, geometry escaped this stifling straitjacket description, an urban legend formulation of Felix Klein's Erlangen Program. Nonetheless, there is a valuable ge(r)m of truth in this vision of geometry. Arithmetic and geometry have been intertwined since Euclid's development of arithmetic from geometric constructions. A group, in the abstract, is a set of elements, devoid of concrete form, with just one operation satisfying a minimalist set of axioms. Representation theory is the study of how such an abstract group appears in different avatars as symmetries of geometries over number fields or more general fields of scalars. This book is an initiating journey into this subject.

A large part of the route we take passes through the representation theory of semisimple algebras. We will also make a day-tour out of the realm of finite groups to look at the representation theory of unitary groups. These are infinite, continuous groups, but their representation theory is intricately interlinked with the representation theory of permutation groups, and hence this detour from the main route of the book seems worthwhile.

Our navigation system is set to avoiding speedways as well as slick shortcuts. Efficiency and speed are not high priorities in this journey. For many of the ideas we view the same set of results from several vantage points. Sometimes we pause to look back at the territory covered or to peer into what lies ahead. We stop to examine glittering objects - specific examples - up close.

The role played by the characteristic of the field underlying a representation is described carefully in each result. We stay almost always within the semisimple territory, etched out by the requirement that the characteristic of the field does not divide the number of elements of the group. By not making any special choice for the field $\mathbb{F}$ we are able to see the role of semisimplicity at every stage and in every result.

Authors generally threaten readers with the admonishment that they *must* do the exercises to appreciate the text. This could give rise to insomnia if one wishes to peruse parts of this text at bedtime. However, for daytime readers, there are several exercises to engage in, some of which may call for breaking intellectual sweat, if the eyes glaze over from simply reading.

The style of presentation I have used is unconventional in some ways. Aside from the very informal tone, I have departed from rigid mathematical custom by repeating definitions instead of sending the reader scurrying back and forth to consult them. I have also included all hypotheses (such as those on the ground field $\mathbb{F}$ of a representation) in the statement of every result, instead of stating them at the beginnings of sections or chapters. This should help the reader who wishes to take just a quick look at some result or sees the statement on a sample page online.

For whom is this book? For students, graduate and undergraduate, for teachers, researchers, and also those who want to simply explore this beautiful subject for itself. This book is an introduction to the subject; at the end, or even part way through, the reader will have enough equipment and experience to take up more specialized monographs to pursue roads not traveled here.

A disclaimer on originality needs to be stated. To the best of my knowledge, there is no result in this book not already "known." Mathematical results evolve in form, from original discovery through mutations and cultural forces, and I have added historical remarks or references only for some of the major results. The reader interested in a more thorough historical analysis should consult works by historians of the subject.

Acknowledgment for much is due to many. To friends, family, strangers, colleagues, students, and a large number of fellow travelers in life and mathematics, I owe thanks for comments, corrections, criticism, encouragement and discouragement. Many discussions with Thierry Lévy have influenced my view of topics in representation theory. I have enjoyed many anecdotes shared with me by Hui-Hsiung Kuo on the frustrations and rewards of writing a book. I am grateful to William Adkins, Daniel Cohen, Subhash Chaturvedi, and Thierry Lévy for specific comments and corrections. It is a pleasure to thank Sergio Albeverio for his kind hospitality at the University of Bonn where this work was completed. Comments by referees, ranging from the insightful to the infuriating, led to innumerable improvements in presentation and content. Vaishali Damle, my editor at Springer, was a calm and steady guide all through the process of turning the original rough notes to the final form of the book. Financial support for my research program from Louisiana State University, a Mercator Guest Professorship at the University of Bonn, and US National Science Foundation Grant DMS-0601141 is gratefully acknowledged. Here I need to add the required disclaimer: Any opinions, findings and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation. Beyond all this, I thank Ingeborg for support that can neither be quantified in numbers nor articulated in words.

Chapter 1

Concepts and Constructs

A group is an abstract mathematical object, a set with elements and an operation satisfying certain axioms. A representation of a group realizes the elements of the group concretely as geometric symmetries. The same group may have many different such representations. A group that arises naturally as a specific set of symmetries may have representations as geometric symmetries at different levels.

In quantum physics the group of rotations in three dimensional space gives rise to symmetries of a complex Hilbert space whose rays represent states of a physical system; the same abstract group appears once, classically, in the avatar of rotations in space and then expresses itself at the level of a more 'implicate order' [6] in the quantum theory as unitary transformations on Hilbert spaces.

In this chapter we acquaint ourselves with the basic concepts, defining group representations, irreducibility and characters. We work through certain useful standard constructions with representations, and explore a few results that follow very quickly from the basic notions.

All through this chapter G denotes a group, and $\mathbb{F}$ a field. We will work with vector spaces, usually denoted V, W, or Z, over the field $\mathbb{F}$. *There are no standing hypotheses on G or $\mathbb{F}$*, and any conditions needed will be stated where needed.

1.1 Representations of Groups

A representation ρ of a group G on a vector space V associates to each element $g \in G$ a linear map

$$\rho(g) : V \to V : v \mapsto \rho(g)v$$

such that

$$\begin{aligned} \rho(gh) &= \rho(g)\rho(h) \qquad \text{for all } g, h \in G, \text{ and} \\ \rho(e) &= I, \end{aligned} \tag{1.1}$$

where $I : V \to V$ is the identity map and e is the identity element in G. Here our vector space V is over a field $\mathbb{F}$, and we denote by

$$\mathrm{End}_{\mathbb{F}}(V)$$

the set of all endomorphisms of V. A representation ρ of G on V is thus a map

$$\rho : G \to \mathrm{End}_{\mathbb{F}}(V)$$

satisfying (1.1). The homomorphism condition (1.1), applied with $h = g^{-1}$, implies that each $\rho(g)$ is invertible and

$$\rho(g^{-1}) = \rho(g)^{-1} \quad \text{for all } g \in G.$$

A representation ρ of G on V is said to be *faithful* if $\rho(g) \neq I$ when g is not the identity element in G. Thus, a faithful representation ρ provides an isomorphic copy $\rho(G)$ of G sitting inside $\mathrm{End}_{\mathbb{F}}(V)$.

A *complex representation* is a representation on a vector space over the field $\mathbb{C}$ of complex numbers.

Bibliography

[1] Alperin, J. L., and Bell, Rowen, B., *Groups and Representations.* Springer (1995).

[2] Artin, Emil, *Geometric Algebra.* Interscience Publishers (1957).

[3] Berkovic, Ya. G., and Zhmud', E. M., *Characters of Finite Groups.* Part I. Translated by P. Shumyatsky and V. Zobina. American Mathematical Society (1997).

[4] Birkhoff, Garrett, and Neumann, John von, *The Logic of Quantum Mechanics*, Annals of Math. 2nd Series Vol 37 Number 4, pp. 823-843 (1936).

[5] Blokker, Esko, and Flodmark, Stig, *The Arbitrary Finite Group and Its Irreducible Representations*, Int. J. Quant. Chem. **4**, 463-472 (1971).

[6] Bohm, David, *Wholeness and the Implicate Order*, Routledge (2002). (Title phrase borrowed on page 5.)

[7] Brauer, Richard, *On the Representation of a Group of Order g in the Field of g-Th Roots of Unity*, Amer. J. Math. **67** (4), 461-471 (1945).

[8] Bröcker, Theodor, and Dieck, Tammo tom, *Representations of Compact Lie Groups*, Springer (1985).

[9] Burnside, William, *The theory of groups of finite order.* 2nd Edition, Cambridge University Press (1911).

[10] Ceccherini-Silberstein, Tullio, Scarabotti, Fabio, and Tolli, Filippo, *Representation Theory of the Symmetric Groups: The Okounkov-Vershik Approach, Character Formulas, and Partition Algebras*, Cambridge University Press (2010).

[11] Chalabi, Ahmed, *Modules over group algebras and their application in the study of semi-simplicity*, Math. Ann. **201** (1973), 57-63.

[12] Chevalley, Claude, *Fundamental Concepts of Algebra*. Academic Press, New York (1956).

[13] Cohn, Donald L., *Measure Theory*, Birkhäuser (1994).

[14] Curtis, Charles W., *Pioneers of Representation Theory: Frobenius, Burnside, Schur, and Brauer*. American Mathematical Society, London Mathematical Society (1999).

[15] Curtis, Charles W., and Reiner, Irving, *Representation theory of finite groups and associative algebras*. New York, Interscience Publishers (1962).

[16] Dante, Alighieri, *The Inferno*, Transl. Robert Pinsky, Farrar, Straus and Giroux; Bilingual edition (1997). (Alluded to on page **??**.)

[17] Dedekind, Richard, *Über Zerlegungen von Zahlen durch ihre größten gemeinsamen Teiler*, in [20] (pages 103-147).

[18] Dedekind, Richard, *Über die von drei Moduln erzeugte Dualgruppe*, Mathematische Annalen **53** (1900), 371 - 403. Reprinted in [20] (pages 236-271)

[19] Dedekind, Richard, Letters to Frobenius (25th March 1896 and 6th April, 1896), in [20] (pages 420-424).

[20] Dedekind, Richard, *Gesammelte mathematische Werke*,Vol II, editors: Robert Fricke, Emmy Noether, Öystein Ore, Friedr. Vieweg & Sohn Akt.-Ges. (1931).

[21] Diaconis, Persi, *Group representations in probability and statistics*, Lecture Notes–Monograph Series, Volume 11 Hayward, CA: Institute of Mathematical Statistics (1988). Available online.

[22] Dieudonné, Jean Alexandre, and Carrell, James B.: *Invariant Theory Old and New*. Academic Press (1971).

[23] Dixon, John D., *High Speed Computation of Group Characters*, Num Math. 10 (1967), 446-450.

[24] Dixon, John D., *Computing Irreducible Representations of Groups*, Mathematics of Computation **24** (111), 707-712, July 1970.

[25] Duke, William, and Hopkins, Kimberly, *Quadratic reciprocity in a finite group*, American Mathematical Monthly 112 (2005), no. 3, 251256.

[26] Farb, Benson, and Dennis, R. Keith, *Noncommutative Algebra*, Springer-Verlag (1993).

[27] Feit, Walter, *The Representation Theory of Finite Groups*, North-Holland (1982).

[28] Fontane, Theodor, *Effi Briest*, Insel-Verlag (1980); first published 1894-1895. (Alluded to on page **??**.)

[29] Frobenius, Ferdinand Georg, *Über Gruppencharaktere*. Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin, 985-1021 (1896). In the Collected Works: Gesammelte Abhandlungen. Vol III (pages 1-37) Hrsg. von J.-P. Serre. Springer Verlag (1968). indexFrobenius, F. G.

[30] Frobenius, Ferdinand Georg, *Über die Primfactoren der Gruppendeterminante*. Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin, 1343-1382 (1896). In the Collected Works: Gesammelte Abhandlungen. Vol III (pages 38-77) Hrsg. von J.-P. Serre. Springer Verlag (1968).

[31] Frobenius, Ferdinand Georg, *Über Beziehungen zwischen den Primidealen eines algebraischen Körpers und den Substitutionen seiner Gruppe*, Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin, 689703 (1896). In the Collected Works: Gesammelte Abhandlungen. Vol II, (pages 719-733) Hrsg. von J.-P. Serre. Springer Verlag (1968).

[32] Frobenius, Ferdinand Georg, *Über die Darstellungen der endlichen Gruppen durch linearen Substitutionen*. Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin, 944-1015 (1897). In the Collected Works: Gesammelte Abhandlungen, Vol III (pages 82-103) Hrsg. von J.-P. Serre. Springer Verlag (1968).

[33] Frobenius, Ferdinand Georg, *Über Relationen zwischen den Charakteren einer Gruppe und denen ihrer Untergruppen*. Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin, 501-515 (1898). In the Collected Works: Gesammelte Abhandlungen. Vol III (pages 104-118) Hrsg. von J.-P. Serre. Springer Verlag (1968).

[34] Frobenius, Ferdinand Georg, *Über die Charaktere der symmetrischen Gruppe*. Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin, 516-534 (1900). In the Collected Works: Gesammelte Abhandlungen. Vol III (pages 148-166). Hrsg. von J.-P. Serre. Springer Verlag (1968).

[35] Frobenius, Ferdinand Georg, *Über die charakterischen Einheiten der symmetrischen Gruppe*. Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin, 328-358 (1903). In the Collected Works: Gesammelte Abhandlungen, Vol III (pages 244-274) Hrsg. von J.-P. Serre. Springer Verlag (1968).

[36] Frobenius, Ferdinand Georg, and Schur, Issai: *Über die reellen Darstellungen der endlichen Gruppen*. Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin, 186-208 (1906). In the Collected Works: Gesammelte Abhandlungen, Vol III (pages 355-377) Hrsg. von J.-P. Serre. Springer Verlag (1968).

[37] Fulton, William, *Young Tableaux*, Cambridge University Press (1997).

[38] Fulton, William, and Harris, Joe, *Representation Theory, A First Course*, Springer-Verlag (1991).

[39] Glass, Kenneth, and Ng, Chi-Keung, *A Simple Proof of the Hook Length Formula*. The American Mathematical Monthly, Vol. 111, No. 8 (Oct., 2004), pp. 700-704.

[40] Goodman, Roe, and Wallach, Nolan, R., *Representations and Invariants of the Classical Groups*, Cambridge University Press (1998).

[41] Hall, Brian C., *Lie Groups, Lie Algebras, and Representations An Elementary Introduction*. Springer-Verlag (2003).

[42] Hawkins, Thomas, *Emergence of the Theory of Lie Groups: An Essay in the History of Mathematics 1869-1926*. Springer-Verlag (2000).

[43] Hawkins, Thomas, *New light on Frobenius' creation of the theory of group characters*, Arch. History Exact Sci. Vol 12(1974), 217-243.

[44] Hill, Victor, E., *Groups and Characters*, Chaplan & Hall/CRC (2000).

[45] Hora, Akihito, and Obata, Nobuaki, *Quantum Probability and Spectral Analysis of Graphs*, Springer-Verlag (2007).

[46] Humphreys, James E., *Reflection Groups and Coxeter Groups*, Cambridge University Press (1990).

[47] Hungerford, Thomas, W., *Algebra*, Springer-Verlag (1974).

[48] Isaacs, J. Martin, *Character Theory of Finite Groups*, Academic Press (1976).

[49] James, Gordon D., and Liebeck, Martin, *Representations and Characters of Groups*. Cambridge University Press (2001).

[50] James, G. D., *The Representation Theory of the Symmetric Groups*, Springer-Verlag, Lecture Notes in Mathematics 682 (1978)

[51] Kock, Joachim, *Frobenius algebras and 2D topological quantum field theories*, Cambridge University Press (2004).

[52] Lam, T. Y., *A First Course on Noncommutative Rings*, Graduate Texts in Math., Vol. 131, Springer-Verlag, 1991.

[53] Lam, T. Y., *Representations of Finite Groups: A Hundred Years, Part I*. Notices of the American Mathematical Society. March 1998 Volume 45 Issue 3 (361-372).

[54] Lando, Sergei, and Zvonkin, Alexander. *Graphs on Surfaces and their Applications*. Springer Verlag (2004).

[55] Lang, Serge. *Algebra*. Springer 2nd Edition(2002) and 3rd Edition (2004).

[56] Lévy, Thierry, *Schur-Weyl duality and the heat kernel measure on the unitary group*. Adv. Math. 218 (2008), no. 2, 537–575.

[57] Littlewood, Dudley E., *The Theory of Group Characters and Matrix Representations of Groups*. Oxford at the Clarendon Press (1950).

[58] Maschke, Heinrich, *Über den arithmetischen Charakter der Coefficienten der Substitutionen endlicher linearer Substitutionensgruppen*, Math. Ann. **50**(1898), 492-498.

[59] Maschke, Heinrich, *Beweis des Satzes, daß die jenigen endlichen linearen Substitutionensgruppen, in welchen einige durchgehends verschwindende Coefficienten auftreten, intransitiv sind*, Math. Ann. **52**(1899), 363-368.

[60] Molien, Theodor, *Über Systeme höherer complexer Zahlen*, Dissertation (1891), University of Tartu. `http://dspace.utlib.ee/dspace/handle/10062/110`

[61] Mulase, Motohico, and Penkava, Michael, *Volume of Representation Varieties* (2002). Available online.

[62] Neusel, Mara D., *Invariant Theory*. American Mathematical Society (2006).

[63] Orlik, Peter, and Terao, Hiroaki, *Arrangement of Hyperplanes*, Springer-Verlag (1992).

[64] Okounkov, Andrei, and Vershik, Anatoly. *A New Approach to Representation Theory of Symmetric Groups*. Erwin Schrödinger International Institute for Mathematical Physics preprint ESI 333 (1996).

[65] Passman, Donald S., *The algebraic structure of group rings*. John Wiley & Sons, New York, London, Sydney, Toronto (1977).

[66] Puttaswamiah, B. M., and Dixon, John, *Modular Representations of Finite Groups*, Academic Press (1977).

[67] Rudin, Walter, *Functional Analysis*, Second Edition. McGraw-Hill (1991).

[68] Rota, Gian-Carlo, *The Many Lives of Lattice Theory*. Notices of the AMS, Volume 44, Number 11, December 1997, pp. 1440-1445.

[69] Schur, Issai, *Neue Begründing der Theorie der Gruppencharaktere*, Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin, 406-432 (1905). In the Collected Works: *Gesammelte Abhandlungen* Vol I (pages 143-169) Hrsg. von Alfred Brauer u. Hans Rohrbach. Springer (1973).

[70] Schur, Issai, *Über die rationalen Darstellungen der allgemeinen linearen Gruppe*, J. reine angew. Math. **132** (1907) 85-137; in *Gesammelte Abhandlungen* Vol I (pages 198-250) Hrsg. von Alfred Brauer u. Hans Rohrbach. Springer (1973).

[71] Schur, Issai, *Über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionen*, in *Gesammelte Abhandlungen* Vol III (pages 68-85) Hrsg. von Alfred Brauer u. Hans Rohrbach. Springer (1973).

[72] Sengupta, Ambar N., *The volume measure of flat connections as limit of the Yang-Mills measure*, J. Geom. Phys. Vol 47 398-426 (2003).

[73] Serre, Jean-Pierre, *Linear Representations of Finite Groups*. Translated by Leonard L. Scott. (4th Edition) Springer-Verlag (1993).

[74] Simon, Barry, *Representation Theory of Finite and Compact Groups*. American Mathematical Society (1995).

[75] Thomas, Charles B., *Representations of Finite and Lie Groups*. Imperial College Press (2004).

[76] Varadarajan, Veeravalli S., *The Geometry of Quantum Theory*. Springer; 2nd edition (December 27, 2006)

[77] Wedderburn, J. H. M., *On hypercomplex numbers*, Proc. London Math. Soc. (Ser 2) **6** (1908), 77-118.

[78] Weintraub, Steven H., *Representation Theory of Finite Groups: Algebra and Arithmetic*. American Mathematical Society (2003).

[79] Weyl, Hermann, *Group Theory and Quantum Mechanics*. Dover (1956)

[80] Witten, Edward, *On Quantum Gauge Theories in Two Dimensions*, Commun. Math. Phys. **141** (1991), 153-209.

[81] Young, Alfred. *Quantitative Substitutional Analysis I*. Proc. Lond. Math. Soc. (1) 33 (1901). Available in the Collected Works published by University of Toronto Press, c1977.