CDO TRANCHE SENSITIVITIES IN THE GAUSSIAN COPULA MODEL

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Abstract. We derive explicit formulas for CDO tranche sensitivity to parameter variations, and prove results concerning the qualitative behavior of such tranche sensitivities, for a homogeneous portfolio governed by the one-factor Gaussian copula. A Poisson-mixture model is also investigated in a similar vein.

1. Introduction

Collateralized Debt Obligations (CDOs) are of central importance in the credit derivatives market. The size of the CDO market has enormous implications for the broader financial system, as is underlined by the central role of CDOs in the ongoing financial crisis in 2008. Rigorous investigation of the principles of pricing and risk management of CDOs is therefore of great importance. In this paper we present a mathematically rigorous development of some key aspects of the Gaussian copula model, a market standard model for homogeneous CDOs, with some additional results applying to a broader setting.

The Gaussian copula model for managing CDO tranches became popular following the work of Li [13]. It is a widely-used foundational model which displays qualitative characteristics observed in practice and through simulations in other models. Our objective in this paper is to give mathematical proofs for several such features.

A brief summary of the essential notions and terminology we use is presented at the end of this introduction.

There is a large and growing body of literature, both from industry and academia, on various aspects of CDOs. The Gaussian copula model, growing out of the Credit Metrics/KMV model (see [4, page 83] for a description), became an industry standard, especially after the work of Li [13], but it is an idealized first approximation to real default correlations. Much of the literature (a sampling of which is included in the bibliography) focuses on developing, simulating, and evaluating alternative models which better address such issues as the correlation skew and term-structure of defaults. Our objective is more mathematical, in that we focus on rigorous proofs of some of the essential qualitative features of the single-factor Gaussian copula model for tranche pricing and risk management. The work of Cousin and Laurent [6] presents several results which are similar in spirit and substance to some of ours,

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comparing the behavior of expectations of convex functions of portfolio losses under change of copula parameters.

Overview of Results

We shall work in the context of homogeneous portfolios governed by a single-factor Gaussian copula default behavior and with zero recovery. Within this model we shall give mathematical proofs for the following features:

(i) Equity tranches are long correlation and senior tranches are short correlation;
(ii) equity tranche deltas decrease (increase) when the index spread increases (decreases);
(iii) tranche deltas, for index spread movements, form a probability measure on losses;
(iv) the equity tranche is convex with respect to spread movements;
(v) the normalized loss in a size N portfolio converges almost surely to a random variable, of known distribution, as $N \to \infty$.

Some of these results provide mathematical confirmation for observations which are known from simulations in industry practice (for instance, [9] and [11]).

We also obtain explicit formulas for tranche deltas and convexity for this model. In practice these quantities are computed numerically by bumping the relevant parameters and working out derivatives numerically from the observed impact on trade present values. However, closed-form expressions are always of some interest, even if in idealized models.

We then apply our methods to a different model, where losses are modeled with a Poisson distribution with parameter governed by a Gaussian variable. In this model, we can allow a degree of inhomogeneity in that different names can have different spreads. We prove results similar to that for the Gaussian copula model.

Our proofs rely on the specific form of the distributions (Gaussian, binomial, Poisson), and yield results, such as the behavior of tranche deltas, which are true across the full range of tranche cutoffs. This is unlikely to be true for other models. We leave to a future work the task of developing rigorous mathematical methods for examining (i)-(v), and related questions, for a broader class of models.

Terminology and Notions

Before proceeding to the technical results let us briefly specify some of the terminology and notions. For more details on modeling and implementation see, for instance, Bluhm et al. [4], Lando [12] and Schönbucher [18].

We take a CDO to be a portfolio of N defaultable assets, called *names* or *credits* (for example, they may be credit default swaps); a default event for a name results in a loss of 1 unit in the portfolio. The default probability of a name, within a given time horizon, is reflected in its *spread*, which is, very roughly, the premium needed to insure unit notional of the credit against default. In our model, all names in the CDO are assumed to have the same spread. Most of our results will be proven for a *homogeneous* portfolio, with each name having the same default probability.

Losses are divided into *tranches*; for example, an *investor* in the $0-3\%$ *tranche* will have to pay out losses up to 3% of the full portfolio notional in the portfolio and receives, in exchange, a *spread* payment periodically, until the 3% loss level or a specified time horizon is reached.

A tranche running from 0 loss to a particular *detachment* cut-off loss level is called an *equity tranche*, and its complement, the tranche upward of a given loss level will be called a *senior tranche*.

The difficulty in modeling tranche losses arises from *correlation* between the default behavior of different names.

A position, long or short, on a tranche is often hedged by an opposite position on the entire portfolio (which we refer to as the *index*); the notional in the index that hedges the tranche investment, against movement in the spreads, is the *delta* for the tranche (we discuss this with greater precision in section 3).

Often a tranche deal is executed simultaneously with the opposite index hedge position; the sensitivity of the combined hedged portfolio to changes in the index spread is measured by the *gamma* or convexity of the tranche.

When a credit defaults some amount of the notional is recoverable. This recovery factor is usually modeled as a constant or as a factor independent of the other random variables involved. For this reason we set the recovery factor to 0.

2. Sensitivity to Correlation in the Gaussian Model

In this section we will prove that, under the standard one-factor Gaussian copula model, *equity tranches are long correlation and senior tranches are short correlation.* This means, for instance, that an investor in an equity tranche has lower expected loss payments when default correlation rises.

Here, by equity tranche, we mean any tranche of the form $[0, x]$, with attachment point 0; by a senior tranche we mean here any tranche $[x, 1]$ that runs from a given attachment point all the way to the maximum portfolio loss.

More precisely, we will show that the expected loss in any equity tranche (i.e. a tranche running from 0 to some given detachment level) declines with increasing correlation. An investor in such a tranche, with a locked-in spread, receives higher spread payments on the higher-than-expected outstanding notional, and has to pay lower-than-expected losses.

Since a senior tranche is simply the index minus an equity tranche, and since the full portfolio loss is insensitive to correlation, the result for the equity tranche implies the corresponding result for the senior tranche automatically.

These results are supported both by intuition and simulations. If correlation rises, the probability of very few defaults increases (as well as that for many defaults), and this ought to decrease the expected loss for, at least, a low-detachment equity tranche. It is, however, not quite clear intuitively whether this ought to work for all equity tranches. Theorem 2.1 below establishes the result rigorously for the single-factor Gaussian copula model (2.1).

A CDO tranche deal involves periodic payments, of loss and spreads. Our discussion applies to the accumulated loss over *one* such period (typically 3 months). The loss payment for the full life of the CDO is a discounting-weighted sum of the single-period losses, and therefore may be deduced readily from the single-period loss analysis.

Consider a homogeneous portfolio with N names, each with unit notional, with default behavior of the i-th name governed by the factor

$$X_i = \sqrt{\rho} Z + \sqrt{1-\rho}\epsilon_i, \tag{2.1}$$

with $Z, \epsilon_1, ..., \epsilon_N$ independent standard Gaussian variables. We are working with a fixed time horizon. Name i defaults, within the given time horizon, if X_i falls below a threshold c.

The model (2.1) encodes the assumption that the portfolio has a common correlation

$$\mathrm{Corr}(X_i, X_j) = \mathbb{E}[X_i X_j] = \rho \geq 0, \text{for all } i \neq j. \tag{2.2}$$

We use the notation

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \qquad \text{and} \qquad \Phi(x) = \int_{-\infty}^{x} \phi(s)\, ds, \tag{2.3}$$

for the standard Gaussian density and distribution.

The probability that exactly j names default is

$$p_j = \int_{\mathbb{R}} \binom{N}{j} p^j (1-p)^{N-j} \phi(x)\, dx \tag{2.4}$$

where

$$p = \mathbb{P}[X_i \leq c \,|\, Z = x] = \mathbb{P}\left[\epsilon_i \leq \frac{c - \sqrt{\rho} x}{\sqrt{1-\rho}}\right] = \Phi\left(\frac{c - \sqrt{\rho} x}{\sqrt{1-\rho}}\right) \tag{2.5}$$

Consider now the tranche running from 0 loss to a loss of k names. Let l_k^e be the loss in this tranche (with superscript e signifying it is an equity tranche) in the given time horizon.

In our model, the recovery rate, assumed the same for all names, and independent of all other variables, plays no essential role, and so we will set it equal to 0. Thus, default of a name results in loss of one unit to the portfolio. So the tranche loss is

$$l_k^e = \min\{\nu, k\} = \mathbb{1}_{[\nu=1]} + 2\mathbb{1}_{[\nu=2]} + \cdots + (k-1)\mathbb{1}_{[\nu=k-1]} + k\mathbb{1}_{[\nu \geq k]} \tag{2.6}$$

where ν is the number of defaults. The loss, for the senior tranche running from loss level $k+1$ all the way to N is:

$$l_k^s = \nu - \min\{\nu, k\} = \mathbb{1}_{[\nu=k+1]} + 2\mathbb{1}_{[\nu=k+2]} + \cdots + (N-k)\mathbb{1}_{[\nu=N]} \tag{2.7}$$

We can now state our mathematical result:

Theorem 2.1. *Assume that $Z, \epsilon_1, ..., \epsilon_N$ are independent standard Gaussian variables, with $N > 1$, and let*

$$X_i = \sqrt{\rho} Z + \sqrt{1-\rho}\, \epsilon_i, \qquad \text{for } i \in \{1, ..., N\}$$

where $\rho \in (0,1)$. Let $c \in \mathbb{R}$. Let ν be the random variable which counts the number of X_j with value $< c$:

$$\nu = \#\{j \in \{1, ..., N\} : X_j < c\}$$

and, for $k \in \{1, ..., N\}$,

$$l_k^e = \min\{\nu, k\} \tag{2.8}$$

$$l_k^s = \nu - \min\{\nu, k\}. \tag{2.9}$$

Then,

$$\frac{d\mathbb{E}[l_k^e]}{d\rho} < 0, \qquad \text{and} \qquad \frac{d\mathbb{E}[l_k^s]}{d\rho} > 0,$$

for $1 \leq k < N$.

The rest of this section is devoted to proving this result.

Proof. Since

$$l_k^e + l_k^s = \nu,$$

we have

$$\mathbb{E}\left[l_k^e\right] + \mathbb{E}\left[l_k^s\right] = \mathbb{E}\left[\nu\right]$$

Now

$$\begin{aligned}\mathbb{E}\left[\nu\right] &= \mathbb{E}\left[\sum_{j=1}^{N} \mathbb{1}_{[X_j<c]}\right] \\ &= N\mathbb{P}[X_1 < c] = N\Phi(c).\end{aligned}$$

This is independent of ρ. Thus,

$$\frac{d\mathbb{E}\left[l_k^s\right]}{d\rho} = -\frac{d\mathbb{E}\left[l_k^e\right]}{d\rho}$$

So it will suffice to prove that $\frac{d\mathbb{E}[l_k^e]}{d\rho}$ is negative.

The expected equity tranche loss is

$$L_k^e \stackrel{\text{def}}{=} \mathbb{E}\left[l_k^e\right] = p_1 + 2p_2 + \cdots + (k-1)p_{k-1} + k\left[1 - p_0 - \cdots - p_{k-1}\right], \tag{2.10}$$

which can be rewritten as

$$L_k^e = k - \sum_{j=0}^{k}(k-j)p_j. \tag{2.11}$$

From this, and the expression (2.4) for p_j, we have

(2.12)

$$\begin{aligned}\frac{\partial L_k^e}{\partial \rho} &= -\sum_{j=0}^{k}(k-j)\binom{N}{j}\int_{\mathbb{R}}\left[jp^{j-1}(1-p)^{N-j} - (N-j)p^j(1-p)^{N-j-1}\right]\frac{\partial p}{\partial \rho}\phi(x)\,dx \\ &= \int_{\mathbb{R}} I_k(p)\frac{\partial p}{\partial \rho}\phi(x)\,dx,\end{aligned}$$

where

$$I_k(p) \stackrel{\text{def}}{=} -\sum_{j=0}^{k}\binom{N}{j}(k-j)\left[jp^{j-1}(1-p)^{N-j} - (N-j)p^j(1-p)^{N-j-1}\right] \tag{2.13}$$

(Note that the integrand in the expression for $dL_k^e/d\rho$ contains an exponentially decreasing term in x^2, which ensures that $d/d\rho$ and $\int_{\mathbb{R}} \ldots dx$ can be interchanged.)

We can now compute the derivative $\partial p/\partial \rho$ from (2.5):

$$\begin{aligned}\frac{\partial p}{\partial \rho} &= \phi\left(\frac{c-\sqrt{\rho}x}{\sqrt{1-\rho}}\right)\frac{\sqrt{1-\rho}\left\{-\frac{1}{2\sqrt{\rho}}x\right\} - (c-\sqrt{\rho}x)\left\{-\frac{1}{2\sqrt{1-\rho}}\right\}}{1-\rho} \\ &= -\frac{(1-\rho)x - \sqrt{\rho}(c-\sqrt{\rho}x)}{2\sqrt{\rho}(1-\rho)^{3/2}}\phi\left(\frac{c-\sqrt{\rho}x}{\sqrt{1-\rho}}\right) \\ &= -\frac{x - c\sqrt{\rho}}{2\sqrt{\rho}(1-\rho)^{3/2}}\phi\left(\frac{c-\sqrt{\rho}x}{\sqrt{1-\rho}}\right).\end{aligned}$$

So

$$\begin{aligned}
\frac{dL_k^e}{d\rho} &= -\int_{\mathbb{R}} I_k(p)\frac{(x-c\sqrt{\rho})}{2\sqrt{\rho}(1-\rho)^{3/2}}\phi\left(\frac{c-\sqrt{\rho}x}{\sqrt{1-\rho}}\right)\phi(x)\,dx \\
\underbrace{=}_{y=x-c\sqrt{\rho}} & -\int_{\mathbb{R}} I_k(p)\frac{y}{2(1-\rho)^{3/2}\sqrt{\rho}}\phi\left(\frac{c(1-\rho)-\sqrt{\rho}y}{\sqrt{1-\rho}}\right)\phi(y+c\sqrt{\rho})\,dy \\
&= -\int_{\mathbb{R}} I_k(p)\frac{y}{2(1-\rho)^{3/2}\sqrt{\rho}}\frac{1}{2\pi}e^{-\frac{y^2}{2(1-\rho)}-\frac{c^2}{2}}\,dy
\end{aligned}$$

Looking back at (2.5), let us write

$$p(y) = p = \Phi\left(\frac{c-\sqrt{\rho}x}{\sqrt{1-\rho}}\right) = \Phi\left(\frac{c(1-\rho)-\sqrt{\rho}y}{\sqrt{1-\rho}}\right) \tag{2.14}$$

Note that this is clearly monotonically decreasing in y.

Returning again to the derivative $dL_k^e/d\rho$, we have:

$$\frac{dL_k^e}{d\rho} = -\int_0^\infty \left[I_k\big(p(y)\big) - I_k\big(p(-y)\big)\right]\frac{y}{2(1-\rho)^{3/2}\sqrt{\rho}}\frac{1}{2\pi}e^{-\frac{y^2}{2(1-\rho)}-\frac{c^2}{2}}\,dy \tag{2.15}$$

As we prove below in Lemma 2.2, the function $I_k(\cdot)$ is monotonically decreasing. Now, as noted above, for $y > 0$, we have $p(y) < p(-y)$. Hence,

$$I_k\big(p(y)\big) - I_k\big(p(-y)\big) > 0 \qquad \text{for any } y > 0.$$

This implies, from (2.15), that

$$dL_k^e/d\rho < 0,$$

which is the result we had set out to prove. □

We have used the following observation about $I_k(p)$:

Lemma 2.2. *Let*

$$I_k(p) = -\sum_{j=0}^{k}\binom{N}{j}(k-j)\left[jp^{j-1}(1-p)^{N-j} - (N-j)p^j(1-p)^{N-j-1}\right]$$

where N and k are positive integers, with $k \le N$, and $p \in [0,1]$. Then

$$\begin{aligned}
I_k(p) &= N\sum_{j=1}^{k}\binom{N-1}{j-1}p^{j-1}(1-p)^{N-j} \\
&= N - (N-k)k\binom{N}{k}\int_0^p t^{k-1}(1-t)^{N-k-1}\,dt
\end{aligned} \tag{2.16}$$

In particular, $I_k(p)$ is monotonically decreasing with p, if $1 \le k < N$.

Proof. First let us rework the expression for $I_k(p)$:

$$\begin{aligned}
I_k(p) &\stackrel{\text{def}}{=} -\sum_{j=0}^{k}\binom{N}{j}(k-j)\left[jp^{j-1}(1-p)^{N-j}-(N-j)p^j(1-p)^{N-j-1}\right]\\
&= -\sum_{j=0}^{k-1}\left[\binom{N}{j+1}(k-j-1)(j+1)-\binom{N}{j}(k-j)(N-j)\right]p^j(1-p)^{N-j-1}\\
&= \sum_{j=0}^{k-1}\binom{N}{j}(N-j)p^j(1-p)^{N-j-1}\\
&= N\sum_{j=1}^{k}\binom{N-1}{j-1}p^{j-1}(1-p)^{N-j}\\
&= N(1-p)^{N-1}+\sum_{j=1}^{k-1}\binom{N}{j}(N-j)p^j(1-p)^{N-j-1}.
\end{aligned}$$

Taking the derivative, we obtain

$$\begin{aligned}
I_k'(p) &= \sum_{j=1}^{k-1}\binom{N}{j}(N-j)jp^{j-1}(1-p)^{N-j-1}-\sum_{j=0}^{k-1}\binom{N}{j}(N-j)p^j(N-j-1)(1-p)^{N-j-2}\\
&= \sum_{j=0}^{k-2}\underbrace{\left\{\binom{N}{j+1}(N-j-1)(j+1)-\binom{N}{j}(N-j)(N-j-1)\right\}}_{0}p^j(1-p)^{N-j-2}\\
&\qquad -\binom{N}{k-1}(N-k+1)(N-k)p^{k-1}(1-p)^{N-k-1}
\end{aligned}$$

Rewriting the last term, we have

$$I_k'(p) = -(N-k)k\binom{N}{k}p^{k-1}(1-p)^{N-k-1}$$

Integrating, and using the value N for $I_k(0)$, we obtain (2.16). □

The work of Cousin and Laurent [6] contains results in the spirit of those in this section, though their methods are very different. In particular, Proposition 3.9 in [6], which applies to a broad class of copulas, could possibly be used to obtain an alternative proof for our Theorem 2.1.

3. Tranche Deltas

We shall calculate the delta of a tranche with respect to a uniform credit spread movement across the index. Recall that in our homogeneous portfolio, this spread is a function of the default threshold c, and so we shall be concerned with sensitivity of the tranche loss and index loss to variation in c.

As before, L_k^e denotes the expected loss in the $[0, k]$ tranche, and now let L_N be the expected loss for the entire portfolio:

$$L_N = \mathbb{E}\left[l_N\right], \tag{3.1}$$

where

$$l_N = \sum_{j=1}^{N} \mathbb{1}_{[X_j \le c]}. \tag{3.2}$$

By the *delta*, which we denote $\Delta_{k,\text{spread}}$, of the $[0,k]$ tranche we shall mean the factor such that the portfolio

long $[0,k]$ tranche, short $\Delta_{k,\text{spread}}$ times the full index

is stationary to first order, against variation of the default threshold c. Thus,

$$\Delta_{k,\text{spread}} = \frac{\frac{\partial L_k^e}{\partial c}}{\frac{\partial L_N}{\partial c}} \tag{3.3}$$

It is important to note that what we are calling 'delta' is often the displayed 'tranche leverage' scaled by the tranche width.

For the Gaussian copula, the expected loss in the index is

$$L_N = E(l_N) = N\Phi(c), \tag{3.4}$$

and so

$$\frac{\partial L_N}{\partial c} = N\phi(c) = \frac{N}{\sqrt{2\pi}} e^{-\frac{c^2}{2}}. \tag{3.5}$$

Recall that the expected loss L_k^e in the equity $0 - k$ tranche is

$$L_k^e = k - \sum_{j=0}^{k} (k-j)p_j, \tag{3.6}$$

where

$$p_j = \int_{\mathbb{R}} \begin{pmatrix} N \\ j \end{pmatrix} p^j (1-p)^{N-j} \phi(x)\, dx, \tag{3.7}$$

and

$$p = \Phi\left(\frac{c - \sqrt{\rho}x}{\sqrt{1-\rho}}\right). \tag{3.8}$$

Theorem 3.1. *The delta of the* $[0,k]$*-tranche is*

$$\Delta_{k,\text{spread}} = \Delta_{\text{spread}}\big(\{0, 1, ..., k\}\big)$$

where Δ_{spread} *is the measure on subsets of* $\{0, 1, ..., N\}$ *given by*

$$\Delta_{\text{spread}}(S) = \sum_{k \in S} p_{\Delta_s}(k), \tag{3.9}$$

for any $S \subset \{0, ..., N\}$*, where*

$$p_{\Delta_s}(k) = \int_{\mathbb{R}} \binom{N-1}{k-1} p(y)^{k-1} \big(1 - p(y)\big)^{N-1-(k-1)} \frac{1}{\sqrt{2\pi(1-\rho)}} e^{-\frac{y^2}{2(1-\rho)}}\, dy \tag{3.10}$$

for $k \in \{1, ..., N\}$*, and* $p_{\Delta_s}(0)$ *is* 0 *by definition.*

Proof. Let us first work with an equity tranche with losses $\leq k$. From (3.6) we have

$$\frac{\partial L_k^e}{\partial c} = -\sum_{j=0}^{k}(k-j)\binom{N}{j}\int_{\mathbb{R}}\left[jp^{j-1}(1-p)^{N-j}-(N-j)p^j(1-p)^{N-j-1}\right]\frac{\partial p}{\partial c}\phi(x)\,dx \tag{3.11}$$

$$= \int_{\mathbb{R}} I_k(p)\frac{\partial p}{\partial c}\phi(x)\,dx,$$

where, from Lemma 2.2,

$$I_k(p) = N\sum_{j=1}^{k}\binom{N-1}{j-1}p^{j-1}(1-p)^{N-j}. \tag{3.12}$$

Now

$$\frac{\partial p}{\partial c} = \frac{1}{\sqrt{2\pi(1-\rho)}}e^{-\frac{(c-\sqrt{\rho}x)^2}{2(1-\rho)}},$$

and

$$\frac{\partial p}{\partial c}\phi(x) = \frac{1}{2\pi\sqrt{1-\rho}}e^{-\frac{(x-c\sqrt{\rho}))^2}{2(1-\rho)}-\frac{c^2}{2}}.$$

Setting $y = x - c\sqrt{\rho}$, we have

$$\begin{aligned}\frac{dL_k^e}{dc} &= \int_{\mathbb{R}} I_k(p)\frac{\partial p}{\partial c}\phi(x)\,dx\\ &= \int_{\mathbb{R}} I_k(p)\frac{1}{2\pi\sqrt{1-\rho}}e^{-\frac{(x-c\sqrt{\rho}))^2}{2(1-\rho)}-\frac{c^2}{2}}\,dx\\ &= \int_{\mathbb{R}} I_k\big(p(y)\big)\frac{1}{2\pi\sqrt{1-\rho}}e^{-\frac{y^2}{2(1-\rho)}-\frac{c^2}{2}}\,dy\end{aligned}$$

Therefore

$$\begin{aligned}\Delta_{k,\text{spread}} &= \frac{\partial L_k^e}{\partial c}\Big/\frac{\partial L_N}{\partial c}\\ &= \frac{1}{N}\int_{\mathbb{R}} I_k\big(p(y)\big)\frac{1}{\sqrt{2\pi(1-\rho)}}e^{-\frac{y^2}{2(1-\rho)}}\,dy\end{aligned} \tag{3.13}$$

Substituting in the integrand (3.13) the expression for $I(p)$ from (3.12) shows that

$$\Delta_{k,\text{spread}} = \sum_{j\in\{0,1,\ldots,k\}} p_{\Delta_s}(j),$$

where

$$p_{\Delta_s}(j) = \int_{\mathbb{R}}\binom{N-1}{j-1}p(y)^{j-1}\big(1-p(y)\big)^{N-j}\frac{1}{\sqrt{2\pi(1-\rho)}}e^{-\frac{y^2}{2(1-\rho)}}\,dy,$$

understood to be 0 when j is 0. This simplifies to $p_{\Delta_s}(j)$ as given by (3.10). □

This result confirms, for the Gaussian copula model, the generally held view that the delta with respect to index spread movements is a probability measure on the loss levels.

As an immediate consequence of the theorem, we have:

Corollary 3.1.1. *The delta of an equity tranche decreases monotonically with increasing spread. The delta of a senior tranche increases with increasing spread.*

Proof. As $I_k(p)$ is monotonically decreasing with p and $p(y) = \Phi\left(\frac{c(1-\rho)-\sqrt{\rho}y}{\sqrt{1-\rho}}\right)$ is monotonically increasing with c, the delta (3.13) decreases with increasing c. The senior tranche (loss level $> k$) loss is the index loss minus the equity tranche (loss level $\leq k$) loss, and so the delta for the senior tranche is 1 minus the equity delta. □

4. Convexity of Equity Tranches

Consider the portfolio which is long one equity tranche $[0, k]$ and short h units of CDO index (i.e. the notional is h times the full index notional). The expected loss of the portfolio is the negative of

$$V_k(h) = hL_N - L_k^e.$$

We define the *convexity* of the $[0, k]$ tranche to be

$$\Gamma_k = \frac{\partial^2 V_k(h)}{\partial c^2}\Big|_{h=\Delta_{k,\text{spread}}}. \tag{4.1}$$

This gives the second-order increment in the mark-to-market for the investor in the equity tranche who has hedged the tranche investment with protection on the index.

It is important to note that the term ‘gamma’ is used in related but different forms. In particular, we have taken the derivative with respect to the threshold c; it would be more meaningful to compute derivatives with respect to the index spread. However, the index spread is related in a specific monotonic way to c, and one derivative may be computed from the other without difficulty.

The main qualitative observation about Γ_k is that it is positive:

Theorem 4.1. *With notation as above, $\Gamma_k > 0$ for all $k \in \{1, ..., N-1\}$. Thus, for small changes in c, the quantity $V_k(h)$ increases.*

Proof. Recall that

$$\frac{\partial L_k^e}{\partial c} = \int_{\mathbb{R}} I_k\big(p(y)\big)\frac{1}{2\pi\sqrt{1-\rho}}e^{-\frac{y^2}{2(1-\rho)}-\frac{c^2}{2}}\,dy \tag{4.2}$$

and

$$\frac{\partial L_N}{\partial c} = \frac{N}{\sqrt{2\pi}}e^{-\frac{c^2}{2}}, \tag{4.3}$$

where

$$I_k(p) = N - (N-k)k\binom{N}{k}\int_0^p t^{k-1}(1-t)^{N-k-1}\,dt \tag{4.4}$$

and

$$p(y) = \Phi\left(\frac{c(1-\rho)-\sqrt{\rho}y}{\sqrt{1-\rho}}\right). \tag{4.5}$$

As we have seen, the delta of the tranche is:

$$\Delta_{k,\text{spread}} = \frac{1}{N}\int_{\mathbb{R}} I_k\big(p(y)\big)\frac{1}{\sqrt{2\pi(1-\rho)}}e^{-\frac{y^2}{2(1-\rho)}}\,dy. \tag{4.6}$$

Taking the derivative of (4.2), and writing p for $p(y)$, we have

$$\frac{\partial^2 L_k^e}{\partial c^2} = \frac{1}{2\pi\sqrt{1-\rho}}\int_{\mathbb{R}}\left[\frac{\partial I_k(p)}{\partial c}e^{-\frac{y^2}{2(1-\rho)}-\frac{c^2}{2}} + (-c)I(p)e^{-\frac{y^2}{2(1-\rho)}-\frac{c^2}{2}}\right]dy,$$

and

$$\frac{\partial I_k(p)}{\partial c} = -(N-k)k\binom{N}{k}p^{k-1}(1-p)^{N-k-1} \cdot \frac{\sqrt{1-\rho}}{\sqrt{2\pi}}e^{-\frac{(c(1-\rho)-\sqrt{\rho}y)^2}{2(1-\rho)}} \tag{4.7}$$

On the other hand,

$$\frac{\partial^2 L_N}{\partial c^2} \quad = \quad \frac{-cN}{\sqrt{2\pi}}e^{-\frac{c^2}{2}}.$$

Therefore

$$\begin{aligned}
\Gamma_k &= \Delta_{k,\text{spread}}\frac{\partial^2 L_N}{\partial c^2} - \frac{\partial^2 L_k^e}{\partial c^2} \\
&= \left(\frac{-cN}{\sqrt{2\pi}}\right)e^{-c^2/2}\frac{1}{N\sqrt{2\pi(1-\rho)}}\int_{\mathbb{R}} I_k(p)e^{-\frac{y^2}{2(1-\rho)}-}\,dy \\
&\quad -\frac{1}{2\pi\sqrt{1-\rho}}\int_{\mathbb{R}}\left[\frac{\partial I_k(p)}{\partial c}e^{-\frac{y^2}{2(1-\rho)}-\frac{c^2}{2}} + (-c)I_k(p)e^{-\frac{y^2}{2(1-\rho)}-\frac{c^2}{2}}\right]dy \\
&= -\frac{1}{2\pi\sqrt{1-\rho}}\int_{\mathbb{R}}\frac{\partial I_k(p)}{\partial c}e^{-\frac{y^2}{2(1-\rho)}-\frac{c^2}{2}}\,dy.
\end{aligned}$$

From the expression (4.7), we see that, since $k \in \{1, ..., N-1\}$,

$$\tfrac{\partial I_k(p)}{\partial c} < 0 \text{ for all } y \in \mathbb{R},$$

and so we conclude that Γ_k is positive for $k \in \{1, ..., N-1\}$. □

5. The Large-N Limit

As before, we work with the standard Gaussian copula for a portfolio of size N. Since a typical CDO portfolio, at least initially, has over a 100 names, it is of interest to examine the behavior of the portfolio in the limit as $N \to \infty$.

The large-N behavior has been studied through simulations for various copula models, for instance by Schönbucher [17]. Other related works include Andersen and Sidenius [3, Section 3.3], Cousin and Laurent [6], and Frey [8].

As before, default of name i occurs when a random variable X_i falls below a threshold level c. We assume, as model, that there are independent random variables $Z, \epsilon_1, \epsilon_2, ...$, such that

$$X_i = \sqrt{\rho}Z + \sqrt{1-\rho}\epsilon_i$$

where $\rho > 0$ is the correlation between any pair of names in the portfolio.

The portfolio loss, truncated at size N and scaled by N, is

$$L^{(N)} = \frac{1}{N}\sum_{i=1}^{N}\mathbb{1}_{[X_i \le c]}$$

where we have explicitly indicated N on the left. We have then

Theorem 5.1. *The sequence $L^{(N)}/N$ converges with probability 1 to the random variable $\Phi\left(\frac{c-\sqrt{\rho}\,Z}{\sqrt{1-\rho}}\right)$:*

$$\frac{L^{(N)}}{N} \to l^{(\infty)} \stackrel{\text{def}}{=} \Phi\left(\frac{c-\sqrt{\rho}\,Z}{\sqrt{1-\rho}}\right) \quad \textit{almost surely.}$$

Moreover,

$$\frac{L^{(N)}}{N} \to l^{(\infty)}$$

in L^2.

Proof. The variable $L^{(N)}$ is a function of the Gaussian variable $(Z, \epsilon_1, ..., \epsilon_N)$. For each fixed value for Z, it is the average of N independent, identically distributed (bounded) variables. So, by the law of large numbers, for each fixed value z of Z,

$$\lim_{N\to\infty} \frac{L^{(N)}}{N} = \mathbb{P}\left[X_1 \le c \,|\, Z = z\right] = \mathbb{P}\left[\epsilon_1 \le \frac{c - \sqrt{1-\rho}\, z}{\sqrt{\rho}}\right] = \Phi\left(\frac{c - \sqrt{\rho}\, z}{\sqrt{1-\rho}}\right)$$

almost surely in $(\epsilon_1, ..., \epsilon_N)$. Therefore, by Fubini's theorem (which guarantees that a measurable set with all sections of full measure is itself of full measure),

$$\lim_{N\to\infty} \frac{L^{(N)}}{N} = \Phi\left(\frac{c - \sqrt{\rho}\, Z}{\sqrt{1-\rho}}\right)$$

holds almost everywhere.

As for L^2 convergence, denoting $\mathbb{P}[X_i \le c \,|\, Z = z]$ by $p(z)$, we have

$$\mathbb{E}\left[\left\|\frac{1}{N}\sum_{i=1}^{N} 1_{[X_i \le c]} - p(Z)\right\|^2\right] = \mathbb{E}\left[\frac{p(Z)\big(1 - p(Z)\big)}{N}\right] \le \frac{1}{N} \to 0$$

as $N \to \infty$. $\square$

Assuming that Z is also standard Gaussian, the distribution of the limiting average loss $l^{(\infty)}$ is thus

$$\mathbb{P}[l^{(\infty)} \le x] = \Phi\left(\frac{\sqrt{1-\rho}\Phi^{-1}(x) - c}{\sqrt{\rho}}\right) \tag{5.1}$$

This agrees with Schönbucher [17, Eq. (23)] and also the early work of Vasicek [22, page 3]. Similar questions and distributions appear in investigations motivated by physical contexts (see, for instance, Thurner and Hanel [20]).

Now let us consider an extension of this, permitting *a distribution of default thresholds*. To this end suppose that $W_1, W_2, ...$ are independent standard Gaussians, independent of the variables $(Z, \epsilon_1, \epsilon_2, ...)$.

Suppose that the threshold C_i's are all independent random variables with standard Gaussian distribution, and they are also independent of the Gaussian variables $Z, \epsilon_1, \epsilon_2, ...$ Suppose $c \in \mathbb{R}$ and $\sigma > 0$ and assume that the default threshold for name i is

$$C_i = c + \sigma W_i.$$

The proportion of defaulting names, among the first N, is

$$L^{(N)} = \frac{1}{N}\sum_{i=1}^{N} 1_{[X_i \le C_i]}.$$

By the law of large numbers, for each fixed value z of Z,

$$\begin{aligned}
\lim_{N\to\infty} L^{(N)} &= \mathbb{P}[X_1 \le C_1 | Z = z] \\
&= \mathbb{P}[\sqrt{1-\rho}\epsilon_i - \sigma W_i \le c - \sqrt{\rho} z] \\
&= \Phi\left(\frac{c - \sqrt{\rho}\, z}{\sqrt{1-\rho+\sigma^2}}\right).
\end{aligned}$$

Thus,

$$L^{(N)} \to L^{(\infty)} \stackrel{\text{def}}{=} \Phi\left(\frac{c - \sqrt{\rho}\, Z}{\sqrt{1 - \rho + \sigma^2}}\right) \quad \text{almost surely.} \tag{5.2}$$

The special case $\sigma = 0$ yield Theorem 5.1.

6. A Poisson-mixture Model

Variations of the Gaussian copula model include models which use the Poisson distribution (see, for instance, Burtschell [7]). In this section we show that some of the properties proved for the single-factor Gaussian copula model also hold for a 'Poisson mixture' model. A model of this general type is explored in the context of credit defaults in the book of Bluhm et al. [4].

We have again a portfolio of N names. Now, however, we assume that each credit can suffer multi-level 'defaults' (such as downgrades). Conditional on a common global factor, modeled by a standard Gaussian Z, the defaults are independent, and the i-th name's loss distribution is Poisson with mean λ_i given by

$$\lambda_i = \lambda_i(x) = \mathbb{P}\left[X_i \leq c_i \,|\, Z = x\right] = \Phi\left(\frac{c_i - \sqrt{\rho}x}{\sqrt{1-\rho}}\right). \tag{6.1}$$

Here again, we have, for each i, a threshold $c_i \in \mathbb{R}$, and a variable

$$X_i = \sqrt{\rho} Z + \sqrt{1-\rho}\epsilon_i \tag{6.2}$$

where $Z, \epsilon_1, ..., \epsilon_N$ are independent standard Gaussians. Now, however, the threshold is used simply to determine the parameter λ_i.

Note that X_i is standard Gaussian, and so

$$\mathbb{P}[X_i \leq c_i] = \Phi(c_i). \tag{6.3}$$

For small values of λ_i, the probability of multiple losses in an individual name is small, and so the preceding model can be taken as an approximate description for a synthetic CDO of the type we have considered before.

The probability that the portfolio loss ν has value $j \in \{0, 1, 2, ...\}$ is then

$$p_j \stackrel{\text{def}}{=} \mathbb{P}[\nu = j] = \int_{\mathbb{R}} e^{-\lambda} \frac{\lambda^j}{j!} \phi(x)\, dx, \tag{6.4}$$

where

$$\lambda = \lambda(x) = \sum_{i=1}^{N} \lambda_i. \tag{6.5}$$

Then, following the same procedure as for the Gaussian copula, for the expected loss

$$L_k^e = \mathbb{E}\left[\min\{\nu, k\}\right],$$

of the $[0, k]$-tranche, we have

$$\frac{\partial L_k^e}{\partial \rho} = \int_{\mathbb{R}} I_{P,k}(\lambda) \frac{\partial \lambda}{\partial \rho} \phi(x)\, dx \tag{6.6}$$

where

$$I_{P,k}(\lambda) = \sum_{j=0}^{k} (k-j) e^{-\lambda} \left[\frac{\lambda^j}{j!} - \frac{\lambda^{j-1}}{(j-1)!}\right], \tag{6.7}$$

with the second term in $[\cdots]$ here is taken to be 0 when $j = 0$. The subscript P in $I_{P,k}$ is for Poisson.

Lemma 6.1. *For $I_{P,k}$ as in (6.7) we have*

$$I_{P,k}(\lambda) = e^{-\lambda} \sum_{j=0}^{k-1} \frac{\lambda^j}{j!} = 1 - \int_0^\lambda e^{-t} \frac{t^{k-1}}{(k-1)!}\, dt \tag{6.8}$$

In particular, $I_{P,k}(\lambda)$ is monotonically decreasing with λ, if $1 \le k < N$.

Proof. Algebraic simplification gives

$$I_{P,k}(\lambda) = e^{-\lambda} \sum_{j=0}^{k-1} \frac{\lambda^j}{j!} \tag{6.9}$$

The derivative of this is

$$I'_{P,k}(\lambda) = -e^{-\lambda} \sum_{j=0}^{k-1} \frac{\lambda^j}{j!} + e^{-\lambda} \sum_{j=1}^{k-1} \frac{\lambda^{j-1}}{(j-1)!} = -e^{-\lambda} \frac{\lambda^{k-1}}{(k-1)!}.$$

Integrating, and using the value 1 for $I_{P,k}(0)$ by (6.9), we obtain equation (6.8). □

Now by using reasoning similar to that used for the binomial (Gaussian copula) case, we see that

$$\frac{dL_k^e}{d\rho} < 0, \qquad \text{and} \qquad \frac{dL_k^s}{d\rho} > 0. \tag{6.10}$$

For this model we can also study deltas for individual names. We take, as definition,

$$\Delta^i_{\text{spread}}([0,k]) = \frac{\partial L_k^e}{\partial c_i} \Big/ \frac{\partial \Phi(c_i)}{\partial c_i} \tag{6.11}$$

where the denominator is, roughly, the sensitivity of the expected loss for a unit notional credit-default swap on the name i to changes in the threshold c_i.

Using the same method as for the binomial (Gaussian copula) case, but now using Lemma 6.1, we obtain:

$$\Delta^i_{\text{spread}}([0,k]) = \sum_{j \in [0,k]} \int_{\mathbb{R}} e^{-\lambda[i,y]} \frac{\lambda[i,y]^{j-1}}{(j-1)!} \frac{1}{\sqrt{2\pi(1-\rho)}} e^{-\frac{y^2}{2(1-\rho)}}\, dy \tag{6.12}$$

where the summand on the right side is understood to be 0 when $j = 0$, and, also, on the right $\lambda[i,y]$ is λ as a function of y:

$$\lambda[i,y] = \sum_{k=1}^{N} \Phi\left(\frac{(c_k - c_i\rho) - \sqrt{\rho}y}{\sqrt{1-\rho}} \right). \tag{6.13}$$

Notice that $\lambda[i,y]$ is lower for a higher value of c_i (a riskier credit). Then, by Lemma 6.1, $I_{P,k}\big(\lambda[i,y]\big)$ is higher, and so $\Delta^i_{\text{spread}}([0,k])$ is higher; thus,

riskier credits have higher deltas in equity tranches.

This confirms intuition (riskier credits default likely sooner, and therefore impact an equity tranche more) and results of simulations (see, for instance, [11, Chart 8]).

This model permits furthers evaluation of quantities such as iGammas (convexity effect of individual credits), but we leave such questions, in the contexts of more models, as future work.

7. Concluding Remarks

CDOs are credit derivative instruments of great importance in the global financial system. In this paper we have proved in a mathematically rigorous way a variety of key features of CDO tranche prices and tranche risk management characteristics which are often taken for granted in market practice. Our results are mainly, but solely, in the context of the standard single-factor Gaussian copula model, and this may be viewed as an initial step towards a thorough investigation of CDO models.

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