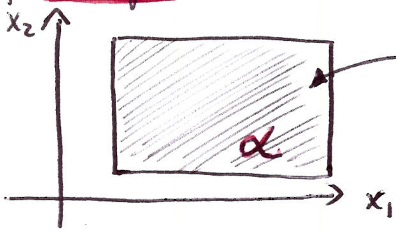
Mathematical Formulation of Thermal Conductivity

I. **A)** Equations of isotropic heat conductivity: Consider a region Γ of conductivity material.



The material is assumed to be isotropic with termal conductivity α, $\alpha \in \mathbb{R}$, $\alpha > 0$.

Let $T(x_1, x_2) = T(x)$ temperature in the conductor (Temperature field)

Then $\nabla T = (\partial_{x_1} T, \partial_{x_2} T)$ is the termal gradient or temperature gradient and the heat flux (vector describing flow of heat) $\sigma = (\sigma_1, \sigma_2)$ is given by:

Fourier's Law of Heat Conduction (1) $\sigma = -\alpha \nabla T$ inside Γ

- If no sources of heat in Γ, then $\operatorname{div} \sigma = 0$

$$\Rightarrow \operatorname{div}(-\alpha \nabla T) = 0$$

$$\Rightarrow -\alpha(\partial_{x_1}^2 T + \partial_{x_2}^2 T) = 0$$

(Laplace Equation)

(2) $T = \xi_1 x_1 + \xi_2 x_2 = \xi \cdot x$ is a linear function on the boundary. Here $\xi = (\xi_1, \xi_2)$; $x = (x_1, x_2)$ and

(3) $-\alpha\left(\partial_{x_1}^2 T + \partial_{x_2}^2 T\right) = 0$ in Γ or in vector notation $\nabla \cdot (-\alpha \nabla T) = 0$

Summing up: The temperature field inside an isotropic conducting region with the boundary temperature given by $T = \xi \cdot x$ solves $\nabla \cdot (-\alpha \nabla T) = 0$ in Γ and it is unique (Schaum's)

Note that $T(x) = \xi \cdot x$ solves the equation and

$$\left.\begin{aligned} \partial_{x_1} T = \xi_1 &\Rightarrow \partial_{x_1}^2 T = 0 \\ \partial_{x_2} T = \xi_2 &\Rightarrow \partial_{x_2}^2 T = 0 \end{aligned}\right\} \nabla \cdot (-\alpha \nabla T) = 0.$$

B) Anisotropic Conductivity: If the material is anisotropic the conductor conducts differently in different directions and so the conductivity is represented by a second order tensor (or matrix) A_{ij}, s.t $\underbrace{A_{ij} = A_{ji}}_{\text{comes from physics}}$ and $A > 0$.

For example, if $A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, then Heat flux $= \sigma = -A\nabla T = \begin{pmatrix} -\alpha & 0 \\ 0 & -\beta \end{pmatrix} \begin{pmatrix} \partial_{x_1} T \\ \partial_{x_2} T \end{pmatrix}$

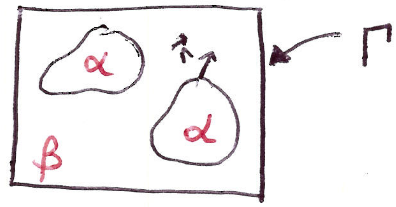
$$= \begin{pmatrix} -\alpha \partial_{x_1} T \\ -\beta \partial_{x_2} T \end{pmatrix} \text{(Fourier's Law)}$$

The temperature field solves (4) $\begin{cases} -\operatorname{div}(A\nabla T) = \nabla \cdot (-A\nabla T) = -\alpha \partial_{x_1}^2 T - \beta \partial_{x_2}^2 T = 0 \\ T = \xi \cdot x \quad \text{on the sides of } \Gamma \end{cases}$

Note that $T(x_1, x_2) = \xi \cdot x$ is a solution of (4), ie we can extend the linear boundary data inside the square as a solution of $-\operatorname{div}(A\nabla T) = 0$.

c) Two phase Conductivity (2 isotropic thermo conductors)

Suppose the conductor is of conductivity $\alpha>0$ in part of Γ and $\beta>0$ in other.



Then given $T(x_1,x_2) = \xi\cdot x$ on sides of Γ, the field T solves

$$(5)\quad \nabla\cdot(-\beta \nabla T) = 0 \quad \text{in } \beta\text{-region}$$

$$(6)\quad \nabla\cdot(-\alpha \nabla T) = 0 \quad \text{in } \alpha\text{-region}$$

$$(7)\quad n\cdot\beta\nabla T\Big|_{\beta\text{-region}} = n\cdot\alpha\nabla T\Big|_{\alpha\text{-region}} \quad \text{at phase boundaries}$$

Note that T is linear in the linear boundary condition $T=\xi\cdot x$, i.e. given T^ξ solving

$$(8)\begin{cases} T^\xi = \xi\cdot x & \text{on } \partial\Gamma \\ \nabla\cdot(-\alpha\nabla T^\xi)=0 \\ \nabla\cdot(-\beta\nabla T^\xi)=0 \\ \alpha\partial_n T^\xi = \beta\partial_n T^\xi & \text{on phase boundary} \end{cases}$$

and T^η, $(\eta=(\eta_1,\eta_2))$ solving

$$(9)\begin{cases} T^\eta = \eta\cdot x & \text{on } \partial\Gamma \\ \nabla\cdot(-\alpha\nabla T^\eta)=0 \\ \nabla\cdot(-\beta\nabla T^\eta)=0 \\ \alpha\partial_n T^\eta = \beta\partial_n T^\eta & \text{on phase boundary} \end{cases}$$

Then $T^\eta + T^\xi = T^{\eta+\xi}$, where $T^{\eta+\xi}$ solves

$$(10)\begin{cases} T^{\eta+\xi} = (\eta+\xi)\cdot x & \text{on } \partial\Gamma \\ \nabla\cdot(-\alpha\nabla T^{\eta+\xi})=0 \\ \nabla\cdot(-\beta\nabla T^{\eta+\xi})=0 \\ \alpha\partial_n T^{\xi+\eta} = \beta\partial_n T^{\xi+\eta} \end{cases}$$

$$\left.\begin{aligned} T^\eta+T^\xi &= \eta\cdot x + \xi\cdot x = (\eta+\xi)\cdot x \quad \text{on } \partial\Gamma \\ 0 = \nabla\cdot(-\alpha\nabla T^\xi) + \nabla\cdot(-\alpha\nabla T^\eta) &= \nabla\cdot(-\alpha\nabla(T^\xi+T^\eta)) \\ 0 = \nabla\cdot(-\beta\nabla T^\xi) + \nabla\cdot(-\beta\nabla T^\eta) &= \nabla\cdot(-\beta\nabla(T^\xi+T^\eta)) \\ \alpha\partial_n(T^\eta+T^\xi) &= \beta\partial_n(T^\eta+T^\xi) \end{aligned}\right\} \quad \therefore T^\xi+T^\eta \text{ solves (10)} \rightarrow T^{\xi+\eta} = T^\xi + T^\eta$$

II. Effective Conductivity of a composite

• Consider Γ containing anisotropic composite of conductivity $A = A_{ij}$ s.t $T(x_1,x_2)=\xi\cdot x$ on $\partial\Gamma$, then $\nabla\cdot(A\nabla T)=0$ in Γ $\Longrightarrow$ $T(x_1,x_2)=\xi\cdot x$ in Γ.

Define

Total Heat Dissipated per Unit time: $\displaystyle\int_\Gamma (A\nabla T\cdot \nabla T)\,dx_1\,dx_2$

$$= \int_\Gamma A\xi\cdot\xi \; dx_1 dx_2$$

$$= A\xi\cdot\xi \int_\Gamma dx_1 dx_2 = A\xi\cdot\xi\,|\Gamma|$$

$$= |\Gamma|\,\underbrace{A\xi\cdot\xi}_{\text{Average total heat dissipated per unit time}}$$

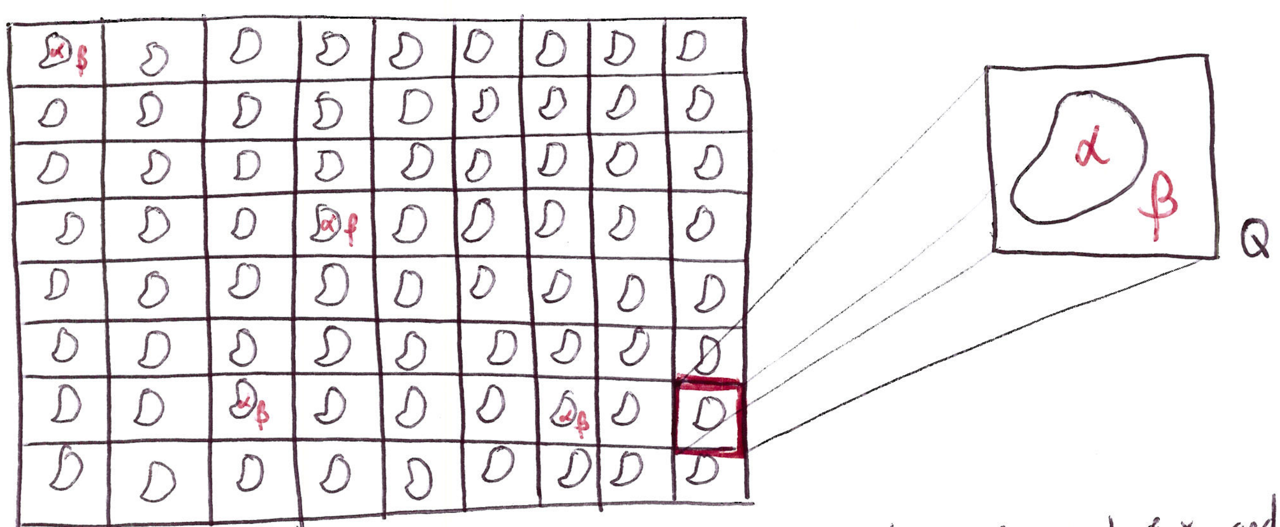
[Here $A\xi\cdot\xi = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}\cdot\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = \begin{pmatrix} A_{11}\xi_1 + A_{12}\xi_2 \\ A_{21}\xi_1 + A_{22}\xi_2\end{pmatrix}\cdot\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}$

$$= A_{11}\xi_1^2 + A_{12}\xi_2\xi_1 + A_{21}\xi_1\xi_2 + A_{22}\xi_2^2$$

$$= \sum_{i,j=1}^{2} A_{ij}\,\xi_i\,\xi_j \quad]$$

• Consider a periodic composite with period cell Q

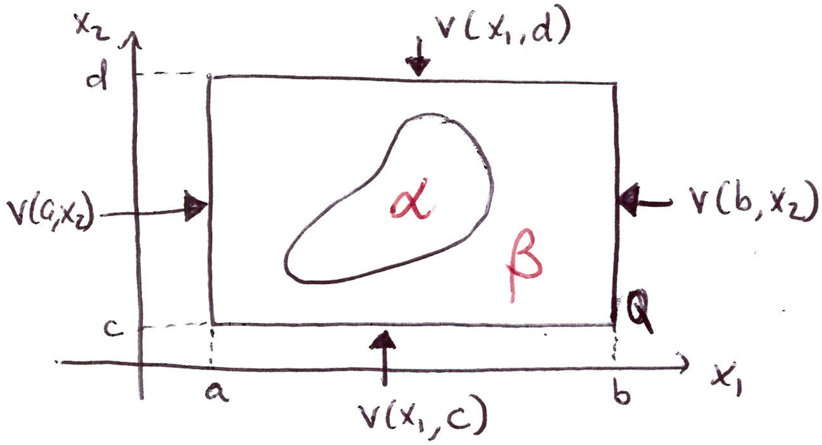
Γ:



For this case we may expect that T has two components: A linear component $\xi\cdot x$ and a Q-periodic one that oscillates with the geometry.

Here $T(x_1,x_2) = \xi\cdot x + v(x_1,x_2)$

$v(x_1,x_2)$ Q-periodic (or satisfies periodic boundary conditions)



The corners don't matter b/c they have measure 0 (they are countable many)

Here $v(x_1, d) = v(x_1, c)$ for $a < x_1 < b$

and $v(a, x_2) = v(b, x_2)$ for $c < x_2 < d$.

Then $v = v^{\xi}$ depends upon the linear component ξ, and

$$(11)\begin{cases} \nabla\cdot(-\alpha \nabla T) = 0 & \text{in } \alpha\text{- region} \\ \nabla\cdot(-\beta \nabla T) = 0 & \text{in } \beta\text{-region} \\ \alpha \partial_{\vec{n}} T = \beta \partial_{\vec{n}} T & \text{on phase boundary.} \end{cases}$$

implies that v^{ξ} solves:

$$(11)'\begin{cases} \nabla\cdot(-\alpha \nabla v^{\xi}) = 0 & \text{in } \alpha\text{-region} \\ \nabla\cdot(-\beta \nabla v^{\xi}) = 0 & \text{in } \beta\text{-region} \\ \alpha n\cdot(\nabla v^{\xi} + \xi) = \beta n\cdot(\nabla v^{\xi} + \xi) & \text{on phase boundary} \\ v^{\xi} \text{ Q-periodic} \end{cases}$$

$$\begin{aligned} 0 = \nabla\cdot(-\alpha \nabla T) &= \nabla\cdot(-\alpha \nabla(\xi\cdot x + v^{\xi})) \\ &= \nabla\cdot(-\alpha(\nabla\xi\cdot x + \nabla v^{\xi})) \\ &= \nabla\cdot(-\alpha(0 + \nabla v^{\xi})) = \nabla\cdot(-\alpha \nabla v^{\xi}) \end{aligned}$$

Remark: It follows immediately that v^{ξ} is linear in ξ (v^{ξ} is unique up to a constant, if we add the assumption $\int_Q v^{\xi}\, dy = 0$ then the constant is fixed and v^{ξ} is unique)

Indeed if v^{η} solves

$$\begin{cases} \nabla\cdot(-\alpha \nabla v^{\eta}) = 0 & \text{in } \alpha \\ \nabla\cdot(-\beta \nabla v^{\eta}) = 0 & \text{in } \beta \\ \alpha n\cdot(\nabla v^{\eta} + \eta) = \beta n\cdot(\nabla v^{\eta} + \eta) & \text{phase boundary} \\ v^{\eta} \text{ Q-periodic} \end{cases}$$

then $v^{\eta} + v^{\xi} = v^{\eta+\xi}$, where $v^{\eta+\xi}$ solves

$$\begin{cases} \nabla\cdot(-\alpha \nabla v^{\eta+\xi}) = 0 & \text{in } \alpha \\ \nabla\cdot(-\beta \nabla v^{\eta+\xi}) = 0 & \text{in } \beta \\ \alpha n\cdot(\nabla v^{\eta+\xi} + (\eta+\xi)) = \beta n\cdot(\nabla v^{\eta+\xi} + (\eta+\xi)) & \text{phase bdry} \\ v^{\eta+\xi} \ \ Q\text{-periodic} \end{cases}$$

$$0 = \nabla\cdot(-\alpha\nabla v^{\eta}) + \nabla\cdot(-\alpha\nabla v^{\xi}) = \nabla\cdot(-\alpha\nabla(v^{\eta}+v^{\xi})) \quad \text{in } \alpha$$

$$0 = \nabla\cdot(-\beta\nabla v^{\eta}) + \nabla\cdot(-\beta\nabla v^{\xi}) = \nabla\cdot(-\beta\nabla(v^{\eta}+v^{\xi})) \quad \text{in } \beta$$

$$\alpha n\cdot(\nabla v^{\xi}+\xi) + \alpha n\cdot(\nabla v^{\eta}+\eta) = \beta n\cdot(\nabla v^{\xi}+\xi) + \beta n\cdot(\nabla v^{\eta}+\eta)$$

$$\Rightarrow \quad \alpha n\cdot(\nabla v^{\xi}+\nabla v^{\eta}+\xi+\eta) = \beta n\cdot(\nabla v^{\xi}+\nabla v^{\eta}+\xi+\eta)$$

so $v^{\xi}+v^{\eta}$ satisfies the above $\Rightarrow$ $v^{\xi}+v^{\eta} = v^{\xi+\eta}$

Given $T(x_1,x_2) = v^{\xi}(x_1,x_2) + \xi\cdot x$ and T solves (11)

The Heat Dissipated $= \displaystyle\int_{\alpha} \alpha \underbrace{(\nabla v^{\xi}+\xi)}_{\nabla T}\cdot\underbrace{(\nabla v^{\xi}+\xi)}_{\nabla T} + \int_{\beta} \beta \underbrace{(\nabla v^{\xi}+\xi)}_{\nabla T}\cdot\underbrace{(\nabla v^{\xi}+\xi)}_{\nabla T}$

$$= \int_{\Gamma} a(x_1,x_2)(\nabla v^{\xi}+\xi)\cdot(\nabla v^{\xi}+\xi)\,dx_1\,dx_2$$

Here $\boxed{a(x_1,x_2) = \alpha\chi_{\alpha}(x_1,x_2) + \beta\chi_{\beta}(x_1,x_2)}$

$$\chi_{\alpha} = \begin{cases} 1 & \text{in } \alpha\text{-region} \\ 0 & \text{outside} \end{cases} \qquad \chi_{\beta} = \begin{cases} 1 & \text{in } \beta\text{-region} \\ 0 & \text{outside} \end{cases}$$

Note: $\chi_{\beta} = 1-\chi_{\alpha}$

$$\boxed{\int_{\Gamma}\chi_{\alpha}(x)\,dx = \theta_{\alpha}, \quad \int_{\Gamma}\chi_{\beta}(x) = \theta_{\beta}, \quad \theta_{\alpha}+\theta_{\beta} = |\Gamma|}$$

Uups... I forgot: v^{ξ} is linear in ξ. Indeed $\gamma v^{\xi} = v^{\gamma\xi}$, γ constant

where $v^{\gamma\xi}$ solves

$$\begin{cases} \nabla\cdot(-\alpha\nabla v^{\gamma\xi}) = 0 & \text{in } \alpha \\ \nabla\cdot(-\beta\nabla v^{\gamma\xi}) = 0 & \text{in } \beta \\ \alpha n\cdot(\nabla v^{\gamma\xi}+\gamma\xi) = \beta n\cdot(\nabla v^{\gamma\xi}+\gamma\xi) & \text{phase bdry} \\ v^{\gamma\xi} \ \ Q\text{-periodic} \end{cases}$$

The proof: SAME THING!

By the linearity of v^{ξ} in ξ, note that

$$\boxed{v^{\xi} = v^{\xi_1 e^1 + \xi_2 e^2} = v^{\xi_1 e^1} + v^{\xi_2 e^2} = \xi_1 v^{e^1} + \xi_2 v^{e^2}}$$

where $e^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

v^{e^i} solves:
(For $i=1,2$)

$$\begin{cases} v^{e^i} \text{ Q-periodic} \\ \nabla\cdot(-\alpha \nabla v^{e^i}) = 0 \text{ in } \alpha \\ \nabla\cdot(-\beta \nabla v^{e^i}) = 0 \text{ in } \beta \\ \alpha n\cdot(\nabla v^{e^i} + e^i) = \beta n\cdot(\nabla v^{e^i} + e^i) \text{ on phase boundary} \end{cases}$$

Then, the Heat Dissipated in Composite: $\int_{\Gamma} a(x_1,x_2)(\nabla v^{\xi} + \xi)\cdot(\nabla v^{\xi} + \xi)\, dx_1 dx_2$

$$= \int_{\Gamma} a(x_1,x_2)\left(\nabla(\xi_1 v^{e^1} + \xi_2 v^{e^2}) + (\xi_1 e^1 + \xi_2 e^2)\right)\cdot\left(\nabla(\xi_1 v^{e^1} + \xi_2 v^{e^2}) + (\xi_1 e^1 + \xi_2 e^2)\right)$$

$$= \int_{\Gamma} a(x_1,x_2)\left[\left(\nabla(\xi_1 v^{e^1}) + \xi_1 e^1\right) + \left(\nabla(\xi_2 v^{e^2}) + \xi_2 e^2\right)\right]\left[\left(\nabla(\xi_1 v^{e^1}) + \xi_1 e^1\right) + \left(\nabla(\xi_2 v^{e^2}) + \xi_2 e^2\right)\right]$$

$$= \int_{\Gamma} a(x_1,x_2)\left[\xi_1(\nabla v^{e^1} + e^1) + \xi_2(\nabla v^{e^2} + e^2)\right]\left[\xi_1(\nabla v^{e^1} + e^1) + \xi_2(\nabla v^{e^2} + e^2)\right] dx_1 dx_2$$

$$= \int_{\Gamma} a(x_1,x_2)\Big[\xi_1^2 (\nabla v^{e^1} + e^1)\cdot(\nabla v^{e^1} + e^1) + \xi_1\xi_2 (\nabla v^{e^1} + e^1)\cdot(\nabla v^{e^2} + e^2) +$$

$$\xi_2\xi_1 (\nabla v^{e^2} + e^2)\cdot(\nabla v^{e^1} + e^1) + \xi_2^2 (\nabla v^{e^2} + e^2)\cdot(\nabla v^{e^2} + e^2)\Big] dx_1 dx_2$$

$$= \int_{\Gamma} a(x_1,x_2)(\nabla v^{e^1} + e^1)\cdot(\nabla v^{e^1} + e^1)\, dx_1 dx_2\, \xi_1^2 + \int_{\Gamma} a(x_1,x_2)(\nabla v^{e^1} + e^1)\cdot(\nabla v^{e^2} + e^2)\, dx_1 dx_2\, \xi_1\xi_2$$

$$+ \int_{\Gamma} a(x_1,x_2)(\nabla v^{e^2} + e^2)\cdot(\nabla v^{e^1} + e^1)\, dx_1 dx_2\, \xi_2\xi_1 + \int_{\Gamma} a(x_1,x_2)(\nabla v^{e^2} + e^2)\cdot(\nabla v^{e^2} + e^2)\, dx_1 dx_2\, \xi_2^2$$

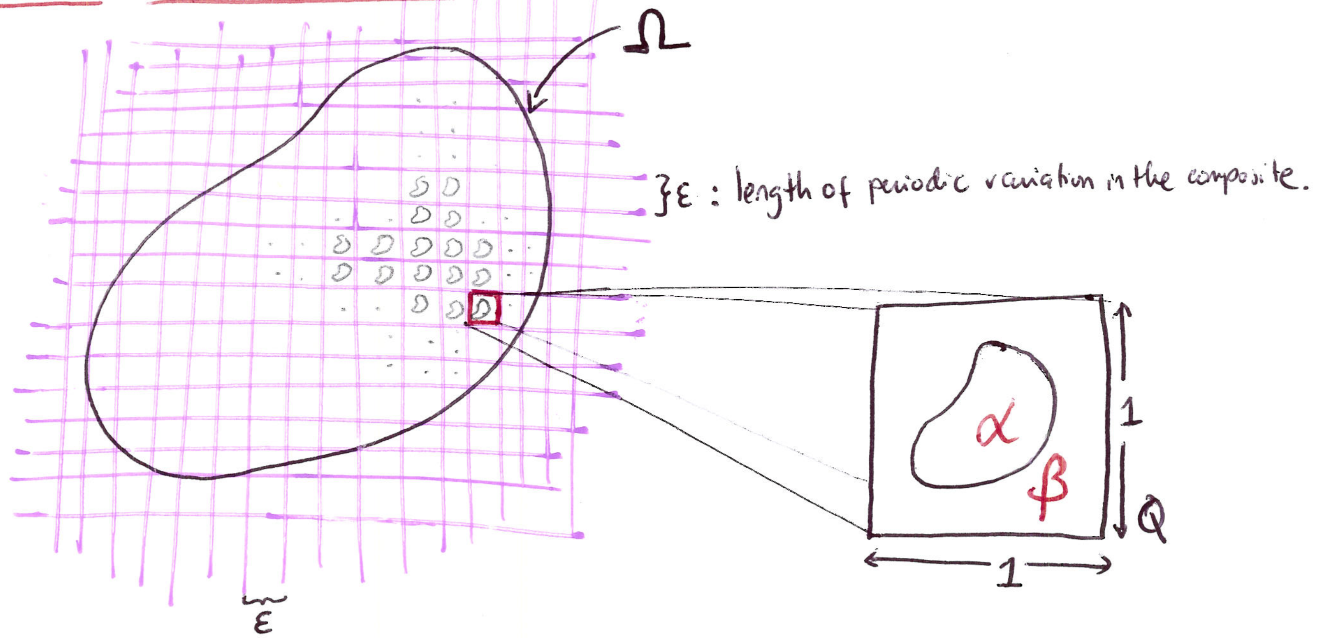
$$= \sum_{i,j=1}^{2}\left[\int_{\Gamma} a(x_1,x_2)(\nabla v^{e^i} + e^i)\cdot(\nabla v^{e^j} + e^j)\, dx_1 dx_2\right]\xi_i\xi_j$$

$$= |\Gamma|\, A^h_{ij}\, \xi_i \xi_j$$

where
$$\boxed{A^h_{ij} = \frac{1}{|\Gamma|}\int_\Gamma a(x_1,x_2)(\nabla v^{e^i}+e^i)\cdot(\nabla v^{e^j}+e^j)\,dx_1dx_2}$$

DEF: The tensor A^h_{ij} is called the effective conductivity and represents the HOMOGENEOUS anisotropic conductor that dissipates heat equal to the composite.

PERIODIC HOMOGENIZATION



Two phase conductor. Conductivity $a(x) = \alpha\chi_\alpha(x) + \beta\chi_\beta(x)$. Here $\chi_\alpha = 1-\chi_\beta$ periodic with period cell Q.

We introduce the parameter ε to denote the scale of the periodic structure. We write
$$\boxed{a^\varepsilon(x) = a\left(\frac{x}{\varepsilon}\right) = \alpha\chi_\alpha\left(\frac{x}{\varepsilon}\right) + \beta\chi_\beta\left(\frac{x}{\varepsilon}\right)}$$

We consider the Dirichlet problem:
$$T^\varepsilon = T^\varepsilon(x_1,x_2) \text{ solves: } \begin{cases} -\nabla\cdot\left(a\left(\frac{x}{\varepsilon}\right)\nabla T^\varepsilon\right) = -\mathrm{div}\left(a\left(\frac{x}{\varepsilon}\right)\nabla T^\varepsilon\right) = f \text{ in } \Omega \\ \left[n\cdot a\left(\frac{x}{\varepsilon}\right)\nabla T^\varepsilon\right]_\alpha^\beta = 0 \text{ on phase boundary} \\ T^\varepsilon = 0 \text{ on } \partial\Omega \end{cases}$$

i.e the heat fluxes are continuous
$$n\cdot\underbrace{a^\varepsilon(x)}_{\alpha}\nabla T^\varepsilon\Big|_{\alpha\text{-side}} = n\cdot\underbrace{a^\varepsilon(x)}_{\beta}\nabla T^\varepsilon\Big|_{\beta\text{-side}}$$

$\Rightarrow \nabla T^\varepsilon$ has to oscillate to create this equality

QUESTION: What happens to the solutions T^ε in the limit as $\varepsilon \to 0$?

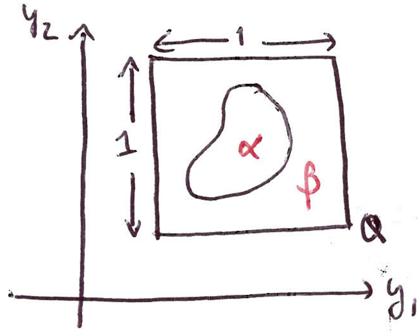
FUNDAMENTAL THEOREM OF PERIODIC HOMOGENIZATION

The solutions T^ε converge to a limit solution T^h, where T^h solves

$$\begin{cases} \nabla \cdot (-A^h \nabla T^h) = f \text{ in } \Omega \\ [n \cdot A^h \nabla T^h]_\alpha^\beta = 0 \quad \text{on phase interphase} \\ T^h = 0 \quad \text{on } \partial\Omega \end{cases}$$

Here A^h is none other than the Effective conductivity defined in page 7.

$$\boxed{A^h_{ij} = \frac{1}{|Q|}\int_Q a(y_1, y_2)\,(\nabla v^{e^i} + e^i)\cdot(\nabla v^{e^j} + e^j)\,dy_1 dy_2 \quad , \; |Q| = 1}$$



where $v^{e^i}(y_1, y_2)$ solves the "cell problem"

$$\begin{cases} \nabla \cdot (-a(y) \nabla v^{e^i}) = 0 \text{ in } Q \\ [n \cdot a(y)\,(\nabla v^{e^i} + e^i)]_\alpha^\beta = 0 \\ v^{e^i} \quad \text{periodic in } Q \end{cases}$$

- The convergence of T^ε to T^h is in "mean square", i.e
$$\int_\Omega |T^\varepsilon - T^h|^2 dx_1 dx_2 \longrightarrow 0 \quad \text{as } \varepsilon \to 0$$
and
- ∇T^ε converge to ∇T^h in the sense of averages, i.e
$$\int_\Omega |\nabla T^\varepsilon|^2 dx_1 dx_2 < C \quad \forall \varepsilon > 0$$
and for all cube $D \subseteq \Omega$ we have
$$\lim_{\varepsilon \to 0} \frac{1}{|D|}\int_D \nabla T^\varepsilon \, dx_1 dx_2 = \frac{1}{|D|}\int_D \nabla T^h \, dx_1 dx_2$$

- Also $$\lim_{\varepsilon \to 0} \frac{1}{|D|}\int_D \underbrace{a^\varepsilon(x) \nabla T^\varepsilon(x)\, dx}_{\text{fluxes.}} = \frac{1}{|D|}\int_D A^h \nabla T^h(x)\, dx$$, i.e the fluxes also converge weakly.

1. Effective Conductivity of a Composite

A

A. Consider Γ containing anisotropic composite of conductivity $A = A_{ij}$ s.t

$$\begin{cases} T(x) = \xi \cdot x & \text{on } \partial\Gamma \\ \nabla \cdot (A\nabla T) = 0 & \text{in } \Gamma \end{cases}$$

Then $T(x) = \xi \cdot x$ in Γ.

Total Heat Dissipated per unit time: $\int_\Gamma (A\nabla T) \cdot \nabla T \, dx$

$$= \int_\Gamma A\xi \cdot \xi \, dx$$

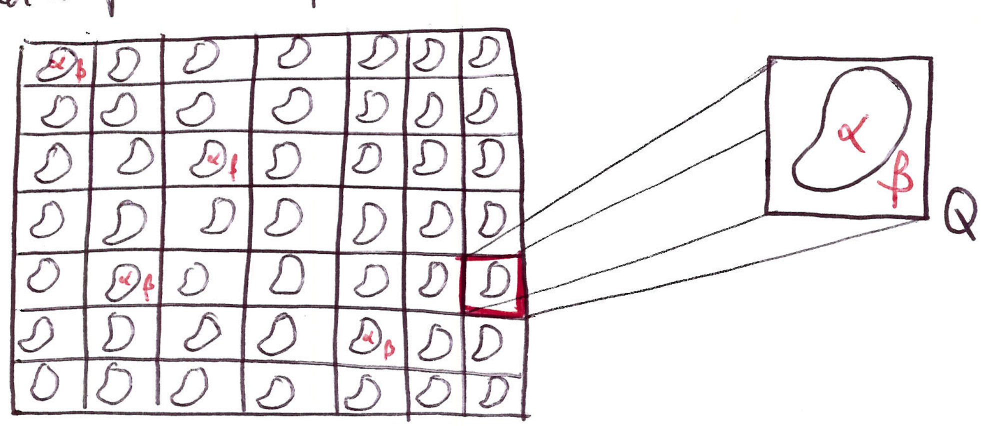
$$= A\xi \cdot \xi \int_\Gamma 1 \, dx = A\xi \cdot \xi \, |\Gamma|$$

$$= |\Gamma| \underbrace{A\xi \cdot \xi}_{\text{Average total heat dissipated per unit time.}}$$

Here
$$A\xi \cdot \xi = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \cdot \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$
$$= \begin{pmatrix} A_{11}\xi_1 + A_{12}\xi_2 \\ A_{21}\xi_1 + A_{22}\xi_2 \end{pmatrix} \cdot \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$
$$= A_{11}\xi_1^2 + A_{12}\xi_2\xi_1 + A_{21}\xi_1\xi_2 + A_{22}\xi_2^2$$
$$= \sum_{i,j=1}^{2} A_{ij}\xi_i\xi_j$$

$$= |\Gamma| \sum_{i,j=1}^{2} A_{ij}\xi_i\xi_j$$

B. Consider a periodic composite with period cell Q



For this case

$$\begin{cases} T(x) = \xi \cdot x \text{ on } \partial\Gamma \\ \nabla \cdot (a(x) \nabla T) = 0 \text{ in } \Gamma \\ \boxed{a(x) = \alpha \chi_\alpha(x) + \beta \chi_\beta(x)} \\ n \cdot a(x) \nabla T \big|_\alpha = n \cdot a(x) \nabla T \end{cases}$$

(B)

- $T(x) = \xi \cdot x + v^\xi(x)$, $v^\xi(x)$ Q-periodic
- v^ξ solves

$$\begin{cases} \nabla \cdot (-\alpha \nabla v^\xi) = 0 & \text{in } \alpha\text{-region} \\ \nabla \cdot (-\beta \nabla v^\xi) = 0 & \text{in } \beta\text{-region} \\ \alpha n \cdot (\nabla v^\xi + \xi) = \beta n \cdot (\nabla v^\xi + \xi) & \text{on phase boundary} \\ v^\xi \text{ Q-periodic} \end{cases}$$

- v^ξ is linear in ξ
  - $v^\eta + v^\xi = v^{\eta+\xi}$ $\qquad \eta, \xi \in \mathbb{R}^2$
  - $v^{\gamma\xi} = \gamma v^\xi$ $\qquad \gamma$ constant, $\xi \in \mathbb{R}^2$

$$\Longrightarrow \boxed{v^\xi = v^{\xi_1 e^1} + v^{\xi_2 e^2} = \xi_1 v^{e^1} + \xi_2 v^{e^2}},$$

where $e^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$

Total Heat Dissipated per unit time in composite $= \int_\Gamma a(x) \nabla T \cdot \nabla T \, dx$

$$= \int_\Gamma a(x) (\nabla v^\xi + \xi) \cdot (\nabla v^\xi + \xi) \, dx$$

$$= \int_\Gamma a(x) (\xi_1 \nabla v^{e^1} + \xi_2 \nabla v^{e^2} + \xi_1 e^1 + \xi_2 e^2) \cdot (\xi_1 \nabla v^{e^1} + \xi_2 \nabla v^{e^2} + \xi_1 e^1 + \xi_2 e^2) dx$$

$$= \sum_{i,j=1}^{2} \left[\int_\Gamma a(x) (\nabla v^{e^i} + e^i) \cdot (\nabla v^{e^j} + e^j) \, dx \right] \xi_i \xi_j$$

$$= |\Gamma| \sum_{i,j=1}^{2} \underbrace{\left[\frac{1}{|\Gamma|} \int_\Gamma a(x_1, x_2) (\nabla v^{e^i} + e^i) \cdot (\nabla v^{e^j} + e^j) \, dx_1 dx_2 \right]}_{A^h_{ij}} \xi_i \xi_j$$

$$= |\Gamma| \sum_{i,j=1}^{2} A^h_{ij} \xi_i \xi_j$$

DEF: A_{ij}^h is called the EFFECTIVE CONDUCTIVITY and represents the HOMOGENEOUS anisotropic conductor that dissipates heat equal to the composite. ⓒ

HOMOGENIZATION THEORY

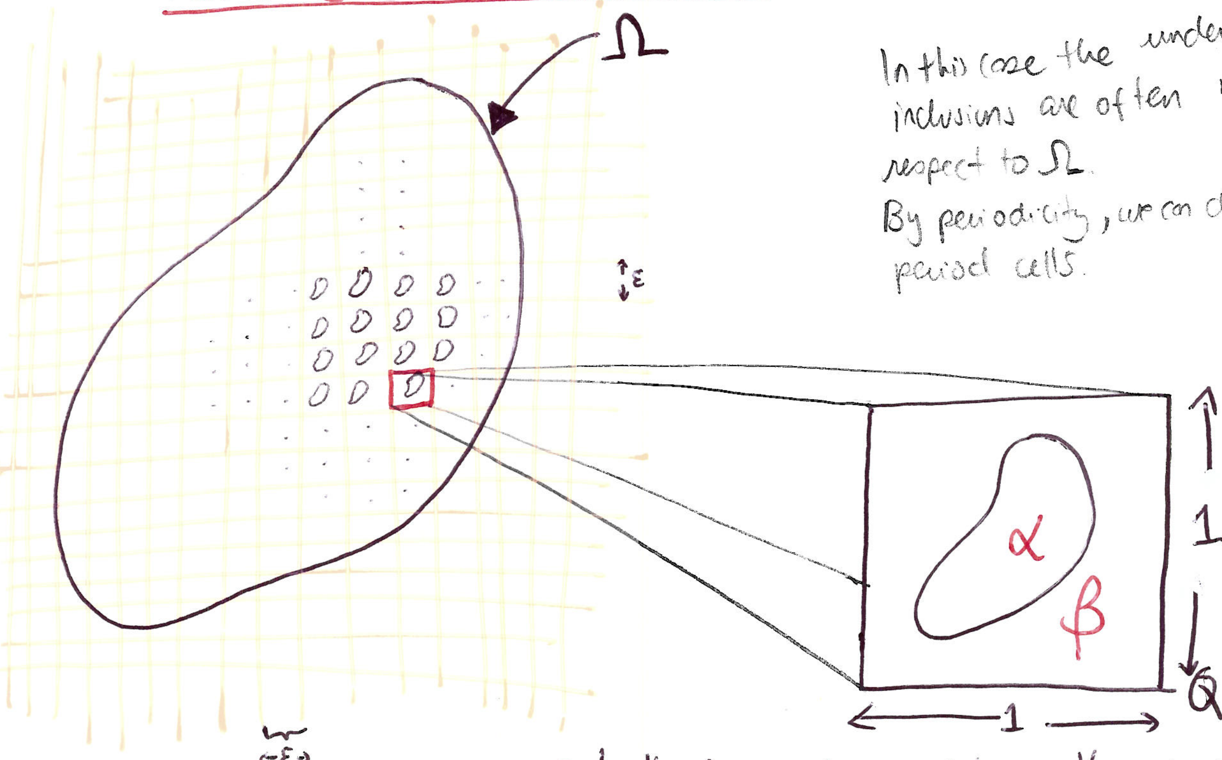
The Theory of homogenization dates back to the late sixties, it has been rapidly developed during the last two decades, and it is now established as a distinct discipline within mathematics.

Composites are materials that have inhomogeneities on length scales that are much larger than the atomic scale (which allows us to use the equations of classical physics at the length scale of the inhomogeneities) but which are essentially (statistically) homogeneous at macroscopic length scales.

A important problem is to determine MACROSCOPIC EFFECTIVE PROPERTIES (for example heat transfer, elasticity, electric conductivity, magnetic permeability, etc) of strongly heterogeneous multiphase materials. A common feature in such problems is that the governing equations involve rapidly oscillating functions due to the heterogeneity of the underlying material, i.e., the physical parameters (such as conductivity, elasticity coefficients, etc...) are discontinuous and oscillate very rapidly b/w the different values characterizing each of the components.

These rapid oscillations render a direct numerical treatment very hard or even impossible. Therefore one has to do some kind of AVERAGING.

PERIODIC HOMOGENIZATION



In this case the underlying periodic inclusions are often microscopic with respect to Ω.
By periodicity, we can divide Ω into period cells.

ε: length of periodic variation in the composite.

Ⓓ

Two phase conductor.

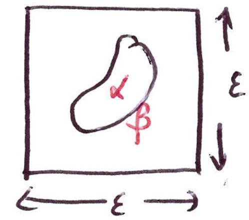
Conductivity $a(x) = \alpha \chi_\alpha(x) + \beta \chi_\beta(x)$. Here $\chi_\alpha = 1 - \chi_\beta$ periodic with period cell Q.

We introduce the parameter $\varepsilon > 0$ to denote the scale of the periodic structure.

We write

$$\boxed{a^\varepsilon(x) = a\left(\frac{x}{\varepsilon}\right) = \alpha \underbrace{\chi_\alpha\left(\frac{x}{\varepsilon}\right) + \beta \chi_\beta\left(\frac{x}{\varepsilon}\right)}_{\text{are } \varepsilon Q\text{-periodic}}}$$

εQ:



Consider the problem: $T^\varepsilon = T^\varepsilon(x_1, x_2)$ solves

$$\begin{cases} -\nabla \cdot \left(a^\varepsilon(x) DT^\varepsilon\right) = -\operatorname{div}\left(a\left(\frac{x}{\varepsilon}\right) DT^\varepsilon(x)\right) = f \text{ in } \Omega \\ n \cdot a^\varepsilon(x) DT^\varepsilon \Big|_\alpha = n \cdot a^\varepsilon(x) DT^\varepsilon \Big|_\beta \quad \text{on phase bdary.} \\ T^\varepsilon(x) = 0 \quad \text{on } \partial\Omega \end{cases}$$

$$\dashrightarrow \quad n \cdot \alpha \nabla T^\varepsilon \Big|_{\alpha\text{-side}} = n \cdot \beta DT^\varepsilon \Big|_{\beta\text{-side}}$$

$\Rightarrow DT^\varepsilon$ has to oscillate a lot! to have this equality

We have a family of problems for each $\varepsilon > 0$.

QUESTION: What happens to the solutions T^ε in the limit as $\varepsilon \to 0$?

the smaller ε gets, the finer the microstructure becomes. For small values of ε, the material macroscopically appears to behave like a homogeneous material, even though the material is strongly heterogeneous on the microscopic level.

FUNDAMENTAL THEOREM OF PERIODIC HOMOGENIZATION

The solutions T^ε converge to a limit solution T^h,

where T^h solves:

$$\begin{cases} \nabla \cdot \left(-A^h \nabla T^h\right) = f \text{ in } \Omega \\ n \cdot A^h DT^h \Big|_\alpha = n \cdot A^h DT^h \Big|_\beta \quad \text{phase bdary.} \\ T^h = 0 \text{ on } \partial\Omega \end{cases}$$

where A^h is none other than the EFFECTIVE CONDUCTIVITY defined in pages Ⓑ & Ⓒ.

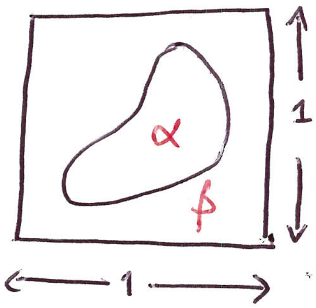
Physical parameters of a homogeneous body, whose behavior is equivalent, from a "macroscopic point of view" to the behavior of the material with the given microstructure.

E

$$\boxed{\begin{aligned} A^h_{ij} &= \frac{1}{|Q|}\int_Q a(y_1,y_2)\,(Dv^{e^i}+e^i)\cdot(Dv^{e^j}+e^j)\,dy_1\,dy_2 \,, \quad \text{since } |Q|=1 \\ &= \int_Q a(y_1,y_2)\,(Dv^{e^i}+e^i)\cdot(Dv^{e^j}+e^j)\,dy_1\,dy_2. \end{aligned}}$$

Where $v^{e^i}(y)$ solves the "cell problem" $\quad y=\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

Q:



$$\begin{cases} -D\cdot\left(a(y)\,Dv^{e^i}\right)=0 \text{ in } Q \\ n\cdot a(y)\,(Dv^{e^i}+e^i)\Big|_\alpha = n\cdot a(y)\,(Dv^{e^i}+e^i)\Big|_\beta \\ v^{e^i} \quad Q\text{-periodic} \end{cases}$$

QUESTION: What kind of convergence do we have?

- The convergence of T^ε to T^h is in "mean square", i.e

$$\boxed{\int_\Omega |T^\varepsilon - T^h|^2\,dx \longrightarrow 0 \quad \text{as } \varepsilon\to 0.}$$

- The convergence of DT^ε to DT^h is in "the sense of averages", i.e

$$\int_\Omega |DT^\varepsilon|^2\,dx < C \quad \text{for all } \varepsilon>0 \text{ and for all cube } D\subseteq\Omega$$

we have

$$\boxed{\lim_{\varepsilon\to 0}\frac{1}{|D|}\int_D DT^\varepsilon\,dx = \frac{1}{|D|}\int_D DT^h\,dx}$$

- The fluxes also converge in the sense of averages

$$\boxed{\lim_{\varepsilon\to 0}\frac{1}{|D|}\int_D a^\varepsilon(x)\,DT^\varepsilon(x)\,dx = \frac{1}{|D|}\int_D A^h\,DT^h(x)\,dx.}$$

(F)

$f(x) = \sin x$

$f^\varepsilon(x) = \sin\left(\frac{x}{\varepsilon}\right) \qquad D = [0, 2\pi]$

$$\lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_0^{2\pi} \sin\left(\frac{x}{\varepsilon}\right) dx = \lim_{\varepsilon \to 0} \frac{1}{2\pi} \left(-\cos\left(\frac{x}{\varepsilon}\right) \cdot \varepsilon\right) \Big|_0^{2\pi}$$

$$= \lim_{\varepsilon \to 0} \frac{-\varepsilon}{2\pi} \left[-\cos\left(\frac{2\pi}{\varepsilon}\right) + 1\right] = 0 = \frac{1}{2\pi} \int_0^{2\pi} 0 \, dx$$

$$0 \leq 1 - \cos\left(\frac{2\pi}{\varepsilon}\right) < 2$$

$$\Rightarrow \frac{-\varepsilon}{\pi} \leq \frac{-\varepsilon}{2\pi}\left(1 - \cos\left(\frac{2\pi}{\varepsilon}\right)\right) \leq 0$$

$$\frac{-\varepsilon}{\pi} \xrightarrow{\varepsilon \to 0} 0 \qquad 0 \xrightarrow{\varepsilon \to 0} 0$$

so $\boxed{f^\varepsilon(x) \text{ cvs to } 0 \text{ in the sense of averages.}}$

Q: If $h^\varepsilon(x) \longrightarrow a(x)$ in the sense of averages and
$g^\varepsilon(x) \longrightarrow b(x)$ in the sense of averages
Does $h^\varepsilon(x) \cdot g^\varepsilon(x) \longrightarrow a(x) b(x)$ in the sense of averages?

A. NO!!

$$f^\varepsilon(x) \cdot f^\varepsilon(x) = \sin^2\left(\frac{x}{\varepsilon}\right)$$

$$\lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_0^{2\pi} \sin^2\left(\frac{x}{\varepsilon}\right) = \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - \cos\left(\frac{2x}{\varepsilon}\right)}{2} dx$$

$$= \lim_{\varepsilon \to 0} \frac{1}{2\pi} \left[\frac{x}{2} - \frac{\sin\left(\frac{2x}{\varepsilon}\right)}{2} \cdot \frac{\varepsilon}{2}\right] \Big|_0^{2\pi} = \lim_{\varepsilon \to 0} \frac{1}{2\pi} \left[\pi - \frac{\varepsilon}{4} \sin\left(\frac{4\pi}{\varepsilon}\right) + 0\right]$$

$$= \lim_{\varepsilon \to 0} \frac{1}{2} - \underbrace{\frac{\varepsilon}{8\pi} \sin\left(\frac{4\pi}{\varepsilon}\right)}_{=0} = \frac{1}{2} \neq 0.$$

called $\boxed{\text{WEAK CONVERGENCE.}}$

Define the corrector matrix

$$P_{ij}(x) = \partial_{x_i} v^{e^j} + e^j_i \qquad e^j_i = \delta_{ij} = \begin{cases} 1 \text{ if } i=j \\ 0 \text{ otherwise} \end{cases}$$

THM:

$$\nabla T^{\varepsilon} = \overbrace{P^{\varepsilon}(x)\nabla T^{h}}^{\text{Corrector!}} + r^{\varepsilon}$$

where if $\partial_{x_i} v^{e^j}$ are nice

$$\int_{\Omega} |r^{\varepsilon}|^2 \, dx = \int_{\Omega} |\nabla T^{\varepsilon} - P^{\varepsilon}(x)\nabla T^{h}|^2 \xrightarrow[\varepsilon \to 0]{} 0$$

$\boxed{\text{POSTER}}$