AN EXPONENTIAL STABILITY CRITERION FOR q-PERIODIC EVOLUTION FAMILIES ON HILBERT SPACES

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Abstract

We consider a mild solution $u_{\mu x}$ of a well-posed inhomogeneous, Cauchy problem

$$\dot{u}(t) = A(t)u(t) + e^{i\mu t}x \quad (t \geq 0), \mu \in \mathbf{R}, u(0) = 0$$

on a separable Hilbert space H, where $A(\cdot)$ is periodic. We prove that if

$$\sup_{\mu \in \mathbf{R}} \sup_{t>0} ||u_{\mu x}(t)|| < \infty, \quad \forall x \in H$$

then the solution of the Cauchy problem

$$\dot{u}(t) = A(t)u(t) \quad (t \geq 0), \quad u(0) = x \in H$$

is uniformly exponentially stable for every $x \in H$. The basic idea for the proof is to apply the spectral mapping theorem for an evolution semigroup associated with a q-periodic evolution family $\mathcal{U}$ on a H-valued almost periodic functions space.

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1. INTRODUCTION

Let X be a Banach space and $\mathcal{L}(X)$ the space of all bounded linear operators on X. We denote by $|| \cdot ||$ the norms of vectors and operators on X. Let A be a linear and bounded operator acting on X. We consider the system

$$\dot{x}(t) = Ax(t) \quad (t \geq 0) \tag{1.1}$$

and the Cauchy problem

$$\dot{u}(t) = Au(t) + e^{i\mu t}x \quad (t \geq 0), \quad x(0) = 0 \tag{1.2}$$

where $\mu \in \mathbf{R}$ and $x \in X$. It is well-known that the system (1.1) is uniformly exponentially stable, that is there exist the constants $N > 0$ and $\nu > 0$ such that $||e^{tA}|| \leq Ne^{-\nu t}$ for all $t \geq 0$, if and only if the solution of the Cauchy problem (1.2) is bounded for every $\mu \in \mathbf{R}$ and any $x \in X$, i.e.

$$\sup_{t>0} ||\int_0^t e^{-i\mu\xi}e^{\xi A}xd\xi|| < \infty, \quad \forall \mu \in \mathbf{R}, \quad \forall x \in X.$$

If A is an unbounded linear operator, the above result is false, see e.g. [12, Example 3.1], but some weakly results hold. For example, a strongly continuous semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$ of bounded linear operators on X is uniformly exponentially stable, i.e. its growth bound

$$\omega_0(\mathbf{T}) := \lim_{t\to\infty} \frac{\ln ||T(t)||}{t}$$

is negative, if and only if for all $\mu \in \mathbf{R}$ and every $f \in \mathcal{P}(q)$ the mild solution of the Cauchy problem

$$\dot{u}(t) = A_T u(t) + e^{i\mu t}f(t) \quad (t \geq 0), \quad u(0) = 0 \tag{1.3}$$

is bounded, i.e.

$$\sup_{t>0} ||\int_0^t e^{-i\mu\xi}T(t-\xi)f(\xi)d\xi|| < \infty, \quad \forall \mu \in \mathbf{R}, \quad \forall f \in \mathcal{P}(q). \tag{1.4}$$

Here $A_{\mathbf{T}}$ is the infinitesimal generator of $\mathbf{T}, \mathcal{P}(q)$ is the space of all X-valued continuous and q-periodic functions defined on $\mathbf{R}_+ = [0, \infty)$ and q is some positive real number. More exactly in [13, Corollary 4.5 and its Reformulation], Vu and Schüler was shown that if (1.4) holds for $\mu = 0$ then $1 \in \rho(T(q))$. With different metods in [4] it has been shown that if (1.4) holds with $\mathcal{P}(q)$ replaced by $\mathcal{P}_0(q) := \{f \in \mathcal{P}(q) : \quad f(0) = 0\}$ and some $\mu \in \mathbf{R}$, then $T(q)$ is power bounded and $e^{i\mu} \in \rho(T(q))$.

Let $x \in X$ and $\omega(x)$ the infimum of all $\omega \in \mathbf{R}$ for which there is $M_\omega > 0$ such that $||T(t)x|| \leq M_\omega e^{\omega t}$ for every $t \geq 0$. Let $\omega_1(\mathbf{T})$ the supremum of all

$\omega(x)$ with $x \in D(A_{\mathbf{T}})$. In [11] J. van Neerven was shown that if there is a constant $M > 0$, independent of μ, such that

$$\sup_{t>0} \|\int_0^t e^{-i\mu\xi} T(\xi) x d\xi\| \leq M\|x\|, \quad \forall \mu \in \mathbf{R}, \quad \forall x \in X. \tag{1.5}$$

then $\omega_1(\mathbf{T}) < 0$, that is, that every solution of the system $\dot{x}(t) = A_{\mathbf{T}}x(t)$ starting in $D(A_{\mathbf{T}})$ is exponentially stable. However may be exist solutions of the system $\dot{x}(t) = A_{\mathbf{T}}x(t)$ starting in $X \setminus D(A_{\mathbf{T}})$ which are not exponentially stable, even if (1.5) holds, see eg [2, Example 2]. Moreover, in [11, Corollary 5 and the Proof of Theorem 4] it is shown that if (1.5) holds then the operator resolvent $R(z, A_{\mathbf{T}})$ exists and is uniformly bounded on $\{\Re(z) > 0\}$. Combining this with a result of L. Gearhart [7], see also F. Huang [8] or G. Weiss [14], it results in the case when X is a complex Hilbert space that $\omega_0(\mathbf{T})$ is negative, i.e. every solution of the system $\dot{x}(t) = A_{\mathbf{T}}x(t) \quad (t \geq 0)$, is exponentially stable.

For a well-posed Cauchy problem

$$\dot{u}(t) = A(t)u(t) \quad (t \geq 0), \quad u(0) = x \in X, \tag{1.6}$$

on a complex Banach space X with (unbounded) linear operators $A(t)$ the solutions of (1.6) lead to an evolution family $\mathcal{U} = \{U(t,s): \quad t \geq s \geq 0\}$ in the space $\mathcal{L}(X)$, that is

- $(\mathbf{e_1})$ $\quad U(t,t) = I, U(t,\tau)U(\tau,s) = U(t,s)$ for $t \geq \tau \geq s \geq 0$;
- $(\mathbf{e_2})$ the map $(t,s) \mapsto U(t,s)x$ is continuous for every $x \in X$.

When the Cauchy problem (1.6) is periodic, i.e. there exists $q > 0$ such that $A(t+q) = A(t)$ for $t \in \mathbf{R}_+$, the corresponding evolution family $\mathcal{U}$ is periodic, that is

- $(\mathbf{e_3})$ $\quad U(t+q, s+q) = U(t,s), \quad \forall t \geq s \geq 0.$

An evolution family which satisfies $(\mathbf{e_1}), (\mathbf{e_2})$ and $(\mathbf{e_3})$ will be called q-periodic evolution family on X. Every q-periodic evolution family $\mathcal{U}$ on X has exponential growth, i.e. there exist $\omega \in \mathbf{R}$ and $M > 0$ such that

$$\|U(t,s)\| \leq Me^{\omega(t-s)} \quad \forall t \geq s \geq 0, \tag{1.7}$$

see [3, Lemma 1] or [6, Theorem 6.6]. We recall that a evolution family $\mathcal{U}$ as above is called uniformly expnentially stable if there are $\omega < 0$ and $M > 0$ such that (1.7) holds. For a locally integrable function $f : \mathbf{R}_+ \to X$ a mild solution of the well-posed inhomogeneous Cauchy problem

$$\dot{u}(t) = Au(t) + f(t), \quad (t \geq 0), \quad u(0) = 0$$

is defined by

$$u_f(t) = \int_0^t U(t,\tau)f(\tau)d\tau \quad (t \geq 0)$$

In particular if $f(t) = e^{i\mu t}x$ with $\mu \in \mathbf{R}$ and $x \in X$ then

$$u_f(t) = u_{\mu x}(t) = \int_0^t e^{i\mu\tau}U(t,\tau)xd\tau. \tag{1.8}$$

The main result of this paper is formulated as follows:

THEOREM 1.1 *Let $\mathcal{U}$ be a q-periodic evolution family on a separable Hilbert space H. If*

$$\sup_{\mu\in\mathbf{R}} \sup_{t>0} ||u_{\mu x}(t)|| < \infty, \quad \forall x \in H \tag{1.9}$$

then $\mathcal{U}$ is uniformly exponentially stable.

2. PRELIMINARY RESULTS

For the remainder of this section $\mathcal{U}$ will be a q-periodic evolution family on a Banach space X, i.e. $(\mathbf{e_1}), (\mathbf{e_2}), (\mathbf{e_3})$ and (1.7) hold. Let $BUC(\mathbf{R}_+, X)$ the Banach space of all X-valued bounded and uniformly continuous functions on $\mathbf{R}_+$ endowed with sup-norm and $AP(\mathbf{R}_+, X)$ the space of all X-valued almost periodic functions in the Bohr's sense, i.e. the linear closed hull in $BUC(\mathbf{R}_+, X)$ of the set of all functions

$$\{e^{i\mu \cdot} : \quad \mu \in \mathbf{R}, \quad x \in X\}$$

Let $AP_0(\mathbf{R}_+, X)$ the set of all $f \in AP(\mathbf{R}_+, X)$ such that $f(0) = 0$. It is clear that $AP_0(\mathbf{R}_+, X)$ is a closed subspace of $AP(\mathbf{R}_+, X)$. Let $\mathcal{P}_q^0(\mathbf{R}_+, X)$ the set of all X-valued continuous functions f defined on $\mathbf{R}_+$ such that $f(t+q) = f(t)$ for every $t \geq 0$ and $f(0) = 0$.

A q-periodic evolution family $\mathcal{U}$ defines an evolution semigroup $E = \{E^t\}_{t\geq 0}$ of bounded linear operators on $AP_0(\mathbf{R}_+, X)$ by setting

$$[E^t f](s) = \begin{cases} U(s, s-t)f(s-t), & s \geq t \\ 0, & 0 \leq s \leq t \end{cases} \tag{1.10}$$

for $f \in AP_0(\mathbf{R}_+, X)$. It can easily seen that E^t acts on $AP_0(\mathbf{R}_+, X)$ for all $t \geq 0$ and E is strongly continuous, see [10, Lemma 2]. Let G be the infinitesimal generator of E and

$$s(G) := \sup\{\Re(z) : z \in \sigma(G)\}$$

It is well-known that $s(G) \leq \omega_0(G)$.

LEMMA 2.1 *Let $u, f \in AP_0(\mathbf{R}_+, X)$. The following statements are equivalent:*

- **(i)** *$u \in D(G)$ and $Gu = -f$;*
- **(ii)** *$u(t) = u_f(t)$ for all $t \geq 0$.*

Proof. It follows as in [9, Lemma 1.1]. We omit the details.

LEMMA 2.2 *A q-periodic evolution family $\mathcal{U}$ is uniformly exponentially stable if and only if the growth bound of the associated evolution semigroup E on $AP_0(\mathbf{R}_+, X)$ is negative.*

Proof. It follows as in [5, Theorem 2.2].

LEMMA 2.3 *Let $\mathcal{U}$ be a q-periodic evolution family on X. The following statements are equivalent:*

- **(i)** *$\mathcal{U}$ is uniformly exponentially stable;*
- **(ii)** *G is invertible;*
- **(iii)** *$u_f(\cdot)$ belongs to $AP_0(\mathbf{R}_+, X)$ for all $f \in AP_0(\mathbf{R}_+, X)$;*
- **(iv)** $\sup\limits_{t>0} ||u_f(t)|| < \infty$ *for all $f \in AP_0(\mathbf{R}_+, X)$;*

- **(v)** *for every* $g \in P_q^0(\mathbf{R}_+, X)$ *and any* $\mu \in \mathbf{R}$ *the solution of the Cauchy problem (1.8) with* $f(t) = e^{i\mu t} g(t)$ *for* $t \geq 0$ *is bounded on* $\mathbf{R}_+$.

Proof. The equivalence of **(i)** anf **(v)** has been shown in [3]. Another way for the proof of (**iv**) $\Rightarrow$ (**i**) is indicated in [5, Theorem 2.5]. A such proof uses the fact the growth bound of arbitrary strongly continuous semigroup $\mathbf{T} = \{T(t)\}_{t \geq 0}$ on X is negative if and only if the map

$$t \mapsto \int_0^t T(\xi) f(t - \xi) d\xi \text{ belongs to } AP_0(\mathbf{R}_+, X) \text{ for all } f \in AP_0(\mathbf{R}_+, X).$$

LEMMA 2.4 *Let* $\mathcal{U}$ *be a q-periodic evolution family on the Banach space* X. *Then the evolution semigroup* E *on* $AP_0(\mathbf{R}_+, X)$ *satisfies the spectral mapping theorem, i.e.,* $\sigma(G) = \{z \in \mathbf{C} : \quad \Re(z) \leq s(G)\}$ *and*

$$e^{t\sigma(G)} = \sigma(E^t) \setminus \{0\}, \quad t \geq 0.$$

Proof. It follows reasoning as in [9, Corollary 2.4] using Lemma 2.1 and Lemma 2.3 above.

3. PROOF OF STABILITY CRITERION

Proof of Theorem 1.1 From (1.9) using the uniformly boundedness principle it follows that there exist a constant $K > 0$ such that

$$\sup_{\nu \in \mathbf{R}} \sup_{s \geq 0} || \int_0^s e^{i\nu\xi} U(s, \xi) x d\xi || \leq K ||x||, \quad \forall x \in H. \tag{3.1}$$

We denote by $PT_0(\mathbf{R}_+, H)$ the set of all functions

$$t \mapsto h_N(t) := \sum_{k=1}^{N} a_k e^{i\mu_k t} x_k : \quad \mathbf{R}_+ \to H,$$

where $N = 1, 2, \cdots, a_k \in \mathbf{C}, \mu_k \in \mathbf{R}, \{x_k : \quad k = 1, 2, \cdots\}$ is a complete ortogonal set in H such that $||x_k|| = 1$ for all k and $\sum_{k=1}^{N} a_k x_k = 0$. It is not hard to see that $PT_0(\mathbf{R}_+, H)$ is dense in $AP_0(\mathbf{R}_+, H)$. Let E the evolution semigroup, defined in (1.10), associated to $\mathcal{U}$ on the space $AP_0(\mathbf{R}_+, H)$. For every $s \geq 0$ and $\mu \in \mathbf{R}$, one has

$$\begin{aligned}(\int_0^\infty e^{-i\mu t} E^t h_N dt)(s) &= \int_0^s e^{-i\mu t} U(s, s-t) h_N(s-t) dt \\ &= \sum_{k=1}^N a_k \int_0^s e^{-i\mu t} e^{i\mu_k(s-t)} U(s, s-t) x_k dt \\ &= e^{-i\mu s} \sum_{k=1}^N a_k \int_0^s e^{i(\mu+\mu_k)\xi} U(s,\xi) x_k d\xi.\end{aligned}$$

Then from (3.1) it results

$$||\int_0^\infty e^{-i\mu t} E^t h_N dt||_{AP_0(\mathbf{R}_+,H)} \leq K \sum_{k=1}^N |a_k|.$$

Moreover using a well-known inequality it follows

$$||\int_0^\infty e^{-i\mu t} E^t h_N dt||^2_{AP_0(\mathbf{R}_+,H)} \leq 2K^2 ||h_N||_{AP_0(\mathbf{R}_+,H)}, \quad \forall h_N \in PT_0(\mathbf{R}_+, H)$$

hence

$$||\int_0^\infty e^{-i\mu t} E^t dt||_{\mathcal{L}(AP_0(\mathbf{R}_+,H))} \leq 2K^2. \tag{3.2}$$

Now it is easy to see that $R(z, G)$ exists and is uniformly bounded in $\{\Re(z) \geq 0\}$, see e.g. [11, Proof of Theorem 4]. From Lemma 2.4 it follows that $\omega_0(E) = \ln(r(E^1))$ is negative, because the spectral radius of (E^1) is given by

$$r(E^1) = \sup\{|z| : \quad z \in \sigma(E^1)\}.$$

The desired resut follows from Lemma 2.2.

REMARK 3.1 If E is an evolution semigroup associated to a strongly continuous and exponentially bounded family $\mathcal{U}$ (which can be not periodic) on a some X-valued functions space and (3.2) holds then $\omega_0(E)$ is negative and $\mathcal{U}$ is uniformly exponentially stable, see e.g. [1, Corollary 2].

COROLLARY 3.2 *Let $\mathcal{U}$ be a q-periodic evolution family on a separable Hilbert space H. If*

$$\sup_{t>0} \int_0^t ||U(t,\xi)x|| d\xi < \infty, \quad \forall x \in H$$

then $\mathcal{U}$ is uniformly exponentially stable.

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