UNIFORM EXPONENTIAL STABILITY OF THE LINEAR ALMOST PERIODIC SYSTEMS IN THE BANACH SPACE.

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Abstract.

This article is devoted to the study of linear non-autonomous dynamical systems, possessing the property of uniform exponential stability. It is proved that if the Cauchy operator of these systems possesses certain compactness property, then the uniform asymptotic stability implies the uniform exponential stability. For recurrent (almost periodic) systems this result is precised. The applications for different classes of linear evolution equations (ordinary linear differential equations in the space of Banach, retarded and neutral functional differential equations, some classes of evolution partial differential equations) are given.

Introduction

Let $A(t)-$ be a continuous $n \times n$ matrix-function and $H(A)$ be a family of all matrix-function $B = \lim\limits_{n\to+\infty} A_{t_n}$, where $\{t_n\} \subset \mathbb{R}, A_{t_n}(t) = A(t_n + t)$ and the convergence $A_{t_n} \to B$ is uniform on every compact from $\mathbb{R}$. It is well-known the following result.

Theorem [25,2,6]. *Let A be a bounded and uniformly continuous on $\mathbb{R}$ matrix-function, then the following conditions are equivalent:*

1. the trivial solution of equation

$$x' = A(t)x \tag{0.1}$$

is uniformly exponentially stable.

2. the trivial solution of equation (0.1) is uniformly asymptotically stable.

3. the trivial solution of equation (0.1) and every equation

$$y' = B(t)y \quad (B \in H(A)) \tag{0.2}$$

is asymptotical stable.

For the equations in the infinite-dimensional spaces the statements 1.-3. , how show the examples [10,24,15], aren't equivalent.

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In the work [10] of author the linear non-autonomous dynamical systems, satisfying the condition 1., or 2., or 3. are studied (see, for example, Theorem 1.1).

It is clear that in the general infinite-dimensional case the condition 1. implies 2. and 2. implies 3.. In this article we show that if the operator Cauchy of equation (0.1) satisfies some compactness condition, then the condition 3. implies 1. (see Theorems 2.3 and 2.4).

For recurrent (almost periodic) systems this result is precised (see Theorems 3.2-3.4). The applications for different classes of linear evolution equations (ordinary linear differential equations in the space of Banach, retarded and neutral functional differential equations, some classes of evolution partial differential equations) are given.

§ 1. Linear non-autonomous dynamical systems.

Assume that X and Y are complete metric spaces, $\mathbb{R}(\mathbb{Z})$ is a group of real (integer) numbers, $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}, \mathbb{T}_+ = \{t \in \mathbb{T} : t \geq 0\}$ and $\mathbb{T}_- = \{t \in \mathbb{T} | t \leq 0\}$. Denote by $(X, \mathbb{T}_+, \pi)$ $\quad ((Y, \mathbb{T}, \sigma))$ a semigroup (group) dynamical system on $X(Y)$. A triple $< (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h >$, where h is a homomorphism of $(X, \mathbb{T}_+, \pi)$ onto $(Y, \mathbb{T}, \sigma)$, is called a non-autonomous dynamical system.

The system $(X, \mathbb{T}_+, \pi)$ is called [7,9]:

- point dissipative, if there is $K \subseteq X$ such that for all $x \in X$

$$\lim_{t \to +\infty} \rho(xt, K) = 0, \tag{1.1}$$

where $xt = \pi^t x = \pi(t, x)$;

- compactly dissipative, if the equality (1.1) takes place uniformly with respect to x on compacts from X ;

- locally dissipative, if for any point $p \in X$ there is $\delta_p > 0$ such that the equality (1.1) takes place uniformly with respect to $x \in B(p, \delta_p) = \{x \in X : \rho(x, p) < \delta_p\}$;

A non-autonomous dynamical system $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h \rangle$ is said to be point (compactly, locally) dissipative, if the autonomous dynamical system $(X, \mathbb{T}_+, \pi)$ is so.

Let (X, h, Y) be a locally trivial Banach fibre bundle over Y[3]. A non-autono mous dynamical system $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h \rangle$ is said to be linear if the mapping $\pi^t : X_y \to X_{yt}$ is linear for every $t \in \mathbb{T}_+$ and $y \in Y$, where $X_y = \{x \in X | h(x) = y\}$ and $yt = \sigma(t, y)$. Let $|\cdot|$ be some norm on the (X, h, Y), and $|\cdot|$ is co-ordinated with the metric ρ on X (that is $\rho(x_1, x_2) = |x_1 - x_2|$ for any $x_1, x_2 \in X$ such that $h(x_1) = h(x_2)$). In the work [10] we received the point (compactly, locally) dissipativity criterion of linear system.

Theorem 1.1 [10]. *Let $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h \rangle$ be a linear non-autonomous dynamical system and Y be compact, then the following assertions hold:*

1. $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h \rangle$ is point dissipative if and only if $\lim_{t \to +\infty} |xt| = 0$ for all $x \in X$;

2. A non-autonomous dynamical system $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h \rangle$ is compactly dissipative if and only if $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h \rangle$ is point dissipative and there exists *a number $M \geq 0$ such that the inequality*

$$|xt| \leq M|x| \tag{1.2}$$

takes place for all $x \in X$ and $t \in \mathbb{T}_+$;

3. A non-autonomous dynamical system $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ is locally dissipative if and only if there exist positive numbers N and ν such that the inequality $|xt| \leq Ne^{-\nu t}|x|$ takes place for all $x \in X$ and $t \in \mathbb{T}_+$.

From the theorem of Banach-Steinhauss follows that point dissipativity and compact dissipativity are equivalent for autonomous linear system. In the work [10] one example of linear autonomous dynamical system which is compactly dissipative, but is not locally dissipative, is constructed.

Theorem 1.2 [7,8]. *Let $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ be a linear non-autonomous dynamical system, Y be a compact , then the following assertions take place:*

1. if $(X, \mathbb{T}_+, \pi)$ is completely continuous (i.e. for all bounded subset $A \subset X$ there exists a positive number $l = l(A)$ such that $\pi^l A$ is precompact), then from point dissipativity of $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ follows its local dissipativity;

2. if $(X, \mathbb{T}_+, \pi)$ is asymptotically compact (i.e. for all bounded sequences $\{x_n\} \subset X$ and $\{t_n\} \to +\infty$ the sequence $\{x_n t_n\}$ is precompact if it is bounded), then from compact dissipativity of $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ results its local dissipativity.

Recall [20,26] that a measure of noncompactness β on a complete metric space X is a function β from the bounded sets of X to the nonnegative real numbers satisfying

(i) $\beta(A) = 0$ for $A \subset X$ if and only if A is precompact ;

(ii) $\beta(A \bigcup B) = \max[\beta(A), \beta(B)]$;

(iii) $\beta(A + B) \leq \beta(A) + \beta(B) \quad \forall A, B \subset X$, if the space X is linear.

The Kuratowski measure of noncompactness α, is defined by

$$\alpha(A) = \inf\{d : A \text{ has a finite cover of diameter } < d\}.$$

The dynamical system $(X, \mathbb{T}_+, \pi)$ is said to be conditionally β -condensing [20] if there exists $t_0 > 0$ such that $\beta(\pi^{t_0} B) < \beta(B)$ for any bounded set B in X, if $\beta(B) > 0$. The dynamical system $(X, \mathbb{T}_+, \pi)$ is said to be β -condensing if it is conditionally β -condensing and a set $\{\pi^{t_0} B\}$ is bounded for all bounded set $B \subseteq X$.

Let $X = E \times Y$ and $A \subset X$. We denote by $A_y = \{x \in A : pr_2 x = y\}$, then $A = \bigcup\{A_y | y \in Y\}$. Let $\tilde{A}_y = pr_1 A_y$ and $\tilde{A} = \bigcup\{\tilde{A}_y | y \in Y\}$. We note that if the space Y is compact, then a set $A \subset X$ is bounded in X if and only if the set $\tilde{A}$ is bounded in E.

Lemma 1.3. *The following equality takes place $\alpha(A) = \alpha(\tilde{A})$ for all bounded set $A \subset X$, where $\alpha(A) \quad (\alpha(\tilde{A}))$ is a Kuratowski measure of noncompactness of set $A \subset X \quad (\tilde{A} \subset E)$.*

Proof. Let $\varepsilon > 0$ and A be a bounded subset in X, then there are sets $A_1, A_2, ..., A_n$ such that $A = \bigcup\{A_i | i = 1, 2, ..., n\}$ and $diam A_i < \alpha(A) + \varepsilon, i = 1, 2, ..., n$. We note that $\tilde{A} = \bigcup\{\tilde{A}_i | i = 1,2,...,n\}$ and $diam\tilde{A}_i \le diam A_i < \alpha(A)+\varepsilon$ and, consequently, $\alpha(\tilde{A}) \le \alpha(A)$.

Let now $\varepsilon > 0, A$ be a bounded set in $X, \tilde{A} = \bigcup\{\tilde{A}_k | k = 1,2,...,m\}$ and $diam\tilde{A}_k < \alpha(\tilde{A}) + \varepsilon$. Since Y is compact, then there are sets $Y_1, Y_2, ..., Y_\ell$ such that $Y_1 \bigcup Y_2 \bigcup ... \bigcup Y_\ell = Y$ and $diam Y_j < \varepsilon \quad (j = 1,2,...,\ell)$. Denote by $A_i = pr_1^{-1}(\tilde{A}_i) \bigcap A$,

$$A_{ij} = pr_2^{-1}(pr_2(pr_1^{-1}(\tilde{A}_i) \bigcap A) \bigcap Y_j) \bigcap A_i$$

and we note that $A_{ij} \subseteq \tilde{A}_i \times Y_j$ and, consequently, $diam A_{ij} \le diam\tilde{A}_i + diam Y_j < \alpha(\tilde{A}) + \varepsilon + \varepsilon = \alpha(A) + 2\varepsilon$. Since $A = \bigcup\{A_{ij} | i = 1,2,...,n, j = 1,2,...,\ell\}$, then from this follows that $\alpha(A) \le \alpha(\tilde{A})$ and, consequently, $\alpha(A) = \alpha(\tilde{A})$. The lemma is proved.

Let E be a Banach space and $\varphi : \mathbb{T}_+ \times E \times Y \mapsto E$ be a continuous mapping with properties: $\varphi(0,u,y) = u$ and $\varphi(t+\tau,u,y) = \varphi(t,\varphi(\tau,u,y),\sigma(\tau,y))$ for all $u \in E$, $y \in Y$ and $t,\tau \in \mathbb{T}_+$. A triplet $\langle E, \varphi, (Y,\mathbb{T},\sigma)\rangle$ is called a continuous cocycle on $(Y,\mathbb{T},\sigma)$ with fibre E.

The dynamical system $(X,\mathbb{T}_+,\pi)$ is called [25] a skew-product system if $X = E \times Y$ and $\pi = (\varphi,\sigma)$ (i.e. $\pi(t,(u,y)) = (\varphi(t,u,y),\sigma(t,y))$ for all $u \in E$, $y \in Y$ and $t,\tau \in \mathbb{T}_+$).

A cocycle φ is called conditionally $\alpha-$condensing, if there exists $t_0 > 0$ such that for any bounded set $B \subseteq E$ the inequality $\alpha(\varphi(t_0,B,Y)) < \alpha(B)$ holds if $\alpha(B) > 0$. The cocycle φ is called $\alpha-$condensing if it is conditionally $\alpha-$condensing cocycle and a set $\varphi(t_0,B,Y) = \bigcup\{\varphi(t_0,u,Y) | u \in B, y \in Y\}$ is bounded for all bounded set $B \subseteq E$.

A cocycle φ is called conditional $\alpha-$ contraction of order $k \in [0,1)$, if there exists $t_0 > 0$ such that for any bounded set $B \subseteq E$ for which $\varphi(t_0,B,Y) = \bigcup\{\varphi(t_0,u,Y) | u \in B, y \in Y\}$ is bounded the inequality $\alpha(\varphi(t_0,B,Y)) \le k\alpha(B)$ holds. The cocycle φ is called $\alpha-$ contraction if it is conditionally $\alpha-$ contraction cocycle and a set $\varphi(t_0,B,Y) = \bigcup\{\varphi(t_0,u,Y) | u \in B, y \in Y\}$ is bounded for all bounded sets $B \subseteq E$.

Lemma 1.4. *Let Y be compact and the cocycle φ be $\alpha-$ condensing, then the skew-product dynamical system $(X,\mathbb{T}_+,\pi)$, generated by cocycle φ, is $\alpha-$ condensing.*

Proof. Let $A \subset X$ be a bounded subset, $t_0 > 0$ and $\alpha(A) > 0$, then

$$\pi(t_0,A) = \bigcup\{\pi(t_0,A_y | y \in Y\}$$

$$= \bigcup\{(\varphi(t_0,A_y,y),yt) | y \in Y\} \subseteq \varphi(t_0,\tilde{A},Y) \times Y. \tag{1.3}$$

Since A is bounded, then $\tilde{A}$ is also bounded in E and according to the condition of lemma the set $\varphi(t_0,\tilde{A},Y)$ is bounded and, consequently, $\pi(t_0,A)$ is bounded. According to lemma 1.3 and (1.3) we have

$$\alpha(\pi(t_0,A)) = \alpha(\bigcup\{(\varphi(t_0,A_y,y),yt_0) | y \in Y\}) \le \alpha(\varphi(t_0,\tilde{A},Y)) < \alpha(\tilde{A}) = \alpha(A).$$

The lemma is proved.

Theorem 1.5. *Let E be a Banach space, φ be a cocycle on $(Y, \mathbb{T}, \sigma)$ with fibre E and the following conditions are fulfilled:*

1. $\varphi(t, u, y) = \psi(t, u, y) + \gamma(t, u, y)$ for all $t \in \mathbb{T}_+, u \in E$ and $y \in Y$;

2. there exists a function $m : \mathbb{R}_+ \mapsto \mathbb{R}_+$ satisfying the condition $m(t) \to 0$ as $t \to +\infty$ such that $|\psi(t, u_1, y) - \psi(t, u_2, y)| \le m(t)|u_1 - u_2|$ for all $t \in \mathbb{T}_+, u_1, u_2 \in E$ and $y \in Y$.

3. $\gamma(t, A, Y)$ is compact for all bounded $A \subset X$ and $t > 0$.

Then the cocycle φ is $\alpha-$ contraction.

Proof. Let $\varepsilon > 0, A$ be a bounded set into E, then there are sets $A_1, A_2, ..., A_n$ such that $A = \bigcup\{A_i | i = 1, 2, ..., n\}$ and $diam A_i < \alpha(A) + \varepsilon (i = 1, 2, ..., n)$. Since Y is compact, then there are a sets $Y_1, Y_2, ..., Y_m$ such that $Y_1 \bigcup Y_2 \bigcup ... \bigcup Y_m = Y$ with condition $diam Y_j < \varepsilon$ for all $j = 1, 2, ..., m$.

Let t_0 be a positive number such that $m(t_0) < 1$. We note that

$$\varphi(t_0, A, Y) \subseteq \psi(t_0, A, Y) + \gamma(t_0, A, Y)$$

$$= \bigcup\{\psi(t_0, A_i, Y_j) | i = 1, 2, ..., n; j = 1, 2, .., m\} + \gamma(t_0, A, Y). \tag{1.4}$$

According to conditions of theorem 1.5 $\alpha(\gamma(t_0, A, Y)) = 0$ and

$$diam \psi(t_0, A_i, y) \le m(t_0) diam A_i$$

for all $y \in Y$. Thus we have

$$|\psi(t_0, u_1, y_1) - \psi(t_0, u_2, y_2)| \le |\psi(t_0, u_1, y_1) - \psi(t_0, u_2, y_1)|$$

$$+ |\psi(t_0, u_2, y_1) - \psi(t_0, u_2, y_2)| \tag{1.5}$$

and, consequently,

$$diam(\psi(t_0, A_i, y) \le m(t_0) diam A_i + diam \psi(t_0, u_2, Y_j) \quad \text{for all} \quad y \in Y_j. \tag{1.6}$$

Since Y is compact, then from (1.5)-(1.6) it follows the inequality

$$diam(\psi(t_0, A_i, Y_j) \le m(t_0) diam A_i \le m(t_0)(\alpha(A) + \varepsilon)$$

and, consequently, $\alpha(\varphi(t_0, A, Y)) \le m(t_0)\alpha(A)$. The theorem is proved.

§ 2. Exponential stable linear non-autonomous dynamical systems.

Lemma 2.1[14]. *Let $m : \mathbb{T}_+ \to \mathbb{T}_+$ satisfies the following conditions:*

1. there exists a positive constant M such that $m(t) \le M$ for all $t \in \mathbb{T}_+$.

2. $m(t) \to 0$ as $t \to +\infty$.

3. $m(t + \tau) \le m(t)m(\tau)$ for all $t, \tau \in \mathbb{T}_+$.

Then there exist two positive constants N and ν such that $m(t) \le Ne^{-\nu t}$ for all $t \in \mathbb{T}_+$.

Theorem 2.2. *Let $\langle (X,\mathbb{T}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ be a linear non-autonomous dynamical system, Y be a compact , then the following conditions are equivalent:*

1. a non-autonomous dynamical system $\langle (X,\mathbb{T}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ is uniformly exponentially stable, i.e. there exist two positive constants N and ν such that $|\pi(t,x)| \le Ne^{-\nu t}|x|$ for all $t\in\mathbb{T}_+$ and $x\in X$.

2. $\|\pi^t\|\to 0$ as $t\to+\infty$, where $\|\pi^t\| = \sup\{|\pi^t x| : x\in X, |x|\le 1\}$.

3. a non-autonomous dynamical system $\langle (X,\mathbb{T}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ is locally dissipative.

Proof. According to Theorem 1.1 the conditions 1. and 3. are equivalent. Now we will prove that the conditions 1. and 2. are equivalent. Really, it is clear that from 1. it follows 2.. According to condition 2. there exists $L>0$ such that

$$\|\pi^t\| \le 1 \tag{1.7}$$

for all $t\ge L$. We claim that the family of operators $\{\pi^t : t\in[0,L]\}$ is uniformly continuous, that is, for any $\varepsilon>0$ there is a $\delta(\varepsilon)>0$ such that $|x|\le\delta$ implies $|xt|\le\varepsilon$ for all $t\in[0,L]$. On the contrary, assume that there are $\varepsilon_0>0$, $\delta_n>0$ with $\delta_n\to 0$, $|x_n|<\delta_n$ and $t_n\in[0,L]$ such that

$$|x_n t_n| \ge \varepsilon_0. \tag{1.8}$$

Since (X,h,Y) is a locally trivial Banach fibre bundle and Y is compact, then the zero section $\Theta=\{\theta_y : y\in Y, \theta_y\in X_y, |\theta_y|=0\}$ of (X,h,Y) is compact and, consequently, we can assume that the sequences $\{x_n\}$ and $\{t_n\}$ are convergent. Put $x_0=\lim\limits_{n\to+\infty} x_n$ and $t_0=\lim\limits_{n\to+\infty} t_n$, then $x_0=\theta_{y_0}$ $(y_0=h(x_0))$. Passing to the limit in (1.8) as $n\to+\infty$, we obtain $0=|x_0t_0|\ge\varepsilon_0$. This last inequality contradicts the choice of ε_0, and hence proves the above assertion. If $\gamma>0$ is such that $|\pi^t x|\le 1$ for all $|x|\le\gamma$ and $t\in[0,L]$, then

$$|xt| \le \frac{1}{\gamma}|x| \tag{1.9}$$

for all $t\in[0,L]$ and $x\in X$. We put $M=\max\{\gamma^{-1},1\}$, then from (1.7) and (1.9) it follows

$$\|\pi^t\| \le M \tag{1.10}$$

for all $t\ge 0$ and $x\in X$. Consider the function $m(t)=\|\pi^t\|$. We note that $m(t+\tau)\le m(t)m(\tau)$ for all $t,\tau\in\mathbb{T}_+$ and $m(t)\le M$ for all $t\in\mathbb{T}_+$ and $m(t)\to 0$ as $t\to+\infty$. According to Lemma 2.1 there exist positive numbers N and ν such that $m(t)\le Ne^{-\nu t}$ for all $t\in\mathbb{T}_+$. Thus $|\pi(t,x)|\le\|\pi^t\||x|\le Ne^{-\nu t}|x|$ for all $t\in\mathbb{T}_+$ and $x\in X$. The theorem is proved.

Theorem 2.3. *Let $\langle (X,\mathbb{T}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ be a linear non-autonomous dynamical system, Y be a compact and $(X,\mathbb{T}_+,\pi)$ be conditionally $\alpha-$ condensing, then the following assertions are equivalent:*

1. a non-autonomous dynamical system $\langle (X,\mathbb{T}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ is point dissipative and this system doesn't admit non-trivial bounded trajectories on $\mathbb{T}$.

2. a non-autonomous dynamical system $\langle (X,\mathbb{T}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ is uniformly exponentially stable.

Proof. Denote by $\Theta=\{\theta_y|\theta_y\in X_y, |\theta_y|=0\}$ a zero section of vectorial fibering (X,h,Y). Since (X,h,Y) is locally trivial and Y is compact, then the zero section Θ is compact and invariant set of dynamical system $(X,\mathbb{T}_+,\pi)$. Taking into account that a dynamical system $(X,\mathbb{T}_+,\pi)$ is conditionally $\alpha-$ condensing according to Theorem 2.4.8 [20] the set Θ is orbitally stable and in particular there exists a positive constant N such that $|xt|\le N|x|$ for all $t\in\mathbb{T}_+$ and $x\in X$. In virtue of Theorem 1.1 a dynamical system $(X,\mathbb{T}_+,\pi)$ is compact dissipative and according to Theorem 1.2 $(X,\mathbb{T}_+,\pi)$ is locally dissipative. It follows from Theorem 2.2 that $(X,\mathbb{T}_+,\pi)$ is uniformly exponentially stable.

Let now a non-autonomous dynamical system $\langle (X,\mathbb{T}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ be uniformly exponentially stable, then according to Theorem 2.2 it is locally dissipative. Let J be its Levinson's centre. We note that according to the linearity of non-autonomous dynamical system $\langle (X,\mathbb{T}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ we have $J=\Theta$. Let φ be a entire bounded trajectory of dynamical system $(X,\mathbb{T}_+,\pi)$. Since a non-autonomous dynamical system $\langle (X,\mathbb{T}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ is conditionally $\alpha-$ condensing then the set $\varphi(\mathbb{T})$ is precompact and $\varphi(\mathbb{T})\subseteq J=\Theta$ because J is a maximal compact invariant set of $(X,\mathbb{T}_+,\pi)$. The theorem is proved.

Theorem 2.4. *Let $\langle (X,\mathbb{T}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ be a linear non-autonomous dynamical system, Y be a compact and $(X,\mathbb{T}_+,\pi)$ be completely continuous, i.e. for any bounded set $A\subseteq X$ there exists a positive number ℓ such that $\pi^{\ell}(A)$ is precompact. Then the following assertions are equivalent:*

1. a non-autonomous dynamical system $\langle (X,\mathbb{T}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ is uniformly exponentially stable.

2. $\lim\limits_{t\to+\infty}|\pi^t x|=0$ for all $x\in X$.

Proof. It is clear that condition 1. implies 2.. Now we will show that from 2. it follows 1.. Really, according to Theorem 1.1 a non-autonomous dynamical system $\langle (X,\mathbb{T}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ is point dissipative. Since a dynamical system $(X,\mathbb{T}_+,\pi)$ is completely continuous in virtue of Theorem 1.2 the dynamical system $(X,\mathbb{T}_+,\pi)$ is locally dissipative. To prove the Theorem it is sufficiently to refer on Theorem 2.2.

§ 3. A linear non-autonomous dynamical system with minimal base.

In this section we study the linear non-autonomous dynamical system $\langle (X,\mathbb{T}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ with compact minimal base $(Y,\mathbb{T},\sigma)$.

Theorem 3.1[4,5]. *Let $\langle (X,\mathbb{T}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ be a linear non-autonomous dynamical system and the following conditions hold:*

1. Y is compact and minimal (i.e. $Y=H(y)=\overline{\{yt:t\in\mathbb{T}\}}$ for all $y\in Y$);

2. for any $x\in X$ there exists $C_x\ge 0$ such that $|xt|\le C_x$ for all $t\in\mathbb{T}_+$;

3. the mapping $y\mapsto\|\pi^t_y\|$ is continuous, where $\|\pi^t_y\|$ is a norm of linear operator $\pi^t_y=\pi^t|_{X_y}$, for every $t\in\mathbb{T}_+$ or $(X,\mathbb{T}_+,\pi)$ is a skew-product dynamical system.

Then there exists $M\ge 0$ such that the inequality

$$|\pi(t,x)|\le M|x| \tag{3.1}$$

holds for all $t \in \mathbb{T}_+$ *and* $x \in X$.

Theorem 3.2. *Let* $\langle (X,\mathbb{T}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ *be a linear non-autonomous dynamical system and the following conditions hold:*

1. Y *is compact and minimal.*

2. a dynamical system $(X,\mathbb{T}_+,\pi)$ *is asymptotically compact.*

3. the mapping $y \mapsto \|\pi^t_y\|$ *is continuous, where* $\|\pi^t_y\|$ *is a norm of linear operator* $\pi^t_y = \pi^t|_{X_y}$*, for every* $t \in \mathbb{T}_+$ *or* $(X,\mathbb{T}_+,\pi)$ *is a skew-product dynamical system.*

Then a non-autonomous dynamical system $\langle (X,\mathbb{T}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ *is uniformly exponentially stable if and only if the following equality holds*

$$\lim_{t\to+\infty} |\pi^t x| = 0 \tag{3.2}$$

for all $x \in X$.

Proof. It is clear that from uniform exponential stability of non-autonomous dynamical system $\langle (X,\mathbb{T}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ it follows the equality (3.2).

From condition (3.2) and minimality of $(Y,\mathbb{T},\sigma)$ in virtue of Theorem 3.1 it follows that for a non-autonomous dynamical system $\langle (X,\mathbb{T}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ the inequality (3.1) holds and according to Theorem 1.1 this system is compactly dissipative. Since a dynamical system $(X,\mathbb{T}_+,\pi)$ is asymptotically compact to finish the proof of the Theorem it is sufficiently to remark that according to Theorem 2.13 [8] every compactly dissipative and asymptotically compact dynamical system is locally dissipative. Thus a linear non-autonomous dynamical system $\langle (X,\mathbb{T}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ is locally dissipative and, consequently, it is uniform exponentially stable. The Theorem is proved.

The entire trajectory of the semigroup dynamical system $(X,\mathbb{T}_+,\pi)$ passing through the point $x \in X$ at $t=0$ is defined to be the continuous map $\gamma : \mathbb{T} \mapsto X$ that satisfies the conditions $\gamma(0) = x$ and $\pi^t\gamma(s) = \gamma(s+t)$ for all $t \in \mathbb{T}_+$ and $s \in \mathbb{T}$. Let Φ_x be the set of all entire trajectories of $(X,\mathbb{T}_+,\pi)$ passing through x at $t=0$ and $\Phi = \bigcup\{\Phi_x : x \in X\}$.

Let $C(\mathbb{T},X)$ be the space of all continuous maps $\varphi : \mathbb{T} \mapsto X$ equipped with the compact-open topology and let $(C(\mathbb{T},X),\mathbb{T},\sigma)$ be the dynamical system of shifts on $C(\mathbb{T},X)$.

Theorem 3.3[5]. *Assume that* $\langle (X,\mathbb{T}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ *is a linear non-autonomous dynamical system, generated by cocycle* φ, $(X,\mathbb{T}_+,\pi)$ *is asymptotically compact, and there is an* $M>0$ *such that* $|\varphi(t,u,y)| \le M|u|$ *for all* $(u,y) \in X = E\times Y$ *and* $t \in \mathbb{T}_+$. *Then the following assertions hold.*

(i) For any $(u,y) \in \mathbb{B} = \{x \in X : \exists \gamma \in \Phi_x$ *such that* $\sup\limits_{t\in\mathbb{T}} |\gamma(t)| < +\infty\}$ *the set* $\Phi_{(u,y)}$ *consists of a single entire recurrent trajectory.*

(ii) $\mathbb{B}$ *is closed in* X.

(iii) $(X,\mathbb{T}_+,\pi)$ *induces a group dynamical system* $(\mathbb{B},\mathbb{T},\pi)$ *on* $\mathbb{B}$.

(iv) $(\mathbb{B},h,Y)$ *is a finite dimensional vectorial subfibering of* (X,h,Y), *i.e.* $\dim \mathbb{B}_y$ *does not depend on* $y \in Y$.

Theorem 3.4. *Suppose that the following conditions are satisfied:*

1. a dynamical system $(Y,\mathbb{T},\sigma)$is compact and minimal.

2. a linear non-autonomous dynamical system $\langle (X,\mathbb{T}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ is generated by cocycle φ (i.e. $X = E\times Y$, $\pi = (\varphi,\sigma)$ and $h = pr_2 : X \mapsto Y$).

3. a dynamical system $(X,\mathbb{T}_+,\pi)$ is conditionally $\alpha-$ condensing.

4. there exists a positive number M such that $|\varphi(t,u,y)| \le M|u|$ for all $t\in Y$ and $t\in\mathbb{T}_+$.

Then there are two vectorial positive invariant subfibering (X^0,h,Y) and (X^s,h,Y) of (X,h,Y) such that:

a. $X_y = X^0_y + X^s_y$ and $X^0_y \bigcap X^s_y = 0_y$ for all $y\in Y$, where $0_y = (0,y)\in X = E\times Y$ and 0 is a zero in the Banach space E.

b. a vectorial subfibering (X^0,h,Y) is finite dimensional, invariant (i.e. $\pi^t X^0 = X^0$ for all $t\in\mathbb{T}_+$) and every trajectory of a dynamical system $(X,\mathbb{T}_+,\pi)$ belonging to X^0 is recurrent.

c. there exist two positive numbers N and ν such that $|\varphi(t,u,y)| \le Ne^{-t}|u|$ for all $(u,y)\in X^s$ and $t\in\mathbb{T}_+$.

Proof. Let $X^0 = \mathbb{B}$, then according to Theorem 3.3 the statement b. holds. Denote by P_y the projection of $X_y = h^{-1}(y)$ to $\mathbb{B}_y = \mathbb{B}\bigcap h^{-1}(y)$, then $P_y(u,y) = (\mathcal{P}(y)u,y)$ for all $u\in E, \mathcal{P}^2(y) = \mathcal{P}(y)$ and the mapping $\mathcal{P} : Y \mapsto [E]$ ($y\mapsto \mathcal{P}(y)$) is continuous, where by $[E]$ is denoted a set of all linear continuous operators acting into E. Now we denote by $X^s_y = \mathcal{Q}(y)X_y$ and $X^s = \bigcup\{X^s_y : y\in Y\}$, where $\mathcal{Q}(y) = Id_E - \mathcal{P}(y)$. We will show that X^s is closed in X. Really, let $\{x_n\} = (u_n,y_n)\subseteq X^s$ and $x_0 = (u_0,y_0) = \lim\limits_{n\to\infty} x_n$. Note that $P_{y_0}(x_0) = (\mathcal{P}(y_0)u_0,y_0) = (\lim\limits_{n\to\infty}\mathcal{P}(y_n)u_n,y_0) = (0,y_0) = 0_{y_0}$ and, consequently, $x_0\in X^s_{y_0}\subseteq X^s$.

Let $(X^s,\mathbb{T}_+,\pi)$ be a dynamical system induced by $(X,\mathbb{T}_+,\pi)$. It is clear that under conditions of Theorem 4.5 a dynamical system $(X^s,\mathbb{T}_+,\pi)$ is asymptotically compact and every positive semi-trajectory is precompact and, consequently, $\lim\limits_{t\to\infty}|\pi(t,x)| = 0$ for all $x\in X^s$ because a dynamical system $(X^s,\mathbb{T}_+,\pi)$ doesn't have a non-trivial entire trajectory bounded on $\mathbb{T}$. Thus we can apply the Theorem 2.3 according which there exist two positive constants N and ν such that $|\varphi(t,u,y)| \le Ne^{-t}|u|$ for all $(u,y)\in X^s$ and $t\in\mathbb{T}_+$. The theorem is proved.

§ 4. Some classes of linear uniformly exponentially stable non-autonomous differential equations.

Let Λ be a complete metric space of linear operators that act on Banach space E and $C(\mathbb{R},\Lambda)$ be a space of all continuous operator-functions $A:\mathbb{R}\to\Lambda$ equipped with open-compact topology and $(C(\mathbb{R},\Lambda),\mathbb{R},\sigma)$ be a dynamical system of shifts on $C(\mathbb{R},\Lambda)$.

4.1. Ordinary linear differential equations. Let $\Lambda = [E]$ and consider the linear differential equation

$$u' = \mathcal{A}(t)u \quad , \tag{4.1}$$

where $\mathcal{A}\in C(\mathbb{R},\Lambda)$. Along with equation (4.1), we shall also consider its $H-$class, that is, the family of equations

$$v' = \mathcal{B}(t)v \quad , \tag{4.2}$$

where $\mathcal{B} \in H(\mathcal{A}) = \overline{\{\mathcal{A}_\tau : \tau \in \mathbb{R}\}}, \mathcal{A}_\tau(t) = \mathcal{A}(t+\tau) \quad (t \in \mathbb{R})$ and the bar denotes closure in $C(\mathbb{R}, \Lambda)$. Let $\varphi(t, u, \mathcal{B})$ be the solution of equation (4.2) that satisfies the condition $\varphi(0, v, \mathcal{B}) = v$. We put $Y = H(\mathcal{A})$ and denote the dynamical system of shifts on $H(\mathcal{A})$ by $(Y, \mathbb{R}, \sigma)$, then the triple $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle$ is a linear non-autonomous dynamical system, where $X = E \times Y, \pi = (\varphi, \sigma) \quad$ (i.e. $\pi((v, \mathcal{B}), \tau) = (\varphi(\tau, v, \mathcal{B}), \mathcal{B}_\tau)$ and $h = pr_2 : X \to Y$).

Lemma 4.1[11,13]. *(i) The mapping $(t, u, \mathcal{A}) \mapsto \varphi(t, u, \mathcal{A})$ of $\mathbb{R} \times E \times C(\mathbb{R}, [E])$ to E is continuous, and*

(ii) the mapping $\mathcal{A} \mapsto U(\cdot, \mathcal{A})$ of $C(\mathbb{R}, [E])$ to $C(\mathbb{R}, [E])$ is continuous, where $U(\cdot, \mathcal{A})$ is the Cauchy operator [17] of equation (4.1).

Theorem 4.2. *Let $\mathcal{A} \in C(\mathbb{R}, \Lambda)$ be compact (i.e. $H(\mathcal{A})$ is a compact set of $(C(\mathbb{R}, \Lambda), \mathbb{R}, \sigma)$), then the following conditions are equivalent:*

1. a trivial solution of equation (4.1) is uniformly exponentially stable, i.e. there exist positive numbers N and ν such that $\|U(t, \mathcal{A})U(\tau, \mathcal{A})^{-1}\| \le Ne^{-(t-\tau)}$ for all $t \ge \tau$.

2. there exist positive numbers N and ν such that $\|U(t, \mathcal{B})U(\tau, \mathcal{B})^{-1}\| \le Ne^{-(t-\tau)}$ for all $t \ge \tau$ and $\mathcal{B} \in H(\mathcal{A})$.

3. $\lim\limits_{t \to +\infty} \sup\{\|U(t, \mathcal{B})\| : \mathcal{B} \in H(\mathcal{A})\} = 0$.

Proof. Applying Theorem 2.2 to non-autonomous system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle$, generated by equation (4.1) we obtain the equivalence of conditions 2. and 3.. According to Lemma 3 [11] the conditions 1. and 2. are equivalent. The theorem is proved.

Theorem 4.3. *Let $\mathcal{A} \in C(\mathbb{R}, \Lambda)$ be recurrent with respect to $t \in \mathbb{T}$ (i.e. $H(\mathcal{A})$ is a compact and minimal set of $(C(\mathbb{R}, \Lambda), \mathbb{R}, \sigma)$), a non-autonomous system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle$ generated by equation (4.1) is asymptotically compact, then the following conditions are equivalent:*

1. a trivial solution of equation (4.1) is uniformly exponentially stable, i.e. there exist positive numbers N and ν such that $\|U(t, \mathcal{A})U(\tau, \mathcal{A})^{-1}\| \le Ne^{-(t-\tau)}$ for all $t \ge \tau$.

2. $\lim\limits_{t \to +\infty} \sup |\varphi(t, u, \mathcal{B})| = 0$ for every $u \in E$ and $\mathcal{B} \in H(\mathcal{A})$.

Proof. According to Lemma 4.1 the mapping $U(t, \cdot) : [E] \mapsto [E]$ is continuous and, consequently, the mapping $\mathcal{B} \mapsto \|U(t, \cdot)\|$ is also continuous for every $t \in \mathbb{T}$. Now applying Theorem 3.2 to non-autonomous system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle$ generated by equation (4.1), we obtain the equivalence of conditions 1. and 2.. The theorem is proved.

We now formulate some sufficient conditions for the $\alpha-$ condensedness (in particular, asymptotically compactness) of the linear non-autonomous dynamical system generated by equation (4.1).

Theorem 4.4. *Let $\mathcal{A} \in C(\mathbb{R}, [E]), \mathcal{A}(t) = \mathcal{A}_1(t) + \mathcal{A}_2(t)$ for all $t \in \mathbb{R}$, and assume that $H(\mathcal{A}_i)(i = 1, 2)$ are compact and the following conditions hold:*

(i) The zero solution of the equation

$$u' = \mathcal{A}_1(t)u \tag{4.3}$$

is uniformly asymptotically stable, that is, there are positive numbers N and ν such that

$$||U(t,\mathcal{A}_1)U^{-1}(\tau,\mathcal{A}_1) \le Ne^{-\nu(t-\tau)} \tag{4.4}$$

for all $t \ge \tau (t,\tau \in \mathbb{R})$, where $U(t,\mathcal{A}_1)$ is the Cauchy operator of equation (4.3).

(ii) The family of operators $\{\mathcal{A}_2(t) : t \in \mathbb{R}_+\}$ is uniformly completely continuous, that is, for any bounded set $A \subset E$ the set $\{\mathcal{A}_2(t)A : t \in \mathbb{R}_+\}$ is precompact.

Then the linear non-autonomous dynamical system generated by equation (4.1) is $\alpha-$ contraction.

Proof. First of all we note that a set $\varphi(t,A,Y)$ is bounded for every $t > 0$ and bounded set $A \subseteq E$. Let $\mathcal{B} \in H(\mathcal{A})$. Then there are $\{t_n\} \subset \mathbb{T}$ such that $\mathcal{B}(t) = \mathcal{B}_1(t) + \mathcal{B}_2(t)$ and $\mathcal{B}_i(t) = \lim\limits_{t\to+\infty} \mathcal{A}_i(t+t_n)$. Note that

$$\varphi(t,v,\mathcal{B}) = U(t,\mathcal{B}_1)v + \int_0^t U(t,\mathcal{B}_1)U^{-1}(\tau,\mathcal{B}_1)\mathcal{B}_2(\tau)\varphi(\tau,v,\mathcal{B})d\tau. \tag{4.5}$$

By Lemma 4.3,

$$U(t,\mathcal{B}_i) = \lim_{t\to+\infty} U(t,\mathcal{A}_{it_n}), \quad \mathcal{A}_{it_n}(t) = \mathcal{A}_i(t+t_n),$$

and the equality

$$U(t,\mathcal{A}_{1t_n})U^{-1}(\tau,\mathcal{A}_{1t_n}) = U(t+t_n,\mathcal{A}_1)U^{-1}(\tau+t_n,\mathcal{A}_1)$$

and inequality (4.4) imply that

$$||U(t,\mathcal{B}_1)U^{-1}(\tau,\mathcal{B}_1) \le Ne^{-\nu(t-\tau)} \tag{4.6}$$

for all $t \ge \tau$ and $\mathcal{B}_1 \in H(\mathcal{A}_1)$. By Theorem 1.5, Theorem 4.4 will be proved if we can prove that the set

$$\{\int_0^t U(t,\mathcal{B}_1)U^{-1}(\tau,\mathcal{B}_1)\mathcal{B}_2(\tau)\varphi(\tau,v,\mathcal{B})d\tau : (v,\mathcal{B}) \in A\}$$

is precompact for every $t > 0$ and every bounded positively invariant set $A \subseteq E\times Y$. We put

$$K_A^t = \overline{\{\mathcal{B}_2(\tau)\varphi(\tau,v,\mathcal{B}) : \tau \in [0,t], (v,\mathcal{B}) \in A\}}$$

and we note that a set K_A^t is compact. Really, the set $\varphi([0,t],A] = \bigcup\{\varphi(\tau,v,\mathcal{B}) : \tau \in [0,t], (v,\mathcal{B}) \in A\}$ is bounded because $|\varphi(\tau,v,\mathcal{B})| \le e^{Mt}r$ for all $\tau \in [0,t]$ and $(v,\mathcal{B}) \in A$, where $r = \sup\{|v| : \exists\mathcal{B} \in H(\mathcal{A}),$ such that $(v.\mathcal{B}) \in A\}$ and $M = \sup\{||\mathcal{A}(t)|| : t \in \mathbb{T}\}$. Then

$$\int_0^t U(t,\mathcal{B}_1)U^{-1}(\tau,\mathcal{B}_1)\mathcal{B}_2(\tau)\varphi(\tau,v,\mathcal{B})d\tau$$

$$\in t\overline{conv}\{U(t,\mathcal{B}_1)U^{-1}(\tau,\mathcal{B}_1)w : 0 \le \tau \le t, \mathcal{B}_1 \in H(\mathcal{A}_1), w \in K_A^t\}. \tag{4.7}$$

Since $H(\mathcal{A}_1), H(\mathcal{A})$, and K_A^t are compact sets, then formula (4.7), condition (ii) of Theorem 4.4 , and Lemma 4.3 imply that $\{U(t,\mathcal{B}_1)U^{-1}(\tau,\mathcal{B}_1)w : 0 \le \tau \le t, \mathcal{B}_1 \in H(\mathcal{A}_1), w \in K_A^t\}$ is compact, which completes the proof of the theorem.

Theorem 4.5. *Let $H(\mathcal{A})$ be compact and assume that there is a finite-dimensional projection $P \in [E]$ such that*

(i) the family of projections $\{P(t) : t \in \mathbb{R}\}$, where $P(t) = U(t, \mathcal{A})PU^{-1}(t, \mathcal{A})$, is precompact in $[E]$, and

(ii) there are positive numbers N and ν such that

$$\|U(t, \mathcal{A})QU^{-1}(\tau, \mathcal{A})\| \leq Ne^{-\nu(t-\tau)}$$

for all $t \geq \tau$, where $Q = I - P$.

Then the linear non-autonomous dynamical system generated by equation (4.1) is $\alpha-$ contraction.

Proof. Since the family of projections $P(t) = U(, \mathcal{A})PU^{-1}(t, \mathcal{A})$ is precompact in $[E]$, then the family $\mathbb{H} = \overline{\{P(t) : t \in \mathbb{R}\}}$ is uniformly completely continuous, where the bar denotes closure in $[H]$. Indeed, let A be a bounded subset of E, $\{x_n\} \subseteq \{QA : Q \in \mathbb{H}\}$, and $\varepsilon_n \downarrow 0$. Then there are $tn \in \mathbb{R}$ and $v_n \in A$ such that $\rho(x_n, P(t_n)v_n) \leq \varepsilon_n$. Since the sequence $\{P(t_n)\}$ is precompact, we can assume that it converges. Let $L = \lim\limits_{n\to+\infty} P(t_n)$. Then L is completely continuous, which implies that the sequence $\{x'_n\} = \{Lv_n\}$ is precompact. Note that

$$\rho(x_n, x'_n) \leq \rho(x_n, P(t_n)v_n) + \rho(P(t_n)v_n, Lv_n) \leq \varepsilon_n + \|P(t_n) - L\||v_n|,$$

which implies that $\rho(x_n, x'_n) \to 0$ as $n \to +\infty$. Hence, $\{x_n\}$ is precompact.

Assume that $\mathcal{B} \in H(\mathcal{A})$ and $\{t_n\} \subset \mathbb{R}$ are such that

$$\mathcal{B} = \lim_{n\to+\infty} \mathcal{A}_{t_n}, P(\mathcal{B}) = \lim_{n\to+\infty} P(\mathcal{A}_{t_n}),$$

where $P(\mathcal{A}_{t_n}) = U(t_n, \mathcal{A})PU^{-1}(t_n, \mathcal{A})$. The assertions proved above imply that the family $\{P(\mathcal{B}) : \mathcal{B} \in H(\mathcal{A})\}$ is uniformly completely continuous. Note that $Q(\mathcal{B}) = \lim\limits_{n\to+\infty} Q(\mathcal{A}_{t_n})$, where $Q(\mathcal{B}) = I - P(\mathcal{B})$ and $Q(\mathcal{A}_{t_n}) = I - P(\mathcal{A}_{t_n})$. Moreover, condition (ii) of Theorem 4.5 implies that

$$\|U(t, \mathcal{A}_{t_n})QU^{-1}(\tau, \mathcal{A}_{t_n})\| \leq Ne^{-\nu(t-\tau)} \tag{4.7}$$

for all $t \geq \tau$. Passing to the limit in (4.7) as $n \to +\infty$ and taking into account Lemma 4.1, we obtain that

$$\|U(t, \mathcal{B})QU^{-1}(\tau, \mathcal{B})\| \leq Ne^{-\nu(t-\tau)}$$

for all $t \geq \tau$ and $\mathcal{B} \in H(\mathcal{A})$. We complete the proof of the theorem by observing that $U(t, \mathcal{B})Q(\mathcal{B}) + U(t, \mathcal{B})P(\mathcal{B}) = U(t, \mathcal{B})$ and applying Theorem 1.5.

Theorem 4.6. *Suppose that the following conditions are satisfied:*

1. the operator-function $\mathcal{A} \in C(\mathbb{R}, [E])$ is recurrent with respect to $t \in \mathbb{R}$.

2. a linear non-autonomous dynamical system $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ generated by equation (4.1) is conditionally $\alpha-$ condensing.

3. the trivial solution of equation (4.1) is uniform stable in the positive direction, i.e. there exist a positive number M such that

$$\|U(t,\mathcal{A})U^{-1}(\tau,\mathcal{A})\| \le M \tag{4.8}$$

for all $t \ge \tau$.

Then there are two vectorial positively invariant subfiberings (X^0,h,Y) and (X^s,h,Y) of (X,h,Y) such that:

a. $X_y = X_y^0 + X_y^s$ and $X_y^0 \bigcap X_y^s = 0_y$ for all $y \in Y$, where $0_y = (0,y) \in X = E \times Y$ and 0 is a zero in the Banach space E.

b. a vectorial subfibering (X^0,h,Y) is finite dimensional, invariant (i.e. $\pi^t X^0 = X^0$ for all $t \in \mathbb{T}_+$) and every trajectory of a dynamical system $(X,\mathbb{T}_+,\pi)$ belonging to X^0 is recurrent.

c. there exist two positive numbers N and ν such that $|\varphi(t,u,\mathcal{B})| \le Ne^{-\nu t}|u|$ for all $(u,\mathcal{B}) \in X^s$ and $t \in \mathbb{T}_+$, where $\varphi(t,u,\mathcal{B}) = U(t,\mathcal{B})u$.

Proof. Assume that $\mathcal{B} \in H(\mathcal{A})$ and $\{t_n\} \subset \mathbb{R}$ are such that $\mathcal{B} = \lim\limits_{n\to+\infty} \mathcal{A}_{t_n}$, then condition (ii) of Theorem 4.6 implies that

$$\|U(t,\mathcal{A}_{t_n})U^{-1}(\tau,\mathcal{A}_{t_n})\| \le N \tag{4.8}$$

for all $t \ge \tau$. Passing to the limit in (4.8) as $n \to +\infty$ and taking into account Lemma 4.3, we obtain that

$$\|U(t,\mathcal{B})U^{-1}(\tau,\mathcal{B})\| \le N$$

for all $t \ge \tau$ and $\mathcal{B} \in H(\mathcal{A})$ and, consequently,

$$\|U(t,\mathcal{B})\| \le N$$

for all $t \ge 0$ and $\mathcal{B} \in H(\mathcal{A})$. Now to finish the proof of Theorem 4.6 it is sufficiently to refer on Theorem 3.5.

4.2. Partial linear differential equations. Let Λ be some complete metric space of linear closed operators acting into Banach space E (for example $\Lambda = \{A_0 + B | B \in [E]\}$, where A_0 is a closed operator that acts on E). We assume that the following conditions are fulfilled for equation (4.1) and its $H-$ class (4.2):

a. for any $v \in E$ and $\mathcal{B} \in H(\mathcal{A})$ equation (4.2) has exactly one solution defined on $\mathbb{R}_+$ and satisfies the condition $\varphi(0,v,\mathcal{B}) = v$;

b. the mapping $\varphi : (t,v,\mathcal{B}) \to \varphi(t,v,\mathcal{B})$ is continuous in the topology of $\mathbb{R}_+ \times E \times C(\mathbb{R};\Lambda)$;

c. for every $t \in \mathbb{R}_+$ the mapping $U(t,\cdot) : H(\mathcal{A}) \to [E]$ is continuous, where $U(t,\cdot)$ is the Cauchy operator of equation (4.2), i.e. $U(t,\mathcal{B})v = \varphi(t,v,\mathcal{B})$ $(t \in \mathbb{R}_+, v \in E$ and $\in H(\mathcal{A})$).

Under the above assumptions the equation (4.1) generates a linear non-autono mous dynamical system $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$, where $X = E \times Y, \pi = (\varphi,\sigma)$ and $h = pr_2 : X \to Y$. Applying the results from § 2 and § 3 to this system, we will obtain the analogous assertions for different classes of partial differential equations.

We will consider examples of partial differential equations which satisfy the above conditions a.-c. .

Example 4.7. A closed linear operator $\mathcal{A} : D(\mathcal{A}) \mapsto E$ with dense domain $D(\mathcal{A})$ is said [21] to be sectorial if one can find a $\varphi \in (0, \frac{\pi}{2})$, an $M \geq 1$, and a real number a such that the sector

$$S_{a,\varphi} = \{\lambda : \varphi \leq |arg(\lambda - a)| \leq \pi, \lambda \neq a\}$$

lies in the resolvent set $\rho(\mathcal{A})$ of $\mathcal{A}$ and $\|(\lambda I - \mathcal{A})^{-1}\| \leq M|\lambda - a|^{-1}$ for all $\lambda \in S_{a,\varphi}$. An important class of sectorial operators is formed by elliptic operators [21,22].

Consider the differential equation

$$u' = (\mathcal{A}_1 + \mathcal{A}_2(t))u, \tag{4.9}$$

where $\mathcal{A}_1$ is a sectorial operator that does not depend on $t \in \mathbb{R}$, and $\mathcal{A}_2 \in C(\mathbb{R}, [E])$. The results of [21,23] imply that equation (4.9) satisfies conditions a.-c..

Example 4.8. Let $\mathcal{H}$ be a Hilbertian space with a scalar product $\langle \cdot, \cdot \rangle = |\cdot|^2, \mathcal{D}(\mathbb{R}_+, \mathcal{H})$ be a set of all infinite differentiable and finite into $\mathbb{R}_+$ functions with values into $\mathcal{H}$.

Denote by $(C(\mathbb{R}, [\mathcal{H}]), \mathbb{R}, \sigma)$ a dynamical system of shifts on $C(\mathbb{R}, [\mathcal{H}])$. Consider the equation

$$\int_{\mathbb{R}_+} [< u(t), \varphi'(t) > + < \mathcal{A}(t)u(t), \varphi(t) >]dt = 0, \tag{4.10}$$

along with the family of equations

$$\int_{\mathbb{R}_+} [< u(t), \varphi'(t) > + < \mathcal{B}(t)u(t), \varphi(t) >]dt = 0, \tag{4.11}$$

where $\mathcal{B} \in H(\mathcal{A}) = \overline{\{\mathcal{A}_\tau | \tau \in \mathbb{R}\}}, \mathcal{A}_\tau(t) = (t + \tau)$ and the bar denotes closure in $C(\mathbb{R}, [\mathcal{H}])$.

The function $u \in C(\mathbb{R}_+, \mathcal{H})$ is called a solution of equation (4.10), if (4.10) takes place for all $\varphi \in \mathcal{D}(\mathbb{R}_+, \mathcal{H})$.

Assume that the operator $\mathcal{A}(t)$ is self-adjoint and negative defined, i.e. $\mathcal{A}(t) = -\mathcal{A}_1(t) + i\mathcal{A}_2(t)$ for all $t \in \mathbb{R}$, where $\mathcal{A}_1(t)$ and $\mathcal{A}_2(t)$ are self-adjoined and

$$< \mathcal{A}_1(t)u, u > \geq \alpha |u|^2 \tag{4.12}$$

for all $t \in \mathbb{R}$ and $u \in \mathcal{H}$, where $\alpha > 0$. Let $(H(\mathcal{A}), \mathbb{R}, \sigma)$ be a dynamical system of shifts on $H(\mathcal{A}), \varphi(t, v, \mathcal{B})$ be a solution of equation (4.11) with condition $\varphi(0, v, \mathcal{B}) = v, \overline{X} = \mathcal{H} \times H(\mathcal{A}), X$ be a set of all the points $\langle u, \mathcal{B} \rangle \in \overline{X}$ such that through point $u \in \mathcal{H}$ passes a solution $\varphi(t, u, \mathcal{A})$ of equation (4.10) defined on $\mathbb{R}_+$. According to Lemma 2.21 [12] the set X is closed in $\overline{X}$. In virtue of Lemma 2.22 [12] the triple $(X, \mathbb{R}_+, \pi)$ is a dynamical system on X (where $\pi = (\varphi, \sigma)$) and $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle$ is a linear non-autonomous dynamical system, where $h = pr_2 : X \to Y = H(\mathcal{A})$. Applying the results from [1] it is possible to show that for every t the mapping $\mathcal{B} \mapsto U(t, \mathcal{B})$ (where $U(t, \mathcal{B})v = \varphi(t, v, \mathcal{B})$) from $H(\mathcal{A})$ into $[\mathcal{H}]$ is continuous and, consequently, for this system is applicable Theorem 2.2 . Thus the following assertion takes place.

Theorem 4.9. *Let $\mathcal{A} \in C(\mathbb{R}, [\mathcal{H}])$ be compact then the following assertion hold:*

1. a trivial solution of equation (4.1) is uniformly exponentially stable, i.e. there exist positive numbers N and ν such that $\|U(t, \lfloor)\| \le Ne^{-\nu t}$ for all $t \ge 0$ and $\mathcal{B} \in H(\mathcal{A})$.

2. $\lim\limits_{t\to+\infty} \sup\{\|U(t,\mathcal{B})\| : \mathcal{B} \in H(\mathcal{A})\} = 0$.

Proof. According to Lemma 2.23 [12] from inequality (4.12) follows the condition 1. In virtue of Theorem 2.2 the cionditions 1. and 2. are equivalent. The Theorem is proved.

We will give the example of limit problem reducing to equation of type (4.10). Let Ω be a bounded domain in $\mathbb{R}^n, \Gamma$ be frontier of $\Omega, Q = \mathbb{R}_+ \times \Omega$ and $S = \mathbb{R}_+ \times \Gamma$. Consider in Q the first initial limit problem for equation

$$\frac{\partial u}{\partial t} = L(t)u \qquad (u|_{t=0} = \varphi, u|_S = 0), \tag{4.13}$$

where

$$L(t)u = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(t,x)\frac{\partial u}{\partial x_j}) - a(t,x)u$$

The operator $\mathcal{A}(t)$ according to theorem of Riesz is defined by equality

$$\langle \mathcal{A}(t)u, \varphi\rangle = -\int_\Omega [\sum_{i,j=1}^{n} a_{ij}(t,x)\frac{\partial u}{\partial x_j}\frac{\partial \varphi}{\partial x_i} + a(t,x)u\varphi]dx.$$

If $a_{ij}(t,x) = a_{ji}(t,x)$ and the functions $a_{ij}(t,x)$ and $a(t,x)$ are bounded and uniformly continuous with respect to $t \in \mathbb{R}$ uniformly with respect to $x \in \Omega$, then for equation (4.13) is applicable Theorem 2.2, if $\mathcal{H} = \dot{W}_2^1(\Omega)$

4.3. Linear functional-differential equations.

Let $r > 0, C([a,b], \mathbb{R}^n)$ be a Banach space of all continuous functions $\varphi : [a,b] \to \mathbb{R}^n$ with sup-norm. If $[a,b] = [-r,0]$, then we put $\mathcal{C} = C([-r,0], \mathbb{R}^n)$. Let $\sigma \in \mathbb{R}, \alpha \ge 0$ and $u \in C([\sigma - r, \sigma + \alpha], \mathbb{R}^n)$. For any $t \in [\sigma, \sigma + \alpha]$ we define $u_t \in \mathcal{C}$ by equality $u_t(\theta) = u(t+\theta), -r \le \theta \le 0$. Denote by $\mathfrak{A} = \mathfrak{A}(C, \mathbb{R}^n)$ a Banach space of all linear continuous operators acting from $\mathcal{C}$ into $\mathbb{R}^n$, equipped by operational norm. Consider the equation

$$u' = \mathcal{A}(t)u_t \quad , \tag{4.14}$$

where $\mathcal{A} \in C(\mathbb{R}, \mathfrak{A})$. We put $H(\mathcal{A}) = \overline{\{\mathcal{A}_\tau : \tau \in \mathbb{R}\}}, \mathcal{A}_\tau(t) = \mathcal{A}(t+\tau)$ and the bar denotes closure in the topology of uniform convergence on every compact from $\mathbb{R}$.

Along with equation (4.14) we also consider the family of equations

$$u' = \mathcal{B}(t)u_t \quad , \tag{4.15}$$

where $\mathcal{B} \in H(\mathcal{A})$. Let $\varphi_t(v, \mathcal{B})$ be a solution of equation (4.15) with condition $\varphi_0(v, \mathcal{B}) = v$ defined on $\mathbb{R}_+$. We put $Y = H(\mathcal{A})$ and denote by $(Y, \mathbb{R}, \sigma)$ a dynamical system of shifts on $H(\mathcal{A})$. Let $X = C \times Y$ and $\pi = (\varphi, \sigma)$ a dynamical system on X, defined by equality $\pi(\tau, (v, \mathcal{B})) = (\varphi_\tau(v, \mathcal{B}), \mathcal{B}_\tau)$. A non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$ $(h = pr_2 : X \to Y)$ is linear. The following assertion takes place.

Lemma 4.10[4]. *Let $H(\mathcal{A})$ be compact in $C(\mathbb{R},\mathfrak{A})$, then the non-autonomous dynamical system $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$ generated by equation (4.14) is completely continuous.*

Theorem 4.11. *Let $H(\mathcal{A})$ be compact. Then the following assertions are equivalent:*

1. for any $\mathcal{B}\in H(\mathcal{A})$ the zero solution of equation (4.15) is asymptotically stable, i.e. $\lim\limits_{t\to+\infty}|\varphi_t(v,\mathcal{B})|=0$ for all $v\in C$ and $\mathcal{B}\in H(\mathcal{A})$;

2. the zero solution of equation (4.14) is uniformly exponentially stable, i.e. there are positive numbers N and ν such that $|\varphi_t(v,\mathcal{B})|\le Ne^{-\nu t}|v|$ for all $t\ge 0, v\in C$ and $\mathcal{B}\in H(\mathcal{A})$.

Proof. Let $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$ be a linear non-autonomous dynamical system, generated by equation (4.14). According to Lemma 4.10 this system is completely continuous and to finish the proof it is sufficiently to refer on Theorems 2.4.

Consider the neutral functional differential equation

$$\frac{d}{dt}Dx_t=\mathcal{A}(t)x_t \quad , \tag{4.16}$$

where $\mathcal{A}\in C(\mathbb{R},\mathfrak{A})$ and $D\in\mathfrak{A}$ is nonatomic at zero operator [19, p.67]. As well as in the case of equation (4.14), the equation (4.16) generates a linear non-autonomous dynamical system $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$, where $X=C\times Y, Y=H(\mathcal{A})$ and $\pi=(\varphi,\sigma)$. The following statement takes place.

Lemma 4.12. *Let $H(\mathcal{A})$ be compact and the operator D is stable, i.e. the zero solution of homogeneous difference equation $Dy_t=0$ is uniformly asymptotically stable. Then a linear non-autonomous dynamical system $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$, generated by equation (4.16), is conditionally $\alpha-$ condensing.*

Proof. According to [20, p.119, formula (5.18)] the mapping $\varphi_t(\cdot,\mathcal{B}):\mathcal{C}\to\mathcal{C}$ can be written as

$$\varphi_t(\cdot,\mathcal{B})=S_t(\cdot)+U_t(\cdot,\mathcal{B})$$

for all $\mathcal{B}\in H(\mathcal{A})$, where $U_t(\cdot,\mathcal{B})$ is conditionally completely continuous for $t\ge r$. Also there exist positive constants N,ν such that $\|S_t\|\le Ne^{-\nu t}(t\ge 0)$. Then the proof is completed by referring to Theorem 1.5.

Theorem 4.13. *Let $\mathcal{A}\in C(\mathbb{R},\mathfrak{A})$ be recurrent (i.e. $H(\mathcal{A})$ is compact minimal set in the dynamical system of shifts $(C(\mathbb{R},\mathfrak{A}),\mathbb{R},\sigma)$) and D is stable, then the following assertions are equivalent:*

1. the zero solution of equation (4.14) and all equation

$$\frac{d}{dt}Dx_t=\mathcal{B}(t)x_t \quad , \tag{4.17}$$

where $\mathcal{B}\in H(\mathcal{A})$, is asymptotically stable, i.e. $\lim\limits_{t\to+\infty}|\varphi_t(v,\mathcal{B})|=0$ for all $v\in\mathcal{C}$ and $\mathcal{B}\in H(\mathcal{A})$ ($\varphi_t(v,\mathcal{B})$ is a solution of equation (4.17) with condition $\varphi_0(v,\mathcal{B})=v$);

2. the zero solution of equation (4.16) is uniformly exponentially stable, i.e. there are positive numbers N and ν such that $|\varphi_t(v,\mathcal{B}|\le Ne^{-\nu t}|v|$ for all $t\ge 0, v\in\mathcal{C}$ and $\mathcal{B}\in H(\mathcal{A})$.

3. all solutions of all equations (4.17) are bounded on $\mathbb{R}_+$ *and they don't have non-trivial solutions bounded on* $\mathbb{R}$ *.*

Proof. Let $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle$ be a linear non-autonomous dynamical system, generated by equation (4.16). According to Lemma 4.12 this system is conditionally $\alpha-$ condensing. To finish the proof of Theorem 4.13 it is sufficiently to refer on Theorems 2.3 and 3.3. The theorem is proved.

Theorem 4.14. *Let* $\mathcal{A} \in C(\mathbb{R}, \mathfrak{A})$ *be recurrent (i.e.* $H(\mathcal{A})$ *is compact minimal in the dynamical system of shifts* $(C(\mathbb{R}, \mathfrak{A}), \mathbb{R}, \sigma)$ *), D is stable, and all solutions of all equations (4.17) are bounded on* $\mathbb{R}_+$*.*

Let $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h \rangle$ *be a linear non-autonomous dynamical system generated by equation (4.16).*

Then there are two vectorial positive invariant subfiberings (X^0, h, Y) *and* (X^s, h, Y) *of* (X, h, Y) *such that:*

a. $X_y = X_y^0 + X_y^s$ *and* $X_y^0 \bigcap X_y^s = 0_y$ *for all* $y \in Y$*, where* $0_y = (0, y) \in X = E \times Y$ *and* 0 *is a zero in the Banach space* E*.*

b. a vectorial subfibering (X^0, h, Y) *is finite dimensional, invariant (i.e.*$\pi^t X^0 = X^0$ *for all* $t \in \mathbb{T}_+$*) and every trajectory of a dynamical system* $(X, \mathbb{T}_+, \pi)$ *belonging to* X^0 *is recurrent.*

c. there exist two positive numbers N *and* ν *such that* $|\varphi_t(v, \mathcal{B}| \leq N e^{-t}|u|$ *for all* $(u, \mathcal{B}) \in X^s$*, where* $\varphi_t(v, \mathcal{B} = U(t, \mathcal{B})u$ *and* $U(t, \mathcal{B})$ *is the Cauchy operator of equation (4.17).*

Proof. Let $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h \rangle$ be a linear non-autonomous dynamical system generated by equation (4.16). In virtue of Lemma 4.12 this non-autonomous dynamical system is conditionally $\alpha-$ condensing. According to Theorem 1.6 [5] there exists a positive number M such that $|\varphi_t(v, \mathcal{B})| \leq M|v|$ for all $t \geq 0, v \in C$ and $\mathcal{B} \in H(\mathcal{A})$. To finish the proof of Theorem 4.14 it is sufficiently to refer to Theorem 3.5.

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